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ABSTRACT

This is the teacher's edition of a text for the second year of a two-year high school geometry course. The course bases plane and solid geometry and trigonometry on the fact that the translations of a Euclidean space constitute a vector space which has an inner product. Congruence is a geometric topic reserved for Volume 2. Volume 2 opens with an analysis of basic properties of perpendicularity and distance which leads to the introduction of an inner product of translations and to the development of Euclidean geometry and trigonometry. The basic facts concerning volume-measures of solids are dealt with in an appendix to Volume 2. The commentary contains answers to all exercises and questions raised in the text, sample (or suggested) quizzes, keys to the chapter tests, suggestions to the teacher, and a great deal of mathematical and logical background material which has proved to be helpful in orienting teachers in preparation for teaching the course. (Author/MK)

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In memory of Max Beberman, a colleague, mathematics teacher, and above all, a friend

SE 030 756

A Vector Approach to Euclidean Geometry

Inner Product Spaces,
Euclidean Geometry and Trigonometry

HERBERT E. VAUGHAN & STEVEN SZABO

Volume 2

TEACHER'S EDITION

University of Illinois Committee on School Mathematics
The Macmillan Company, New York, New York
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INTRODUCTION

Volume 2 of *A Vector Approach to Euclidean Geometry* is the text for the second year of a two-year high school mathematics course which bases plane and solid geometry and trigonometry on the fact that the translations of a euclidean space constitute a vector space which has an inner product. Volume 1 deals largely with affine geometry, and the notion of dimension is introduced in the final chapter, Chapter 10, of that volume. Up to this point, although the students may be thinking in terms of 3-dimensional space, the space under consideration might be of any dimension. The principal geometric topics treated in Volume 1 deal with parallelism of lines and planes and with ratios, making it possible to treat a good deal of the geometry of triangles and quadrilaterals. Congruence is a geometric topic reserved for Volume 2, after the introduction of an inner (or, dot) product into the formal operational scheme. Volume 2 opens with an analysis of basic properties of perpendicularity and distance which leads to the introduction of an inner product of translations and to the development of Euclidean geometry and trigonometry. The basic facts concerning volume-measures of solids are dealt with in an appendix to Volume 2. By the time students complete Volume 2, they will have become acquainted with the theorems dealt with in standard American geometry and trigonometry texts and with numerous additional geometry theorems, both analytic and synthetic. They will also have a good foundation of knowledge of a subject of contemporary mathematical importance, namely, that of linear vector spaces.

The preparation of the present course began in 1963 and has been carried out with support from the National Science Foundation. Beginning with the 1963-64 school year, the course has been taught in an ever-growing number of schools throughout the country. It has also been the basis for teacher education courses in NSF institutes. In most schools using the course it has been taught in grades 10 and 11, but it has also been used successfully in grades 11 and 12 and in grades 9 and 10.

In common with other UICSM courses, this one lays stress on students' discovery of concepts through doing appropriate exercises. The course also stresses logic and proof, and students are aided in developing the ability to write proofs, preferably in the form of paragraphs. To this end, some of the beginning chapters in Volume 1 deal at some length with rules of logic in the contexts of the proofs of early theorems. No additional rules of logic are introduced in Volume 2. However, the students will find many opportunities to make use of their knowledge of such rules in working with the mathematical ideas developed in Volume 2.

By the nature of the course much of the development of geometry is algebraic in nature and the algebra resembles [and includes] that of the real numbers to such an extent that previously acquired algebraic skills are maintained and developed further. Lest the reader thinks that this algebraization leads to the loss of much of the beauty of conventional geometry, it bears pointing out that after a sufficient number of geometrical results have been established algebraically other theorems can be derived synthetically from them. It should also be noted that, beginning in Chapter 10 and to a greater extent in Volume 2, it becomes easy to introduce coordinate methods and deal with geometric matters analytically. As becomes obvious, however, the algebra of analytic geometry is in many instances less efficient than is the algebra of points and translations which is developed in this course.

In each of the volumes some attempt has been made in the Background Topics, which close many of the chapters, to familiarize the students with topics in algebra with which they may not be acquainted. Nevertheless, there are some standard topics of high school algebra which are not covered in either a first year algebra course or the two volumes of text for this course. Consequently, in order to complete the student's high school mathematics program, a second year algebra course should be offered. This course should include topics such as radicals, exponents, logarithms, and polynomial functions, and may be offered to the students as a fourth-year course after they complete their study of Volume 2 or as a second-year course between first-year algebra and Volume 1. Each of these alternatives has been successfully employed with classes of students in public school trials of the course.

This commentary contains answers to all exercises and questions raised in the text, sample (or, suggested) quizzes, keys to the chapter tests, suggestions to the teacher, and a great deal of mathematical and logical background material which has proved to be helpful in orienting teachers in preparation for teaching the course.

The following is a brief lesson guide for Volume 2. It gives suggestions on time allotments for covering sections of the text together with some comments on the kinds of activities and assignments which are appropriate in those sections. The guide suggests the administering of "sample quizzes" from time to time, and allows full lessons for chapter tests. Sample quizzes have been included in the commentary on the following pages:

TC20 TC33 TC41(3) TC57(3) TC62(2) TC81(2)
TC85(3) TC90 TC93 TC160(2) TC169 TC177
TC221(1) TC227(2) TC239(3) TC271(2) TC281(2) TC300
TC305 TC326 TC328 TC349(2) TC361 TC388(2)
TC397 TC415(2) TC419 TC426(2) TC439(2) TC456

Suggested Teaching Schedule for Volume 2

Lesson	Section	Possible activities
1	Introduction	Do most or all of A, B in class; assign rest of A, B, C1-3.
2	Introduction	Discuss C4-5; assign C6, A.
3	Introduction	Discuss some of B in class; assign rest of B. [See note in commentary on assignments.]
4	11.01	Discuss text preceding and following A; do A in class; assign B.
5	11.01	Discuss text preceding C; do C in class; C3(c) good problem for review of tree-diagram; discuss text following C; assign D, asking for informal arguments. [See note in commentary on notation.]
6	11.02	Discuss text; do 1-2 in class; sample quiz; assign 3 - 7.
7	11.03	Discuss text preceding A; do A in class; discuss ideas in text between A and B [A model of a 3-dimensional coordinate system is helpful here.]; assign B.
8	11.03	Go through derivations in text; review Notions 1-4; assign C. Long term assignment: Begin looking into Background Topic in section 11.12. All solutions due by the time the Chapter Test is given (about 3 weeks).
9	11.04	Discuss Notions 5-7 and do A in class; review sense of vector; assign B.
10	11.05	Go through derivation; do A in class; do B in class; assign answering question in text preceding C or C.
11	11.05-.06	Discuss page 34-5 briefly; text pages 35-8 summarize preceding work; do some of 1-2 in class, using sentences from summary [See note in commentary.]; sample quiz; assign 3-5.
12	11.07	Discuss text; do as many of exercises in class as time permits; assign rest.
13	11.08	Discuss text; do A in class; divide proofs in B among class, having each do a few; sample quiz. [Alternately, see note in commentary on handling this section.]
14	11.08	Discuss text; do A1-2, B1-3 in class; assign A3, B4-6.

Lesson	Section	Possible activities
15	11.08-.09	Do C1, 4a-c in class; study Theorem 11-3; do A1-3 in class; assign C2-3, 4d-f and A4-6.
16	11.09	Do B1-4 in class; do some of C in class; assign rest of B, C.
17	11.09	Do D, E1-4, and F1 in class; assign rest of E, F.
18	11.10	Sketch proofs of Theorem 11-5b-e in class; discuss Definition 11-2, Theorem 11-6; do as much of A as time permits; assign rest of A.
19	11.10	Do B1, 2, 4 in class; do C1-4 in class; assign B3, 5, C5; use D as extra-credit work.
20	11.11	Discuss Lemma and Theorem 11-9; do 3-4 in class; sample quiz, assign 1-4. [Alternately, see note in commentary.]
21	11.11	Do A1-3, B1-3 in class; assign A4, B4-6; quiz.
22	11.12	Use Chapter Test as hourly exam or as a take-home exam. If used as a take-home exam, spend the period discussing the Background Topic.
23	12.01	Discuss theorems, sketching proofs [See note in the commentary.]; assign B. Long-term assignment: Begin looking into Background Topic in section 12.05. Have all solutions complete before Chapter Test (about 2 weeks).
24	12.01	Discuss Definition 12-2, Theorem 12-8; do A1-2, 4a in class; assign A3, 4b, 5-6. [Alternately, see note in commentary.]
25	12.01	Do B, C in class; assign D; sample quiz
26	12.02	Discuss text, answering questions [This completes A1-4.]; assign A5, B1-4. [Alternately, see note in commentary.]
27	12.02	Do B5-8, C1-2 in class; assign B9-11, C3-4; quiz.
28	12.02	Do D1-4 in class; assign D5-8.
29	12.03	Study Definition 12-4, 12-5; give informal proofs for Theorems 12-18 to 12-23 as time permits; assign A2, 3, 5, 6 [Alternately, see note in commentary.]; sample quiz.
30	12.03	Do B1-4, C1-4 in class; assign B5, C5-8; quiz.

Lesson	Section	Possible activities
31	12.04	Discuss theorems, example in text; do A1-2, B1-4. C1-2 in class; assign A3-4, B5-6, C3-4.
32	12.05	Use Chapter Text as hourly exam or as take-home exam.
33	13.01	[See note in commentary on using Chapter 10 test as combined class discussion and homework assignment.] Sentences (1)-(9) serve as a summary of some of what we learned about equations of planes and lines. Center the discussion around the derivation of (10), the example following, and the exercises in Part A.
34	13.01	Discuss some of B, C; assign rest. Long-term assignment: Background Topic in section 13.05. Have solutions completed before Chapter Test (about 2 weeks).
35	13.01	Do D in class; Do E in class and tie result in with text following it; assign F.
36	13.01	Discuss G1-4; assign rest of G.
37	13.02	Discuss A, B in class; assign C.
38	13.02	Discuss D and following text in class, using E for examples; assign F.
39	13.03	Discuss the process of generating an orthonormal basis as done in the text; note Theorem 13-4 and its corollary; assign A. [Use B, C as extra-credit assignments or have several of the better students prepare to lead short discussions about these exercises.]
40	13.04	Discuss A, B and related text in class; assign C1-5.
41	13.04	Discuss C6-9 and following text in class (this takes care of D); assign E.
42	13.04	Discuss F and following text in class; assign G.
43	13.05	Use Chapter Test as hourly exam or as take-home test.
44	14.01	Write Schwartz inequality on chalkboard; discuss A, Lemma, B, D1, 2a in class; assign C1-3, D2b, 3, 4; C4 might be used as extra-credit problem.
45	14.01-.02	Do E as class work; discuss Definition 14-1 and Theorems 14-2 and 14-3 and their proofs; Do A3 in class or assign as extra-credit problem; assign B1-4. [Alternately, on 14.02, see commentary.]

Lesson	Section	Possible activities
46	14.02	Discuss text pages preceding C in class; do some of proofs in class; assign C1-4; use C5 as extra-credit problem.
47	14.02	Discuss definitions and theorems in text; do D1-3, E1-3 in class; assign D4, E4-7; sample quiz.
48	14.02	Do F, G1-2 in class; assign G3-5.
49	14.02	Discuss terms and Theorems 14-12, 14-13; do some of H in class; assign rest of H; [optional] might choose one or two students to work on I so that they could explain the topic discussed to the rest of the class.
50	14.02	Begin J in class; have at least one part of 5 done in detail in class; assign those problems not discussed in class; sample quiz.
51	14.03	Discuss text through Theorem 14-16 [Note. Text preceding A outlines the proof of Theorem 14-16(b). Don't expect all to go through this.]; do A, B1, 2 in class; assign B3-6.
52	14.03	Get at explanations of (8) and (9); do C1-4 in class; assign C5-6.
53	14.03	Discuss text through Theorem 14-17; Do D1-2, E1-2 in class; assign D3-6, E3-5; use E6 as extra-credit problem.
54	14.04	Discuss definitions and theorems; do A, B1-3 in class; assign B4-5, C [Alternately, see teacher's commentary.]; sample quiz.
55	14.04	Do D, E in class; assign F.
56	14.04	Do G in class; discuss Lemma, Theorem 14-24; do H1, I1 in class; assign H2, I2-5.
57	14.05	Discuss text and do A1-3 in class; assign A4-10.
58	14.05	Discuss proof of Lemma; do B in class; get started on C and assign at least through C4.
59	14.05	Do D1-4 and E1 in class; assign rest.
60	14.05	Discuss Theorem 14-29; read Theorem 14-30 and its corollary; discuss text leading to Theorem 14-31; assign F.
61	14.06	Use Chapter Test as hourly exam or as a take-home test.
62	14.06	Discuss notion of mathematical induction in terms of text material; assign representative problems.
63	15.01	Discuss new terms; do A, B1 in class; assign rest of B.
64	15.02	Discuss new terms; prove Theorem 15-3; do A in class [Alternately, see note in commentary.]; assign B3 and any two problems in C.
65	15.02	Go over results in D, E in view of text discussion; note (1)-(3) following E and how Theorem 15-5 follows from them; assign appropriate parts of D, E.
66	15.03	Introduce cosine; do A in class; discuss text leading to Theorem 15-7; assign B.
67	15.03	Do C1-4, 6, D1, 2(a) in class; assign rest of C, D.
68	15.03	Discuss Theorem 15-8; assign E.
69	15.04	Discuss terms and theorems; do some of A in class; assign rest of A [Alternately, see note in commentary.]; sample quiz.
70	15.04	Do B, C1 in class; assign C2-4.
71	15.05	Introduce terms; do A1 in class; [Alternately, see note in commentary.]; assign A2-6.
72	15.05	Prove Theorem 15-14; do C1-3 in class; assign B3, C4-6; sample quiz.
73	15.06	Introduce terms; discuss Theorems 15-15 and 15-16; assign A. [Alternately, see note in commentary.]
74	15.06	Discuss terms and Theorem 15-17; review Ceva's theorem and discuss Theorems 15-18 to 15-20; do B1; assign B2-5.
75	15.06	Do parts of C, D, E in class; assign rest.
76	15.07	Discuss terms; do A, B1 in class; assign B2-5. [Alternately, see note in commentary.]
77	15.07	Study Definition 15-10; do C1-4, D1 in class; assign C5-6, D2-3.
78	15.08	Discuss Definition 15-11; do A, B in class; assign C; sample quiz.
79	15.09	Study discussion leading to Definition 15-12; do A1-3, B1-3 in class; assign rest of A and B.

Lesson	Section	Possible activities
80	15.09	Do C1-2, D1-3 in class; assign rest of C, D.
81	15.10	Do A1-3, B1-3 in class; assign rest of A, B.
82	15.11	Use Chapter Test as hourly exam or as a take-home test.
83	15.11	Discuss Theorem and its corollary; review mathematical induction; do A in class; assign B.
84	16.01	Prove Theorems 16-1 and 16-2; assign A3, B1-2. [Alternately, see note in commentary.]
85	16.01	Do B3-5 in class; assign C, allowing students to work in teams.
86	16.01	Do D1-2 in class; assign D3-6, E1-2.
87	16.01	Do E3-6, F1 in class; assign F2-4.
88	16.01	Do some of each of G, H, I in class; assign rest.
89	16.02	Introduce new terms; do A1-2 in class; assign A3-7.
90	16.02	Introduce new terms; do B, C1-2 in class; assign C3-10.
91	16.02	Do D in class; study the corollaries; assign E.
92	16.03	Assign A [Alternately, see note in commentary.]; sample quiz.
93	16.03	Note Theorem 16-8 [This is one whose proof is quite complex and is discussed in the text between Parts B and C. Shouldn't be required of all students.]; study Theorems 16-9, 16-10; assign E, F.
94	16.04	Study text preceding A; do A1-3 in class; assign A4-8.
95	16.04	Introduce new terms; do B1-2 in class; assign B3-5, C.
96	16.05	Do A, B1-2 in class; assign B3-4, C; sample quiz.
97	16.05	Do D1-2 in class; assign D3-4, E.
98	16.06	Do A in class; assign B.
99	16.06	Do C1, D1-2 in class; assign rest of C, D.
100	16.07	Do A1-2 in class; assign rest of A. [Alternately, see note in commentary.]

Lesson	Section	Possible activities
101	16.07	Do B1, 3 and C1-4 in class; assign rest of B, C.
102	16.07	Discuss new terms and theorems; assign D.
103	16.08	Have Exploration Exercises done at beginning of period; discuss new terms and Theorem 16-25; assign A.
104	16.08	Discuss Theorems 16-27, 16-28, 16,29; do B, C in class; assign D1-3.
105	16.08	Do D4-5, E1 in class; assign E2-3.
106	16.09	Study text preceding A; do several from A, B in class; assign rest of A, B; sample quiz.
107	16.09	Do some of C, D in class; assign rest of C, D; quiz.
108	16.10	Use Chapter Test as take-home test.
109	16.10	Discuss signum and greatest integer functions and least upper bound principle; assign problems not done in class.
110	17.01	Introduce new terms; do A1-3, B1-5 in class; assign A4-6, B6-8.
111	17.01	Do C, D1-3 in class; assign D4-7.
112	17.01	Do E1-4 in class; assign E5-7.
113	17.02	Introduce new terms; do A1-3, B1-3 in class; assign rest of A, B.
114	17.03	Discuss Theorems 17-8, 17-9, 17-10; introduce new terms; do A1-2, B1-2 in class; assign A3-5, B3-4; sample quiz.
115	17.03	Do as much of C, D in class as time permits; assign rest; sample quiz.
116	17.04	Introduce new terms; do A1-3 in class; assign A4-5, B.
117	17.04	Review sgn function; discuss C1-6 in class; assign C7-11.
118	17.04	Do as much of D, E in class as time permits; assign rest.
119	17.04	Discuss text preceding F; do F1 in class; assign F2-4.
120	17.05	Do G in class; discuss least upper bound property; do Exercises 1, 2 in class; assign 3-6.

Lesson	Section	Possible activities
121	17.06	Discuss text preceding A; assign A.
122	17.06	Discuss text preceding B; do B1-2, C1-2 in class; assign rest of B, C.
123	17.06	Study theorems; assign D; sample quiz.
124	17.07	Introduce new terms; do A1-2, B1 in class; assign rest of A, B.
125	17.07	Do C1-4, D1-4 in class; introduce new terms; assign C5-6, D6-10; use D5 as extra-credit problem.
126	17.07	Study theorems, text preceding F; do E in class; assign F.
127	17.08	Do A1-4, B1-2 in class; assign A5-8, B3-4, C.
128	17.09	Discuss terms; assign A [Alternately, see note in commentary.]; sample quiz.
129	17.09	Discuss text preceding B; do B1-2, C in class; assign rest of B, C.
130	17.10	Use Chapter Test as hourly exam or as take home test.
131	17.10	Introduce terms; do some of A, B in class; assign rest.
132	18.01-.02	Introduce terms; do Exercises in class; discuss text preceding A; do A1-3, B1, 3 in class; assign A4-5, rest of B.
133	18.02	Do C1 in class; assign C2-6.
134	18.03	Study Definition 18-1; do A1-3, B1-2 in class; assign rest of A, B.
135	18.03	Study text preceding C; assign C.
136	18.04	Study text preceding A; do A1-3 in class; assign A4-6.
137	18.04	Note Theorem 18-4; do some of B1-2, C1-3; assign rest of B, C; sample quiz.
138	18.05	Introduce terms; do A1-3 in class; assign A4-6.
139	18.05	Introduce terms; do B1-3, C1-2 in class; assign rest of B, C.
140	18.05	Do D1-4, E1-2 in class; assign rest of D, E.
141	18.05	Study text preceding F; do some of F1-2; assign rest of F.
142	18.06	Introduce terms; assign A; sample quiz.
143	18.06	Do B1-2; assign rest of B.
144	18.07	Use Chapter Test as take-home exam.
145	18.07	Discuss text; do A, B in class; assign C.
146	18.07	Do some parts of D, E, F; assign rest.
147	19.01	Discuss Definition 19-1 and lemmas; assign A.
148	19.01	Prove Theorems 19-1, 19-2; assign B3-5.
149	19.02	Discuss text preceding A; start A1 [Alternately, see note in commentary.]; assign A.
150	19.02	Do B in class; assign C; sample quiz.
151	19.03	Discuss text preceding A; do A1, 3-6 in class; assign A2, 7-11. [Alternately, see note in commentary.]
152	19.03	Study theorems listed preceding B; discuss parts (a) and (b) of Theorem 19-13, 19-14; assign (c) and (d) of these theorems; sample quiz.
153	19.03	Discuss text preceding C; assign C.
154	19.03	Discuss text preceding D; do D1-3 in class; assign D4-6.
155	19.03	Study text preceding E; assign E.
156	19.03	Study text preceding F; assign F.
157	19.04	Introduce terms; assign A [Alternately, see note in commentary.]; sample quiz.
158	19.04	Do some of B, C in class; assign rest.
159	19.04	Do some of D, E in class; assign rest of D, E and Exploration Exercises.
160	19.05	Discuss text preceding A [Alternately, see note in commentary.]; assign A.
161	19.05	Do some parts of B in class; assign rest.
162	19.05	Study text and theorems preceding C; assign C.
163	19.05	Do some parts of D in class; assign rest of D; sample quiz.

Lesson	Section	Possible activities
164	19.06	Study text and theorems preceding A; assign A; use B as extra-credit part. [Alternately, see note in commentary.]
165	19.07	Discuss summary given in text. [See note in commentary.]
166	19.08	Study text preceding A; do A1-3 in class; assign A4-5. [Alternately, see note in commentary.]
167	19.08	Discuss B, C in class; assign D.
168	19.08	Do as much of E in class as time permits; assign selected problems from rest of E.
169	19.08	Do as much of F in class as time permits; assign selected problems from rest of F.
170	19.08	Do G1 in class; assign G2-4; sample quiz.
171	19.09	Discuss text; do A1-2, B1, C1 in class; assign rest of A, B, C. [Alternately, see note in commentary.]
172	19.09	Study text preceding D; do some of D, E in class; assign rest of D and E.
173	19.10	Study text; introduce terms; do some parts of A, B in class; assign rest.
174	19.10	Discuss definitions, theorems preceding C; do parts of C; assign rest.
175	19.10	Do part (a) of each exercise in D in class; assign rest of D.
176	19.11	Discuss sample solutions given in text; do A in class; assign B1-8.
177	19.11	Assign B9-18.
178	19.11	Use Chapter Test as hourly exam or as take-home test.

TO THE STUDENTS:

This book is the text for the second year of the two-year course in mathematics which you started on last year. You will find the postulates, definitions, and other theorems which you studied in Volume 1 listed at the end of this book. Reading them over and thinking about them would be a good review and give you a running start on this year's work. You will also review some of the more important ideas from last year when you study the Introduction to this book.

If you think about it, you will probably agree that two of the most important notions in Volume 1 are those of parallelism and ratio. These ideas are basic for the kind of geometry which is called *affine geometry*, and it is this kind of geometry which you studied in Volume 1. You will find that the most important new ideas in this volume are those of perpendicularity and distance. They, together with parallelism and ratio, are basic for a more special kind of geometry called *Euclidean* [*metric*] *geometry*. We shall define perpendicularity and distance in terms of a new operation on translations. We shall adopt additional postulates for this operation in Chapter 11. [Since the two volumes make up one course, we have numbered the chapters consecutively in Volumes 1 and 2. Chapter 11 is, after the Introduction, the first chapter of this book.] In Euclidean geometry we can study many more properties of triangles, quadrilaterals, etc., as well as properties of circles and spheres.

The new operation on vectors which is introduced in Chapter 11 makes it possible to define some important functions which are related to angles. You will study these in Chapter 15 and, at more length, in Chapter 19. The study of these functions is what is usually called "trigonometry" [triangle measuring].

Chapter 13 contains some more of the work in "analytic geometry" you were introduced to when you studied about coordinate systems in Chapter 10. In this chapter you will learn to solve some geometry problems analytically by taking advantage of your knowledge about perpendicularity. And you will continue to learn more about such matters as you proceed in your study of this volume.

Several of the chapters of this book are followed by "background topics" most of which are intended to enlarge your understanding of the algebra of the real numbers. Two of them deal with another important number system — that of the complex numbers.

As was the case with Volume 1, this is a book to be read and thought about, and you are given chances in the exercises to discover for yourself. We hope you make use of these.

Herbert E. Vaughan & Steven Szabo

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Introduction

In our study of the geometry of Euclidean space we covered [in Volume 1] how translations act on points, how translations react with one another, and how real numbers operate on translations. This has led us to adopt five postulates concerning the set $\mathscr{E}$ of points, the set $\mathscr{T}$ of translations, and the set $\mathscr{R}$ of real numbers. These are:

Postulate 1. (a) $B - A \in \mathscr{T}$ (b) $A + \overrightarrow{a} \in \mathscr{E}$
Postulate 2. (a) $A + (B - A) = B$ (b) $\overrightarrow{a} = (A + \overrightarrow{a}) - A$
Postulate 3. $(B - A) + (C - B) = C - A$
Postulate 4′. $\mathscr{T}$ is a 3-dimensional vector space over $\mathscr{R}$.
Postulate 5′. $\mathscr{R}$ is an ordered field.

Each of the last two postulates consists of several parts. These are given on page 501 of this book.

Using the notions of linear dependence and linear independence of vectors [Definitions 6-2 and 6-3 on page 502] we have been able to define 'line' and 'plane' and to prove various theorems concerning lines and planes. These theorems, and the definitions adopted in Volume 1, are given on pages 501–513. Looking back you will see that much of our work dealt with the notions of parallelism and of ratio.

In this volume we wish to complete our study of Euclidean geometry. In addition to parallelism of lines and planes we shall deal with perpendicularity—of planes to lines, of lines to lines, and of planes to planes. In addition to ratio we shall deal with the notion of the distance between two points. In Volume 1 we used ratio to compare parallel intervals as to size; distance will enable us to compare nonparallel intervals. We shall see how, using perpendicularity and distance, we can compare sizes of angles, deal with similar and with congruent triangles, and prove theorems concerning special kinds of triangles and quadrilaterals—for examples, isosceles triangles and rectangles. In dealing with distances and with measures of angles we shall need to learn more about the real numbers.

Before we begin to deal with these matters, let's review some of the concepts we dealt with in Volume 1. We do this in the following exercises.

The geometry developed in the preceding volume dealt principally with the notions of parallelism and ratio and, so, with the affine properties of Euclidean space. In the present volume we introduce notions of perpendicularity and distance and, so, deal with Euclidean metric geometry. [The word 'metric' indicates that we shall assume given a particular unit of distance. Euclidean geometry, properly so-called, deals with a family of proportional distance functions and treats any one as on a par with any other.]

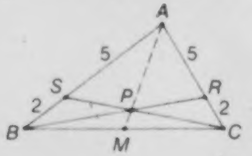
For most of this volume we need to add to Postulate 5′ only the assumption that each nonnegative real number has a principal square root; this is done implicitly on page 6. However, for the introduction of arc length and angle measure we shall need the assumption that each nonempty set of real numbers which has an upper bound has a *least* upper bound. Since, to save space and time, we shall assume a good many of the theorems which we could prove about arc measure, we shall not linger long over this ''completeness property'' of the real number system. A more adequate treatment, including a proof of the existence of square roots, can be found in Chapters 12 and 13 of Beberman and Vaughan, *High School Mathematics, Course 3*.

Note that many of the concepts from Volume 1 are reviewed in the proof of Theorem 1 on page 4.

Exercises

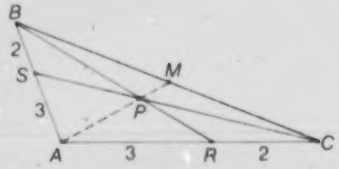
Part A

1. Suppose that C, D, and E are points of a line $\overleftrightarrow{AB}$ such that $C = A + (B - A)\tfrac{3}{2}$, $D = A + (B - A)3$, and $E = A + (B - A)\tfrac{7}{2}$. Draw an appropriate picture for these conditions, and compute the following ratios. [Recall that, for proper translations $\vec{a}$ and $\vec{b}$ in the same direction, $\vec{a} : \vec{b} = c$ if and only if $\vec{a} = \vec{b}c$.]
   (a) $(C - A) : (B - A)$ (b) $(B - A) : (C - B)$
   (c) $[(C - A) : (B - A)][(B - A) : (C - B)]$ (d) $(C - A) : (C - B)$
   (e) $(B - A) : (D - C)$ (f) $(D - C) : (A - C)$
   (g) $(B - E) : (E - C)$ (h) $(C - A) : (D - B)$
2. Given the information indicated in the pictures, compute the required ratios. [See Sections 8.01 and 8.02 of Volume 1 for help.]
   (a)



   Compute: $(R - S) : (C - B)$
   $(P - S) : (C - P)$
   $(P - R) : (B - R)$

   (b)



   Compute: $(R - B) : (R - P)$
   $(P - C) : (S - P)$
   $(P - A) : (M - A)$

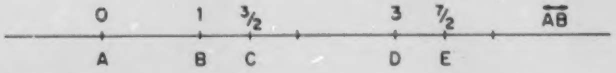
3. In each of parts (a) and (b) of Exercise 2, show that M is the midpoint of $\overline{BC}$.

Part B

1. (a) Suppose that π and σ are two intersecting planes. What kind of set is $\pi \cap \sigma$?
   (b) Suppose that π_1 and σ are two intersecting planes and that $\pi_2 \parallel \pi_1$. What kind of set is $\pi_2 \cap \sigma$? [Can $\pi_2 \cap \sigma$ be empty?]
   (c) If $\pi_1 \parallel \pi_2$ and $\pi_1 \cap \sigma \neq \varnothing$, what can you say about $\pi_1 \cap \sigma$ and $\pi_2 \cap \sigma$?
2. Suppose that l is a transversal of π—that is, suppose that $\pi \cap l$ consists of a single point.
   (a) If $m \parallel l$, does it follow that m is a transversal of π?
   (b) If $\sigma \parallel \pi$, does it follow that l is a transversal of σ?
3. Given lines l and l', how many planes contain l' and are parallel to l
   (a) if $l \parallel l'$? (b) if $l \nparallel l'$?

Answers for Part A

1. Here is an appropriate picture for this exercise:



   (a) 3/2 (b) 2 (c) 3 (d) 3
   (e) 2/3 (f) −1 (g) −5/4 (h) 3/4
2. (a) 5/7; 5/7; 5/12
   (b) 8/3; 5/3; 3/4
3. Since $\overline{AM}$, $\overline{BR}$, and $\overline{CS}$ are concurrent, $\frac{AS}{SB} \cdot \frac{BM}{MC} \cdot \frac{CR}{RA} = 1$. Since $\frac{AS}{SB} = \frac{RA}{CR}$, $\frac{BM}{MC} = 1$. That is, M is the midpoint of $\overleftrightarrow{BC}$.

Answers for Part B

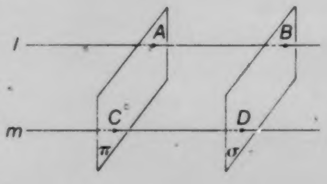
1. (a) A line. (b) A line. [No.] (c) They are parallel lines.
2. (a) Yes. (b) Yes.
3. (a) Many. (b) Exactly one.

4. Recall that, for any subset $\mathscr{X}$ of $\mathscr{E}$, $\mathscr{X} + \vec{a}$ is the image of $\mathscr{X}$ under the translation $\vec{a}$.
   (a) If $A \in \mathscr{X}$, what point is sure to be in $\mathscr{X} + (B - A)$?
   (b) Given a plane π, what kind of set is $\pi + \vec{a}$?
   (c) What can you say about π and $\pi + \vec{a}$?
   (d) If $\pi_2 \parallel \pi_1$, $A \in \pi_1$ and $B \in \pi_2$, what can you say about $\pi_1 + (B - A)$?

5. (a) Draw a triangle—say, $\triangle ABC$—and draw a second triangle, $\triangle A'B'C'$, such that $\overline{A'B'} \parallel \overline{AB}$, $\overline{B'C'} \parallel \overline{BC}$, and $\overline{C'A'} \parallel \overline{CA}$.
   (b) What can you say about the ratios $(B' - A') : (B - A)$, $(C' - B') : (C - B)$, and $(A' - C') : (A - C)$?

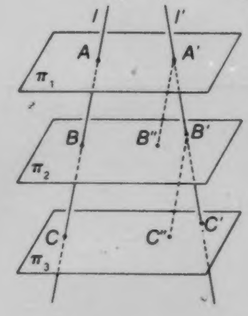
Part C

1. Suppose that l and m are parallel transversals of parallel planes π and σ, as shown in the picture.



   (a) Given that $l \neq m$ and $\pi \neq \sigma$, what kind of figure is $ABDC$? Explain your answer. [*Hint*: See Exercise 1 of Part B.]
   (b) Show that $B - A = D - C$. [Consider three cases: $l \neq m$ and $\pi \neq \sigma$; $l = m$; $\pi = \sigma$.]

2. Suppose that l and l' are transversals of parallel planes π_1, π_2, and π_3, where $\pi_1 \neq \pi_2$ and $\pi_2 \neq \pi_3$.



   (a) Show that if $l \parallel l'$ then
   $$(*)\ (C' - B' : (B' - A') = (C - B) : (B - A).$$
   [*Hint*: Use Exercise 1.]

   In parts (b) – (g) we shall assume that $l \nparallel l'$ and show that (*) holds in this case, also. To do so, let $B'' = A' + (B - A)$ and $C'' = B' + (C - B)$.
   (b) Show [easily] that $(C'' - B') : (B'' - A') = (C - B) : (B - A)$.
   (c) Show that $B'' \in \pi_2$ and that $C'' \in \pi_3$. [*Hint:* You can use Exercise 4 of Part B.]
   (d) Show that $\overline{A'B'B''} = \overline{B'C'C''}$. [*Hint:* Recall Exercise 3 of Part B.]
   (e) Show that $\overline{B'B''} \parallel \overline{C'C''}$. [*Hint:* See Exercise 1 of Part B.]
   (f) Show that $(C' - B') : (B' - A') = (C'' - B') : (B'' - A')$. [*Hint:* Consider $\triangle AB'B''$ and $\triangle B'C'C''$.]

Answers for Part B [cont.]

4. (a) B (b) A plane. (c) They are parallel.
   (d) $\pi_1 + (B - A) = \pi_2$
5. (a) [Many answers.]
   (b) The ratios are the same. [See Theorem 8-11.]

Answers for Part C

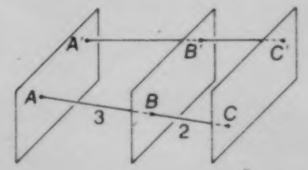
1. (a) If $\ell \neq m$ and $\pi \neq \sigma$ then ABDC is a parallelogram. For, $\overline{AB} \parallel \overline{CD}$ and the plane containing ℓ and m intersects π and σ in parallel lines containing $\overline{AC}$ and $\overline{DB}$.
   (b) If $\ell \neq m$ and $\pi \neq \sigma$ then $B - A = D - C$ because ABDC is a parallelogram. [See Theorem 8-16.] If $\ell = m$ then $C = A$ and $D = B$ and, so, $B - A = D - C$. If $\pi = \sigma$ then $B = A$ and $D = C$ and, so, $B - A = \vec{0} = D - C$.

2. (a) Assuming that $\ell \parallel \ell'$ it follows by Exercise 1 that $C' - B' = C - B$ and $B' - A' = B - A$. So, under this assumption $(C' - B'):(B' - A') = (C - B):(B - A)$.
   (b) Since $C'' - B' = C - B$ and $B'' - A' = B - A$ it follows that $(C'' - B'):(B'' - A') = (C - B):(B - A)$.
   (c) Since $A \in \pi_1$ and π_2 is the plane through B which is parallel to π_1, $\pi_2 = \pi_1 + (B - A)$. So, since $A' \in \pi_1$, $A' + (B - A) \in \pi_2$. Similarly, $B' + (C - B) \in \pi_3$.
   (d) $\overline{A'B'B''}$ contains ℓ' and the line $\overleftrightarrow{A'B''}$ which is parallel to ℓ. Since $\ell \nparallel \ell'$ it follows that $\overline{A'B'B''}$ is the plane containing ℓ' and parallel to ℓ. So is $\overline{B'C'C''}$.
   (e) $\overline{B'B''}$ and $\overline{C'C''}$ are subsets of the intersections of the parallel planes π_2 and π_3 with the plane containing ℓ' and parallel to ℓ.
   (f) $\overline{A'B'} \parallel \overline{B'C'}$ since both intervals are contained in ℓ'; $\overline{B'B''} \parallel \overline{C'C''}$ by part (c); $\overline{B''A'} \parallel \overline{C''B'}$ since both are parallel to ℓ. So, by Exercise 5 of Part B, $(C' - B'):(B' - A') = (C'' - B'):(B'' - A')$ $[= (C'' - C'):(B'' - B')]$.

(g) Complete the proof that (*) holds in case $l \nparallel l'$. [Incidentally, where did you need this assumption? Also, we did not assume that, as in the figure, $\pi_1 \neq \pi_3$. Did we need to?]

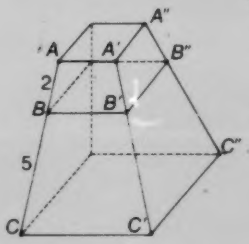
3. The results in Exercise 2 tell us that the ratio of segments on a transversal intercepted by three parallel planes is the same as the ratio of corresponding intercepted segments on any other transversal of those planes. Use this notion together with the information given in the figures to compute the indicated ratios.

(a)



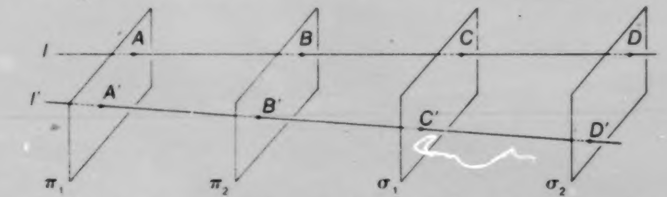
Compute:
$(B' - A') : (C' - B')$
$(B' - A') : (C' - A')$
$(A' - C') : (C' - B')$

(b)



Compute:
$(B' - A') : (C' - A')$
$(C'' - B'') : (A'' - B'')$
$(C' - A') : (C' - B')$

4. Suppose that l and l' are transversals of parallel planes π_1, π_2,



σ_1, and σ_2, where $\pi_1 \neq \pi_2$ and $\sigma_1 \neq \sigma_2$. Show that $(D' - C') : (B' - A') = (D - C) : (B - A)$. [*Hint*: In case $\pi_2 = \sigma_1$, this follows from Exercise 2. In case $\pi_2 \neq \sigma_1$, use Exercise 2 twice. Recall that $(\vec{d} : \vec{c})(\vec{c} : \vec{b}) = \vec{d} : \vec{b}$.]

5. We shall use the following extension of Exercise 4 in the next chapter:

> Theorem I Suppose that l_1, l_2, and l' are transversals of parallel planes π_1, π_2, σ_1, and σ_2 such that $l_1 \parallel l_2$, $\pi_1 \neq \pi_2$, and $\sigma_1 \neq \sigma_2$. If $l_1 \cap \pi_1 = \{A\}$, $l_1 \cap \pi_2 = \{B\}$, $l_2 \cap \sigma_1 = \{C\}$, $l_2 \cap \sigma_2 = \{D\}$, and l' intersects π_1, π_2, σ_1, and σ_2 at A', B', C', and D', respectively, then
> $$(D' - C') : (B' - A') = (D - C) : (B - A).$$

Answers for Part C [cont.]

2. (g) (*) is obtained by combining the results in part (b) and part (f). [The assumption that $l \nparallel l'$ was required in part (d). None of the argument requires that $\pi_1 \neq \pi_3$.]

3. (a) 3/2; 3/5; −5/2
   (b) 2/7; −5/2; 7/5

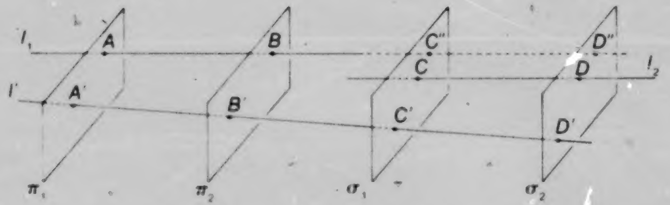
4. If $\sigma_1 = \pi_2$ this result is merely a re-lettering of that in Exercise 2. Suppose that $\sigma_1 \neq \pi_2$. It follows by Exercise 2 that

$$(D' - C'):(C' - B') = (D - C):(C - B)$$

and that

$$(C' - B'):(B' - A') = (C - B):(B - A).$$

The result now follows by multiplication as suggested in the hint.



Prove this theorem. [*Hint*: As in the figure, suppose that l_1 intersects σ_1 and σ_2 at C'' and D'', respectively. Use Exercise 4 and Exercise 1.]

6. Theorem I, of Exercise 5, includes the result of Exercise 4 and also the result of Exercise 2. [Explain.] It also includes most of the result of Exercise 1:

> Corollary Suppose that l and m are parallel transversals of parallel planes π and σ. If $l \cap \pi = \{A\}$, $l \cap \sigma = \{B\}$, $m \cap \pi = \{C\}$, and $m \cap \sigma = \{D\}$ then $D - C = B - A$.

(a) Show that the case of the corollary in which $\pi \neq \sigma$ follows from the theorem.

(b) Prove the case in which $\pi = \sigma$.

Background Topic

We shall have considerable use in this course for three operations on real numbers—squaring, absolute valuing, and square rooting. By definition,

$$(1) \qquad a^2 = aa.$$

Using this and Postulate 5′ it is not difficult to prove:

$$(2) \qquad a^2 \geq 0$$

and:

$$(3) \qquad a^2 = b^2 \longrightarrow a = b \quad [a \geq 0, b \geq 0]$$

It is also easy to show that, given any real number, there is at least one nonnegative real number which has the same square. By (3) there is at most one such number. This number is, by definition, the absolute value of the given number:

$$(4) \qquad |a| \geq 0 \text{ and } |a|^2 = a^2$$

Answers for Part C [cont.]

5. By Exercise 4, $(D' - C'):(B' - A') = (D'' - C''):(B - A)$. By Exercise 1, $D'' - C'' = D - C$. Hence, $(D' - C'):(B' - A') = (D - C):(B - A)$. [Incidentally, the theorem is called 'Theorem Eye', not 'Theorem One'.]

6. (a) In case $\pi \neq \sigma$ let, in Theorem I, $\pi_1 = \sigma_1 = \pi$, $\pi_2 = \sigma_2 = \sigma$, $l' = l_1 = l$, and $l_2 = m$. With these identifications, $C' = A'$ and $D' = B'$ and, so, $(D' - C'):(B' - A') = 1$. Hence, by Theorem I, $(D - C):(B - A) = 1$ and, so, $D - C = B - A$.

   (b) In case $\pi = \sigma$, $D - C = \vec{0} = B - A$.

*

In preparation for the background topic it may be well to review, very briefly, the order postulates for real numbers from section 4.04 of volume 1:

5_8. $a > b$ or $b > a$ $[a \neq b]$

5_9. $a \not> a$

5_{10}. $(a > b \text{ and } b > c) \Longrightarrow a > c$

5_{11}. $a < b \Longleftrightarrow b > a$

5_{12}. (a) $a > b \Longrightarrow a + c > b + c$

(b) $a > b \Longrightarrow ac > bc$ $[c > 0]$

and the definition [which might have been labeled '5_{13}']:

$$a \geq b \Longleftrightarrow (a > b \text{ or } a = b)$$

Also, remind students of the theorems proved in the exercises of Part B on page 159 of volume 1 and in Parts B and C on pages 163 and 164.

Sentence (1) is a natural enough definition of squaring and, as students should show in Exercises 1 and 2 of Part B of these exercises, (2) and (3) can be proved by using (1). That there is a nonnegative number whose square is a^2 follows from the fact that $a^2 = a^2$ and $(-a)^2 = a^2$ and that $a \geq 0$ or $-a \geq 0$. The latter can be proved by using 5_8 and Exercise 2 on page 159 of volume 1. Since, for any $a \in \mathcal{R}$, there is one and only one nonnegative real number whose square is a^2, we may speak of the nonnegative real number whose square is a^2. It is this real number which we call the absolute value of a. Sentence (4), then, lists two properties which are characteristic of absolute valuing and we may, and shall, adopt (4) as a "defining principle" for this operation. From (4) and other theorems about real numbers which do not mention absolute valuing it is possible to derive all theorems concerning this operation.

From (4) – which we take as a definition – and (3) we have:

(4′) $(b \geq 0 \text{ and } b^2 = a^2) \longrightarrow b = |a|.$

Using (4′) it is easy to prove:

(5) $[a \geq 0 \longrightarrow |a| = a]$ and $[a \leq 0 \longrightarrow |a| = -a].$

Let's look now at square rooting. By (3), different nonnegative numbers have different squares. In other words, a given real number is the square of at most one nonnegative number. Later we shall adopt a postulate concerning real numbers which will enable us to prove that each nonnegative number is the square of at least one nonnegative number. [By (2), we could not hope for more.] So, any given nonnegative number is the square of exactly one nonnegative number. This number is, by definition, the principal square root of the given number:

(6) $\sqrt{a} \geq 0$ and $(\sqrt{a})^2 = a \quad [a \geq 0]$

From (6) – which we take as a definition – and (3) we have:

(6′) $(b \geq 0 \text{ and } b^2 = a) \longrightarrow b = \sqrt{a}$

Comparing (4) and (6′) we obtain the important result:

(7) $\sqrt{a^2} = |a|$

In Part A of the following exercises you will have an opportunity to become better acquainted with some theorems about absolute valuing and square rooting, and to correct some erroneous ideas which you may have. In Part B you may see how some of the theorems can be proved.

Exercises

Part A

1. Simplify.
   (a) $\sqrt{(a-3)^2}$ (b) $\sqrt{a^2+6a+9}$ (c) $\sqrt{16}+\sqrt{9}$
   (d) $\sqrt{(a+1)^2}-\sqrt{a^2}$ (e) $\sqrt{a^2}-\sqrt{b^2}$ (f) $\sqrt{16+9}$
2. Evaluate the expressions given in 1(a), 1(b), 1(d), and 1(e) for $a = -2$ and $b = 3$.
3. Simplify.
   (a) $\sqrt{9}\cdot\sqrt{16}$ (b) $\sqrt{9\cdot 16}$ (c) $\sqrt{3}\cdot\sqrt{27}$ (d) $\sqrt{-5\cdot -25}$
   (e) $\sqrt{5\cdot 25}$ (f) $\sqrt{8\cdot 2}$ (g) $\sqrt{8}\cdot\sqrt{2}$ (h) $\sqrt{-8\cdot -2}$

For example, from (4) and (3) we can derive (4′) — which says, explicitly that $|a|$ is the only nonnegative number whose square is a^2. To do so, note that, by (3),

$$b^2 = |a|^2 \Longrightarrow b = |a| \quad [b \geq 0]$$

since, by (4), $|a| \geq 0$. Since, by (4), $|a|^2 = a^2$, we have at once

$$b^2 = a^2 \Longrightarrow b = |a| \quad [b \geq 0]$$

which is short for:

$$b \geq 0 \Longrightarrow [b^2 = a^2 \Longrightarrow b = |a|]$$

From this, (4′) follows at once by exportation.

Theorem (4′) can be used in proving (5) — which is a more customary definition of absolute valuing. For the first part merely substitute 'a' for 'b' in (4′) and note that $a^2 = a^2$. For the second part, substitute '−a' for 'b and note that if $a < 0$ then $-a > 0$ and that $(-a)^2 = a^2$.

The defining principle (6) is similar to (4) in that it lists two properties which are characteristic for square rooting. It differs from (4) in that we are not in a position as yet to prove that, for any $a \geq 0$, there is a nonnegative number whose square is a. The adoption of (4) as a definition was quite innocuous. Doing so amounted to no more than adopting a standard notation. In adopting (6) we are sticking our necks out beyond the comparative safety of Postulate 5′.

To obtain the fundamental theorem (7), merely substitute 'a^2' for 'a' and '$|a|$' for 'b' in (6′) and use (4).

Answers for Part A

1. (a) $|a - 3|$ (b) $|a + 3|$ (c) 7
   (d) $|a + 1| - |a|$ (e) $|a| - |b|$ (f) 5
   [You will, of course, call students' attention to the counter-instance of '$\sqrt{a^2 + b^2} = \sqrt{a} + \sqrt{b}$' which is furnished by parts (c) and (f).]
2. (a) 5 (b) 1 (d) −1 (e) −1
   [You may wish to give other examples in class which yield counter-instances to the false '$\sqrt{a^2} = a$'.]
3. (a) 12 $[3 \cdot 4 = 12]$ (b) 12 $[9 \cdot 16 = 144 = 12^2]$
   (c) 9 $[3 \cdot 27 = 81 = 9^2]$ (d) $5\sqrt{5}$ $[-5 \cdot -25 = 25 \cdot 5 = 5^2 \cdot 5]$
   (e) $5\sqrt{5}$ (f) 4 (g) 4 (h) 4

TC 7 (1)

4. (a) $a \geq 0$ (b) $a \leq 0$
   (c) $ab \geq 0$ [or: $(a \geq 0$ and $b \geq 0)$ or $(a \leq 0$ and $b \leq 0)$]
   (d) $a \geq 0$ and $b \geq 0$ (e) $a \geq 0$ and $b > 0$
   (f) $a = 0$ or $b = 0$ $[a^2 + b^2 = (a + b)^2 \Longleftrightarrow (a = 0 \text{ or } b = 0)]$
5. (a) 84
   (b) 12, −6 [$|a - 3| = 9$, so $(a \geq 3$ and $a - 3 = 9)$ or $(a \leq 3$ and $a - 3 = -9)$]
   (c) [no solution]
   (d) −1/2 (e) 1/2 (f) [no solution]

4. Here are some free-variable generalizations. Give conditions under which each is true.
   (a) $\sqrt{a^2} = a$ (b) $\sqrt{a^2} = -a$ (c) $\sqrt{a^2b^2} = ab$
   (d) $\sqrt{a}\sqrt{b} = \sqrt{ab}$ (e) $\sqrt{a}/\sqrt{b} = \sqrt{a/b}$ (f) $\sqrt{a^2 + b^2} = |a + b|$

5. Solve.
   (a) $\sqrt{a - 3} = 9$ (b) $\sqrt{(a - 3)^2} = 9$ (c) $\sqrt{(a - 3)^2} = -9$
   (d) $\sqrt{5 + 2a} = 2$ (e) $\sqrt{5 - 2a} = 2$ (f) $\sqrt{5 - 2a} = a - 3$

Part B

Recall the definitions which were given preceding Part A:

(1) $a^2 = aa$
(2) $|a| \ge 0$ and $|a|^2 = a^2$
(3) $\sqrt{a} \ge 0$ and $(\sqrt{a})^2 = a$ $[a \ge 0]$

The following exercises suggest proofs of theorems which are based on these definitions. Since you are entitled to use theorems you already know concerning real numbers, proofs can be quite short. But, convince yourself that you could prove any things you use in proofs. A quick review of Section 4.04 and 4.05 [in Volume 1] may be of help.

1. Prove:
   (a) $a \ne 0 \longrightarrow a^2 > 0$ (b) $a^2 \ge 0$ (c) $a^2 = 0 \longrightarrow a = 0$
   [*Hint*: For (a) recall Postulates 5_8 and 5_{12}. Show that $0^2 = 0$ and that $(-a)^2 = a^2$.]

2. Prove:
   (a) $a^2 = b^2 \longrightarrow (a = b \text{ or } a = -b)$
   (b) $a^2 = b^2 \longrightarrow a = b$ $[a \ge 0, b \ge 0]$
   [*Hint*: For (a) recall that $a^2 - b^2 = (a - b)(a + b)$. For (b) note that, for $a \ge 0$ and $b \ge 0$, if $a + b = 0$ then ($a = 0$ and $b = 0$).]

3. Prove:

$$(b \ge 0 \text{ and } b^2 = a^2) \longrightarrow b = |a|$$

   [*Hint*: Use definition (2) and Exercise 2(b).]

4. Prove:
   (a) $a \ge 0 \longrightarrow |a| = a$ (b) $a \le 0 \longrightarrow |a| = -a$
   (c) $|ab| = |a| \cdot |b|$ (d) $|a| = 0 \longrightarrow a = 0$
   [*Hint*: The key to (a) and (b) is Exercise 3. For (c) you also need definition (2).]

5. Prove:

$$(b \ge 0 \text{ and } b^2 = a) \longrightarrow b = \sqrt{a}$$

   [*Hint*: Use Exercise 1(b), definition (3), and Exercise 2(b).]

The purpose of Part B is to familiarize students with some important properties of squaring, absolute valuing, and order. However, it is not necessary that each student derive each property in order to gain the needed familiarity. It would be better to have each student prepare a discussion of one of the properties in Part B. Such a discussion may be prepared on a piece of acetate for the overhead projector and should include several instances of the property as well as the derivation. By preparing his discussion on a piece of acetate, a student can more quickly present his discussion to the rest of the class.

Answers for Part B

1. (a) Assuming that $a \ne 0$ it follows that $a > 0$ or $-a > 0$. So, by 5_{12}(b), $a \cdot a > 0$ or $-a \cdot -a > 0$. Since $a \cdot a = -a \cdot -a = a^2$ it follows that if $a \ne 0$ then $a^2 > 0$.
   (b) Since $0^2 = 0$ it follows that if $a = 0$ then $a^2 = 0$. Since $a \ne 0$ or $a = 0$ it follows from part (a) that $a^2 > 0$ or $a^2 = 0$. Hence, by definition, $a^2 \ge 0$.
   (c) Since $a^2 = aa$ it follows that if $a^2 = 0$ then $aa = 0$ and so, by an earlier theorem, $a = 0$ or $a = 0$. Hence, if $a^2 = 0$ then $a = 0$. [The "earlier theorem" is an equivalent of (2) on page 157, volume 1.]

2. (a) Suppose that $a^2 = b^2$. It follows that $a^2 - b^2 = 0$ and, so, that $a - b = 0$ or $a + b = 0$. Hence, if $a^2 = b^2$ then $a = b$ or $a = -b$.
   (b) Suppose that $a^2 = b^2$, $a \ge 0$, and $b \ge 0$. Since $a^2 = b^2$ it follows that $a = b$ or $a = -b$. Now, if $a = -b$ it follows, since $a \ge 0$ and $b \ge 0$, that $-b \ge 0$ and $b \ge 0$ — that is, that $0 \le b$ and $b \ge 0$. Hence, if $a = -b$ then $b = 0$ and, so, $a = 0$. In particular, if $a = -b$ then $a = b$. So, in any case, for $a \ge 0$ and $b \ge 0$, if $a^2 = b^2$ then $a = b$.

3. [This is (4′), and a proof has been given in the commentary for page 6.]

4. (a), (b) [These are the parts of (5) and proofs are indicated in the commentary for page 6.]
   (c) By definition, $|a| \ge 0$ and $|b| \ge 0$ and, so, $|a| \cdot |b| \ge 0$. Also, $(|a| \cdot |b|)^2 = |a|^2 \cdot |b|^2 = a^2 \cdot b^2 = (ab)^2$. Hence, by Exercise 3, $|a| \cdot |b| = |ab|$.
   (d) By definition, $|a|^2 = a^2$. So, if $|a| = 0$ then $a^2 = 0$ and, by Exercise 1(c), $a = 0$.

5. Suppose that $b \ge 0$ and $b^2 = a$. Since $b^2 \ge 0$ it follows that $a \ge 0$. So, by definition (3), $(\sqrt{a})^2 = a$ and, since $b^2 = a$, $b^2 = (\sqrt{a})^2$. Since, also, $\sqrt{a} \ge 0$ and $b \ge 0$ it follows by Exercise 2(b) that $b = \sqrt{a}$. Hence, if $b \ge 0$ and $b^2 = a$ then $b = \sqrt{a}$.

6. Prove:
  (a) $\sqrt{a^2} = |a|$ **(b)** $\sqrt{ab} = \sqrt{a}\sqrt{b}$ $[a \geq 0, b \geq 0]$
  [*Hint*: For (a) use definition (2) and Exercise 5, or Exercise 1(b), definition (3), and Exercise 3. For (b) use definition (3) and Exercise 5.]

7. (a) Suppose that $a < b$. Under what conditions is it the case that $ac < bc$? That $ac > bc$? That $ac = bc$? That $ac \leq bc$? Give examples to support each of your answers.
  (b) Given that $a < b$ does it follow that $a^2 < b^2$? If you think not, give a counterexample; if you think so, give an argument to support your answer.
  (c) Suppose that $a^2 < b^2$. Does it follow that $a < b$? Explain your answer.

8. Prove:
  (a) $0 \leq a < b \longrightarrow a^2 < b^2$ [*Hint*: Assume that $a \geq 0$ and $a < b$. It follows that $a^2 \leq ab$. Explain. Now, can you also show that $ab < b^2$?]
  (b) $a^2 \geq b^2 \longrightarrow a \geq b$ $[a \geq 0]$
  (c) $a \geq b^2 \longrightarrow \sqrt{a} \geq b$ [*Hint*: Use (b) and (3). Why is $a \geq 0$?]
  (d) Use (b) together with (2) on page 7 to prove a theorem like (b) but without any restriction.

9. (a) Show that $|a| \geq a$. [*Hint*: Note that, by definition, $|a|^2 \geq a^2$ and recall Exercise 8(b).]
  (b) Show that $-|a| \leq a \leq |a|$. [*Hint*: Use part (a) twice, recalling that $|-a| = |a|$.]
  (c) Show that if $|a| \leq b$ then $-b \leq a \leq b$.

10. (a) Show that $|a| = a$ or $|a| = -a$. [*Hint*: Consider two cases, $a \geq 0$ or $a \leq 0$.]
  (b) Show that if $-b \leq a \leq b$ then $|a| \leq b$.

Answers for Part B [cont.]

6. (a) By Exercise 5, if $|a| \geq 0$ and $|a|^2 = a^2$ then $|a| = \sqrt{a^2}$. So, by definition (2), $\sqrt{a^2} = |a|$. [Alternatively: By Exercise 3, if $\sqrt{a^2} \geq 0$ and $(\sqrt{a^2})^2 = a^2$ then $\sqrt{a^2} = |a|$. Since $a^2 \geq 0$ it follows by definition (3) that $\sqrt{a^2} \geq 0$ and $(\sqrt{a^2})^2 = a^2$. Hence, $\sqrt{a^2} = |a|$.]
   (b) By definition, for $a \geq 0$ and $b \geq 0$, $\sqrt{a} \geq 0$ and $\sqrt{b} \geq 0$ and, so, $\sqrt{a}\cdot\sqrt{b} \geq 0$. Also, $(\sqrt{a}\cdot\sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2 = ab$. Hence, by Exercise 5, for $a \geq 0$ and $b \geq 0$, $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

7. (a) $c > 0$; $c < 0$; $c = 0$; $c \geq 0$.
   (b) No; $-2 < 0$ but $(-2)^2 \not< 0$.
   (c) No; $0^2 < (-2)^2$ but $0 \not< -2$.

8. (a) Suppose that $a < b$. If $a > 0$ it follows that $a^2 = aa < ab$ and if $a = 0$ then $a^2 = 0 = ab$. So, assuming that $a \geq 0$ and $a < b$ it follows that $a^2 \leq ab$. Since, also, $b > a$ it follows that $b > 0$ and, since $a < b$, that $ab < b^2$. Since $a^2 \leq ab < b^2$ it follows that $a^2 < b^2$. Hence, if $0 \leq a < b$ then $a^2 < b^2$.
   (b) Suppose that $a \not\geq b$. It follows [see Exercise 4 on page 164 of volume 1] that $b > a$. So, for $a \geq 0$ it follows, by part (a) that $b^2 > a^2$ and, so, that $a^2 \not\geq b^2$. Hence, for $a \geq 0$, if $a \not\geq b$ then $a^2 \not\geq b^2$. Consequently, for $a \geq 0$, if $a^2 \geq b^2$ then $a \geq b$.
   (c) Suppose that $a \geq b^2$. Since $b^2 \geq 0$ it follows that $a \geq 0$ and so, by (3) that $\sqrt{a} \geq 0$ and $(\sqrt{a})^2 = a$. Since $a \geq b^2$ it follows that $(\sqrt{a})^2 \geq b^2$ and, since $\sqrt{a} \geq 0$, it follows by (b) that $\sqrt{a} \geq b$. Hence, if $a \geq b^2$ then $\sqrt{a} \geq b$.
   (d) The theorem in question is: $a^2 \geq b^2 \Longrightarrow |a| \geq b$. This follows from part (b) because, by (2), $|a| \geq 0$ and $|a|^2 = a^2$.

9. (a) Since, by definition, $|a|^2 = a^2$ it follows that $|a|^2 \geq a^2$. So, since $|a| \geq 0$, it follows by Exercise 8(b) that $|a| \geq a$.
   (b) By part (a), $|-a| \geq -a$. So, since $|-a| = |a|$, it follows that $|a| \geq -a$ — that is, that $-|a| \leq a$. Combining this with part (a) we have that $-|a| \leq a \leq |a|$.
   (c) Suppose that $|a| \leq b$. It follows that $-b \leq -|a|$ and, combining these two inequations with those of part (c), that $-b < a \leq b$.

10. (a) For $a \geq 0$, $a \geq 0$ and $a^2 = a^2$. Hence, for $a \geq 0$, $|a| = a$. For $a \leq 0$, $-a \geq 0$ and $(-a)^2 = a^2$. Hence, for $a \leq 0$, $|a| = -a$. Since $a \geq 0$ or $a \leq 0$ it follows that $|a| = a$ or $|a| = -a$.
   (b) Suppose that $-b \leq a \leq b$. In case $|a| = a$ it follows that $|a| < b$. In case $|a| = -a$ it follows, since $-a \leq b$, that $|a| \leq b$. So, by part (a), $|a| < b$ in any case. Hence, if $-b \leq a \leq b$ then $|a| \leq b$.

As a consequence of Exercises 9 and 10 we have the very useful real number theorem:

$$|a| \leq b \Longleftrightarrow -b \leq a \leq b$$

Chapter Eleven
Inner Product Spaces

11.01 Some Notions about Perpendicularity

Our aim in this chapter is to discover new properties of translations which we can use to characterize perpendicularity of lines and planes, and distance between points. Once we have discovered these properties we shall describe them in additional parts of Postulate 4. Using these additional postulates we shall be able, in the remainder of the text, to complete our study of the geometry of Euclidean space. As it turns out, what we mostly need are intuitive notions concerning planes perpendicular to lines. To say that a plane π is perpendicular to a line l we shall write '$\pi \perp l$'.

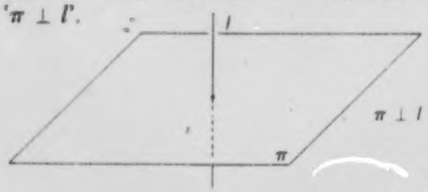


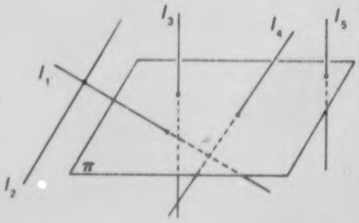
Fig. 11–1

The notion of a plane perpendicular to a line may be one you haven't thought about before. Intuitively, if you think about a horizontal plane, that plane is perpendicular to a vertical line.

Exercises

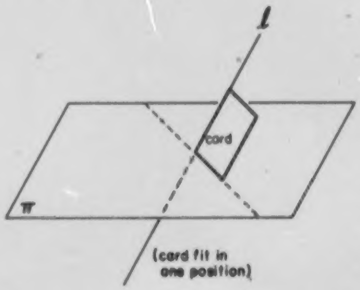
Part A

1. Given the plane π and the lines shown at the right.
   (a) Which ones of these lines are transversals of π?
   (b) Is π perpendicular to any of these lines? If so, which ones?

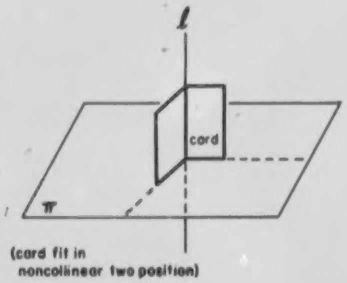


The purpose of this chapter is to motivate the adoption of postulates 4_0(e) and 4_{11} - 4_{14} on page 42 and the use of the multiplication operation introduced there to define perpendicularity and distance. We begin with some intuitive ideas concerning perpendicularity and [later] distance and gradually see how to express these notions algebraically.

We suggest making a model of a line and a plane perpendicular to the line. Use a stick and a piece of cardboard. In your class demonstration hold the model so the cardboard is horizontal at first (as the discussion suggests). After that move the model around so that students can see how the line and a plane appear from other points of view. You might also show how a corner of a 3" × 5" card can be placed at the intersection of the line and plane to test for perpendicularity. When you do this you should make the point that perpendicularity is present only when the card corner fits in two or more non-collinear positions.



(card fit in one position)



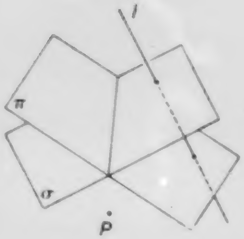
(card fit in noncollinear two position)

The exercises on pages 9-11 provide stimulating discussion if used in class. We suggest that you use these exercises and those on pages 12-14 to vary the activity in your class presentation and to insure that students begin to form correct intuitions about perpendicular lines and planes.

Answers for Part A

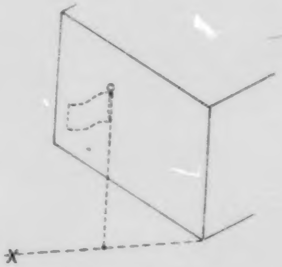
1. (a) All but, perhaps, ℓ_2. [We have tried to draw ℓ_2 so that it looks parallel to π.]
   (b) Yes; ℓ_3 and ℓ_5

2. Given the planes π and σ and line l which is a transversal of both π and σ, as shown at the right.
   (a) Which of the planes appear to be perpendicular to l?
   (b) Given the point P shown in the picture, is there a plane which contains P and is perpendicular to l? How would you describe any such plane?



3. (a) Draw a picture of a plane π and lines l_1 and l_2 such that π is perpendicular to both l_1 and l_2. What can you say about lines l_1 and l_2?
   (b) Now, draw a picture of a line l_3 which is parallel to l_1. What can you say about π and l_3?
   (c) Given that π is perpendicular to l_1, is it the case that l_1 is a transversal of π? Can l_1 have more than one point in common with π?

4. A man is asked to put up a flag pole in front of a building, in the position shown in the picture at the right. A helper is to tell him when the pole is vertical. The helper stands at the spot marked 'x' and tells him that the pole lines up with a vertical edge of the building. Is this information enough to ensure that the pole is vertical? Explain your answer.



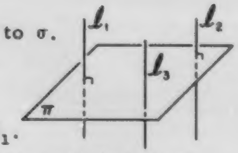
5. (a) Given a line l and a point P not on l, how many planes are there which contain P and are perpendicular to l? Make a sketch to illustrate your answer.
   (b) In part (a), would you change your answer if you were given that P is *on* l?

6. Hold a piece of cardboard and a pencil so that they represent a plane perpendicular to a line. How many planes are there each of which is perpendicular to a given line? What can you say about any two of these planes?

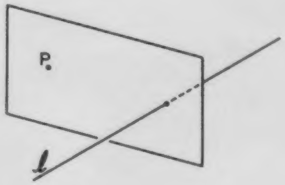
7. Given that a plane π is perpendicular to a line l, what can you say about π and m, where m is any line parallel to l? What can you say about σ and l, where σ is any plane parallel to π?

Answers for Part A [cont.]

2. (a) σ
   (b) Yes; the plane through P which is parallel to σ.

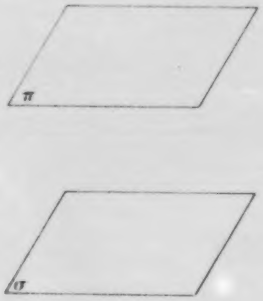


3. (a) $\ell_1 \parallel \ell_2$
   (b) $\pi \perp \ell_3$
   (c) Yes.; No, for if it did, $\ell_1 \subseteq \pi$ so that $\pi \not\perp \ell_1$.
4. No.; At least two different sightings must be made such that the points of these sightings and the foot of the flag pole are not collinear.
5. (a) Precisely one.
   (b) No.



6. Infinitely many; they are parallel.
7. $\pi \perp m$; $\sigma \perp \ell$

8. On your paper, draw a picture of parallel planes π and σ, as shown at the right.
   - (a) Draw a line l such that $\pi \perp l$. What can you say about σ and l?
   - (b) Draw a line m such that $\sigma \perp m$. What can you say about π and m? About l and m?
   - (c) Draw a line n which is parallel to l. What can you say about π and n? About σ and n?



9. Suppose that π is a plane perpendicular to l and that m is any line contained in π.
   - (a) How many planes contain l and also have a point in common with m? Make use of two pieces of cardboard and a pencil to illustrate your answer.
   - (b) Is it always possible to find a plane which contains l and which has no point in common with m? Is it ever possible to do so? Explain your answers.
   - (c) How many of the planes described in (a) are perpendicular to m?

*

The preceding exercises suggest several notions about perpendicularity of planes to lines. Four of these notions will turn out to be especially useful in our search for new postulates. The first is a very obvious one:

Notion 1. $\pi \perp l \longrightarrow l \nparallel \pi$

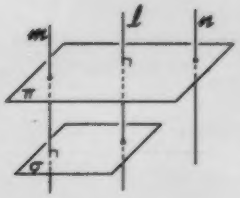
Since $\mathscr{E}$ is 3-dimensional it follows that if π is perpendicular to l then l is a transversal of π—that is, l and π have a single point in common. It also follows that if a line is parallel to a plane then the plane is not perpendicular to the line.

Notion 2. Given a line l and a point P, there is one and only one plane π such that $P \in \pi$ and $\pi \perp l$.

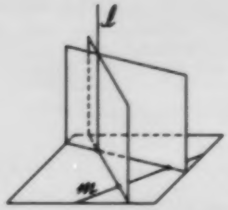
Using these two notions we can define a mapping with $\mathscr{E}$ as its domain and l as its range. For, given any point P, the plane which contains P and is perpendicular to l intersects l in a single point. This point is called *the orthogonal projection of P on l*—for short: $\text{proj}_l\,(P)$. [Read this as 'the projection of P on l'.] The word 'orthogonal' is a sig-

Answers for Part A [cont.]

8. (a) $\sigma \perp l$
   (b) $\pi \perp m$; $l \parallel m$
   (c) $\pi \perp n$; $\sigma \perp n$



9. (a) Infinitely many. [The picture at the right shows two such planes in the case when $l \cap m = \emptyset$.]
   (b) No; if $l \cap m \neq \emptyset$, then any plane containing l intersects m.
   Yes; if $l \cap m = \emptyset$, let σ contain l and be parallel to m.
   (c) Just one.



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The four "Notions" on pages 11, 13, and 14 are our first approximation to postulates concerning perpendicularity. They will be considerably modified in the sequel but will, at any rate, be theorems when we adopt the postulates on page 42 and the definition on page 74.

nal that the projecting on l is carried out through planes which are perpendicular to l.

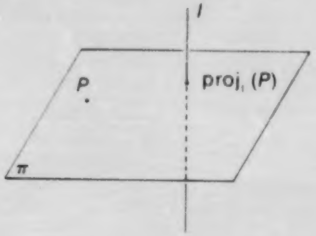


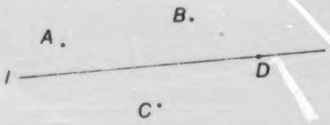
Fig. 11–2

When we have formulated our other notions about perpendicularity we shall have a great deal to say about the function proj_l. At present, note that

$$(1)\qquad \begin{cases} \text{(a) } \mathrm{proj}_l(P) \in l \\ \text{(b) } \mathrm{proj}_l(P) = P \longleftrightarrow P \in l. \end{cases}$$

[Explain.]

Part B



1. On your paper, draw a picture of line l and points A, B, C, and D, as shown at the right.
   - (a) Sketch the planes that determine the orthogonal projections on l of each of the points A, B, C, and D.
   - (b) What is $\mathrm{proj}_l(D)$?
   - (c) Suppose that $\mathrm{proj}_l(A) = A'$. Is A' the projection on l of any other point? Describe all of the points which have A' as their projections on l.
   - (d) Given any point E, describe how one locates $\mathrm{proj}_l(E)$.
2. The function proj_l maps $\mathcal{E}$ into l. Is proj_l a one-to-one function? Does proj_l have an inverse? Is proj_l an onto function? Explain your answers.
3. Making use of Notion 2, give an argument that two planes which are perpendicular to l have no point in common.
4. Given that π and σ are planes each of which is perpendicular to a line l, what can you say about π and σ? Explain your answer.
5. Suppose that $\pi \perp l$ and $\sigma \parallel \pi$. What can you say about σ and l? Give an argument to support your answer.

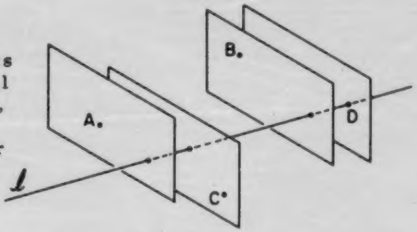
*

Explanation of (a): By definition, $\mathrm{proj}_\ell(P)$ is the point of intersection of ℓ and the plane through P perpendicular to ℓ.

Explanation of (b): If this intersection is P then, of course, $P \in \ell$. And, if $P \in \ell$ then P is the intersection of ℓ and any plane through P which is not parallel to ℓ.

Answers for Part B

1. (a) Here is a sketch of the planes that determine the orthogonal projections on ℓ of A, B, C, and D. The students should have pictures of at most four parallel planes, each of which is perpendicular to ℓ.



   (b) D

   (c) Yes; A' is the orthogonal projection of each point in the plane through A and perpendicular to ℓ.

   (d) $\mathrm{proj}_\ell(E)$ is the point of intersection of ℓ with the plane through E which is perpendicular to ℓ.

2. No. See Exercise 1(c); No, for it is not a one-to-one function; Yes, for each point of ℓ is the image of itself under the function proj_ℓ.

3. Suppose that the two planes do intersect. Let P be one of the points of their intersection. Then each of two planes contains P and is perpendicular to ℓ. This contradicts Notion 2.

4. By Exercise 3, either $\pi = \sigma$ or $\pi \cap \sigma = \emptyset$. In either case $\pi \parallel \sigma$.

5. $\sigma \perp \ell$. Since $\pi \perp \ell$ and $\sigma \parallel \pi$, it follows that σ is a transversal of ℓ — that is, σ intersects ℓ. Let P be the point of intersection of ℓ and σ. Consider the plane σ' which contains P and is perpendicular to ℓ. σ' is parallel to π and contains a point of σ, namely P. So, $\sigma' = \sigma$. Since $\sigma' \perp \ell$, it follows that $\sigma \perp \ell$.

In the exercises just completed, we obtained some consequences of Notion 2. We can combine these into:

$$(*) \qquad \pi \perp l \Longrightarrow [\sigma \perp l \Longleftrightarrow \sigma \parallel \pi]$$

That is, if π is perpendicular to l then the planes perpendicular to l are just those which are parallel to π. This is illustrated in Fig. 11–3, where $\pi_1 \perp l$, π_2 is any plane parallel to π_1 (and, so, $\pi_2 \perp l$), and σ is any plane not parallel to π_1 (and, so, $\sigma \not\perp l$).

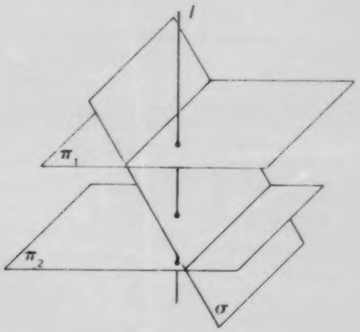


Fig. 11–3

The third intuitive notion we need is rather like (*). It is that if π is perpendicular to l then the lines to which π is perpendicular are just those which are parallel to l:

Notion 3. $\pi \perp l \Longrightarrow [\pi \perp m \Longleftrightarrow m \parallel l]$

According to Notion 2, given a line l, there are planes which are perpendicular to l and these are just those planes which have a certain bidirection. According to Notion 3, this bidirection depends only on the direction of l. So, corresponding to any proper direction $[l]$ there is a proper bidirection $[l]^{\perp}$, which we call *the orthogonal complement of* $[l]$. Each translation in the orthogonal complement of $[l]$ is *orthogonal* to each translation in $[l]$. And, a plane π is perpendicular to a line l if and only if $[\pi]$ *is* the orthogonal complement of $[l]$. In short, this last idea may be stated as:

$$(2) \qquad \pi \perp l \Longleftrightarrow [\pi] = [l]^{\perp}$$

Part C

1. Use (2) to show that
   (a) planes perpendicular to the same line are parallel,
   (b) parallel planes are perpendicular to the same line, and

The bidirection $[\vec{a}]^{\perp}$ is a complement of the direction $[\vec{a}]$ because, as it will turn out, each translation is the sum of uniquely determined translations, one of which is in $[\vec{a}]$ and the other in $[\vec{a}]^{\perp}$. [This is a different use of the word 'complement' than the one you may be accustomed to from the algebra of sets.] The bidirection $[\vec{a}]^{\perp}$ is orthogonal to $[\vec{a}]$ because each translation in $[\vec{a}]^{\perp}$ is orthogonal to each translation in $[\vec{a}]$. [Here, 'orthogonal' is a substitute for 'perpendicular', a word which we wish to save for use in connection with lines and planes.]

Answers for Part C

1. (a) Suppose that $\sigma \perp \ell$ and $\pi \perp \ell$. Then, $[\sigma] = [\ell]^{\perp}$ and $[\pi] = [\ell]^{\perp}$ so that $[\sigma] = [\pi]$. Thus, $\sigma \parallel \pi$. Hence, (a).
   (b) Suppose that $\pi \parallel \sigma$ and that $\sigma \perp \ell$. Then $[\pi] = [\sigma]$ and $[\sigma] = [\ell]^{\perp}$. Thus, $[\pi] = [\ell]^{\perp}$ so that $\pi \perp \ell$. Hence, (b).

(c) a plane which is perpendicular to a given line is also perpendicular to any parallel line.

2. **(a)** Use (2) and the fact that $[l]^\perp$ is a proper bidirection to show that there is a plane which contains P and is perpendicular to l. [*Suggestion*: Theorem 9–11 should suggest a neat way to describe such a plane.]

(b) Use part (a) and one of the parts of Exercise 1 to derive Notion 2.

3. Notice that, by parts (a) and (b) of Exercise 1, (2) implies (*).

(a) Explain why (2) also implies "half" of Notion 3. [*Hint*: Consider Exercise 1(c).]

(b) The "other half" of Notion 3 is:

$$(**) \qquad (\pi \perp l \text{ and } \pi \perp m) \longrightarrow m \parallel l$$

Use (2) and the fact that $[l]^\perp$ and $[m]^\perp$ are proper bidirections to show that (**) implies:

$$(3) \qquad [m]^\perp = [l]^\perp \longrightarrow [m] = [l]$$

[*Hint*: Choose a point P and consider $\overline{P[l]^\perp}$ and $\overline{P[m]^\perp}$. What kind of set is each of these? Why? Assuming that $[m]^\perp = [l]^\perp$, what follows from (2) and (**)?]

(c) Show that (**) follows from (2) and (3).

4. In the preceding three exercises you have seen that Notions 2 and 3 amount, exactly, to the fact that, for any line l, $[l]^\perp$ is a proper bidirection, and that (2) and (3) hold. Try to express what Notion 1 says in terms of '$[l]$' and '$[l]^\perp$'.

*

According to Notion 1, given a line l there are planes which are perpendicular to l. An equally important notion is that, given a plane π, there are lines to which π is perpendicular. Instead of listing this as our fourth notion, we choose to list a notion which will enable us to "construct" such lines.

Notion 4. If l is a subset of some plane which is perpendicular to m then m is a subset of some plane which is perpendicular to l.

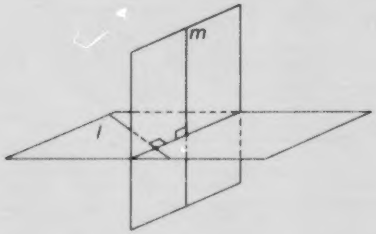


Fig. 11–4

Answers for Part C [cont.]

1. (c) Suppose that $\pi \perp \ell$ and that $\ell \parallel m$. Then, $[\pi] = [\ell]^\perp$ and $[\ell] = [m]$. Thus, $[\pi] = [m]^\perp$ so that $\pi \perp m$. Hence, (c).

2. (a) Let $\pi = \overline{P[\ell]^\perp}$. Since $[\ell]^\perp$ is a bidirection, π is a plane. By Theorem 9-11, $P \in \pi$ and $[\pi] = [\ell]^\perp$. From this last, $\pi \perp \ell$.

(b) By part (a) there is a plane which contains P and is perpendicular to ℓ. By Exercise 1(a), there are not two such planes.

3. (a) By Exercise 1(c) we know that if $\pi \perp \ell$ and $m \parallel \ell$ then $\pi \perp m$. That is, we know:

$$(\pi \perp \ell \text{ and } m \parallel \ell) \Longrightarrow \pi \perp m$$

This is equivalent to:

$$\pi \perp \ell \Longrightarrow [m \parallel \ell \Longrightarrow \pi \perp m]$$

which is "half" of Notion 3.

(b) Suppose that ℓ and m are lines such that $[m]^\perp = [\ell]^\perp$. Since both $[m]^\perp$ and $[\ell]^\perp$ are proper bidirections, each of $\overline{P[\ell]^\perp}$ and $\overline{P[m]^\perp}$ is a plane which contains P. $\overline{P[\ell]^\perp}$ is the plane which contains P and is perpendicular to ℓ. $\overline{P[m]^\perp}$ is the plane which contains P and is perpendicular to m. Since $[m]^\perp = [\ell]^\perp$, $\overline{P[m]^\perp} = \overline{P[\ell]^\perp}$. So, by (**), it follows that $m \parallel \ell$. Thus, by definition, $[m] = [\ell]$. Hence (3).

(c) Suppose that $\pi \perp \ell$ and $\pi \perp m$. By (2), $[\pi] = [\ell]^\perp$ and $[\pi] = [m]^\perp$. So, $[m]^\perp = [\ell]^\perp$. By (3), $[m] = [\ell]$. By Definition 7-6, $m \parallel \ell$. Thus, if $\pi \perp \ell$ and $\pi \perp m$, then $m \parallel \ell$.

4. Notion 1 is: $\pi \perp \ell \Longrightarrow \ell \nparallel \pi$. This is equivalent to saying:

$$[\pi] = [\ell]^\perp \Longrightarrow [\ell] \not\subseteq [\pi]$$

An instance of this last result is:

$$[\ell]^\perp = [\ell]^\perp \Longrightarrow [\ell] \not\subseteq [\ell]^\perp$$

Hence, $[\ell] \not\subseteq [\ell]^\perp$. On the other hand, assume that $[\ell] \not\subseteq [\ell]^\perp$ and that $\pi \perp \ell$. It follows by (2) that $[\pi] = [\ell]^\perp$ and, so, that $[\ell] \not\subseteq [\pi]$. So, by (2), '$[\ell] \not\subseteq [\ell]^\perp$' says what Notion 1 does — that is, that $\ell \nparallel \pi$.

Recall that a line l is parallel to a plane σ if and only if l is a subset of some plane which is parallel to σ. It follows from this that, in view of (*), Notion 4 is equivalent to:

> If l is parallel to some plane which is perpendicular to m then m is parallel to some plane which is perpendicular to l.

[Explain.] In other words, Notion 4 amounts to:

$$[l] \subseteq [m]^{\perp} \longrightarrow [m] \subseteq [l]^{\perp} \tag{4}$$

Using Notion 4 it is not difficult to show that, given a plane π, there is a line l such that $\pi \perp l$. In fact, we shall show:

(5) The plane which contains two given intersecting lines is perpendicular to the line of intersection of any two planes which are perpendicular, respectively, to the given lines.

For, suppose that m_1 and m_2 are lines of a plane π which intersect at O. By Notion 2, there are planes – say, σ_1 and σ_2 – such that $\sigma_1 \perp m_1$ and $\sigma_2 \perp m_2$. By Notions 2 and 3 it follows that, since $m_1 \nparallel m_2$, $\sigma_1 \nparallel \sigma_2$. So, since $\mathscr{E}$ is 3-dimensional, $\sigma_1 \cap \sigma_2$ is a line – say, l. [See Fig. 11–5.]

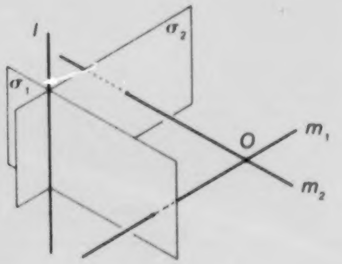


Fig. 11–5

Now, l is contained in σ_1 and $\sigma_1 \perp m_1$. Also, l is contained in σ_2 and $\sigma_2 \perp m_2$. So, it follows by Notion 4 that there is a plane – say, π_1 – such that m_1 is contained in π_1 and $\pi_1 \perp l$. Similarly, there is a plane – say, π_2 – such that m_2 is contained in π_2 and $\pi_2 \perp l$. Note that $O \in m_1$. So, π_1 is the plane containing O which is perpendicular to l. Similarly, since $O \in m_2$, π_2 is this same plane. So, $\pi_1 = \pi_2$. Since π is the only plane which contains both m_1 and m_2, $\pi_1 = \pi_2 = \pi$. Thus, $\pi \perp l$.

It is worth summarizing that Notion 1 - Notion 4 say just what is said by the "more algebraic":

$[\ell]^{\perp}$ is a proper bidirection	
$[\ell] \not\subseteq [\ell]^{\perp}$	[Exercise 4, above]
$\pi \perp \ell \Longleftrightarrow [\pi] = [\ell]^{\perp}$	[(2) on page 13]
$[m]^{\perp} = [\ell]^{\perp} \Longrightarrow [m] = [\ell]$	[(3) on page 14]
$[\ell] \subseteq [m]^{\perp} \Longrightarrow [m] \subseteq [\ell]^{\perp}$	[(4)]

Except for the third — which might serve as a definition of perpendicularity — these sentences state properties of orthogonal complementing. We might take the third as a definition and adopt the others as postulates. Actually, the latter will be theorems once we have adopted the postulates on page 42 and Definition 11-1(b) on page 45.

Since, as is shown on this page, (5) is a consequence of our four Notions, it is also a consequence of the five sentences noted in the commentary for page 15. Since, as indicated there, these five sentences will eventually become theorems, (5) will, at that point, also become a theorem.

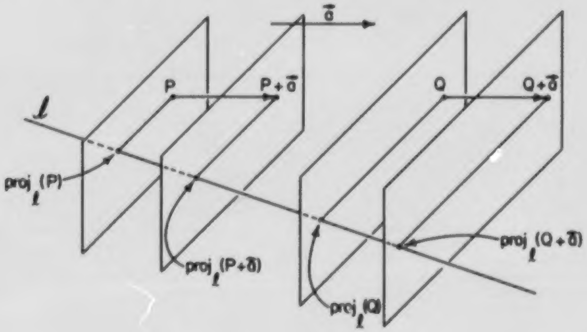
TC 16 (1)

The exercises beginning on this page are discovery exercises for (*) and (**) on page 17.

Anticipate some confusion beginning in Exercise 3. The notation gets a bit cumbersome when specifying the translation determined by the projections of a pair of points. This can be minimized by saying the appropriate word 'point' or 'translation' before reading the various expressions.

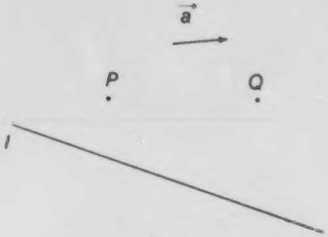
Answers for Part D

Here is a picture for Exercises 1 - 3:



1. They are parallel lines.
2. They are parallel.

Part D



Consider line l, points P and Q, and translation $\vec{a}$, as shown in the picture at the right. Copy the picture on your paper.

1. Locate the points $P + \vec{a}$ and $Q + \vec{a}$. What can you say about $\overleftrightarrow{P(P + \vec{a})}$ and $\overleftrightarrow{Q(Q + \vec{a})}$?
2. Sketch the [four] planes containing the points P, $P + \vec{a}$, Q, and $Q + \vec{a}$, respectively, and perpendicular to l. What can you say about these planes?
3. Locate the projections on l of the points P, $P + \vec{a}$, Q, and $Q + \vec{a}$. Consider the translations $\text{proj}_l\,(P + \vec{a}) - \text{proj}_l\,(P)$ and $\text{proj}_l\,(Q + \vec{a}) - \text{proj}_l\,(Q)$.
   - (a) It may be the case that the projections on l of P and $P + \vec{a}$ are the same point. That is, it may be the case that $\text{proj}_l\,(P) = \text{proj}_l\,(P + \vec{a})$. If this is the case, what can be said of
     - (i) the points $\text{proj}_l\,(Q)$ and $\text{proj}_l\,(Q + \vec{a})$?
     - (ii) the translation $\text{proj}_l\,(P + \vec{a}) - \text{proj}_l\,(P)$?
     - (iii) the translation $\text{proj}_l\,(Q + \vec{a}) - \text{proj}_l\,(Q)$?

     Explain your answers. [*Hint*: What can you say, in this case, about $\vec{a}$ and the direction of the plane through P perpendicular to l?]
   - (b) Show that if $\text{proj}_l\,(P) = \text{proj}_l\,(P + \vec{a})$ then $\text{proj}_l\,(P + \vec{a}) - \text{proj}_l\,(P) = \text{proj}_l\,(Q + \vec{a}) - \text{proj}_l\,(Q)$.
   - (c) Given that $\text{proj}_l\,(P) \neq \text{proj}_l\,(P + \vec{a})$, what can be said of
     - (i) the points $\text{proj}_l\,(Q)$ and $\text{proj}_l\,(Q + \vec{a})$?
     - (ii) the ratio of $\text{proj}_l\,(P + \vec{a}) - \text{proj}_l\,(P)$ to $\text{proj}_l\,(Q + \vec{a}) - \text{proj}_l\,(Q)$?

     Explain your answers. [*Hint*: See Theorem I, page 4.]
4. Show that

   $$(*)\ \text{proj}_l\,(Q + \vec{a}) - \text{proj}_l\,(Q) = \text{proj}_l\,(P + \vec{a}) - \text{proj}_l\,(P).$$

   [*Hint*: See Exercise 3.]
5. Suppose that l and m are parallel lines. Consider any point P and translation $\vec{a}$. Both $\text{proj}_l\,(P)$ and $\text{proj}_l\,(P + \vec{a})$ are points of l. What can be said of
   - (i) $\text{proj}_m\,(P)$ and $\text{proj}_m\,(P + \vec{a})$?
   - (ii) the translations $\text{proj}_l\,(P + \vec{a}) - \text{proj}_l\,(P)$ and $\text{proj}_m\,(P + \vec{a}) - \text{proj}_m\,(P)$?

   Explain your answers. [*Hint*: See Theorem I, page 4.]
6. Show that

   $$(**)\ m \parallel l \longrightarrow \text{proj}_m\,(P + \vec{a}) - \text{proj}_m\,(P) = \text{proj}_l\,(P + \vec{a}) - \text{proj}_l\,(P).$$
7. Discuss the following cases of (**) in Exercise 6.
   (a) $m = l$ (b) $\vec{a} = \vec{0}$ and $m \neq l$ (c) $\vec{a} \in [l]^{\perp}$

Answers for Part D [cont.]

3. (a) $\text{proj}_\ell\,(Q) = \text{proj}_\ell\,(Q + \vec{a})$; $\vec{0}$; $\vec{0}$.

   Given that $\text{proj}_\ell\,(P) = \text{proj}_\ell\,(P + \vec{a})$, P and $P + \vec{a}$ are both in a plane perpendicular to ℓ. So $[\vec{a}] \subseteq [\ell]^{\perp}$.

   So, Q and $Q + \vec{a}$ are in a plane perpendicular to ℓ. That is, $\text{proj}_\ell\,(Q + \vec{a}) = \text{proj}_\ell\,(Q)$. So, each of $\text{proj}_\ell\,(P + \vec{a}) - \text{proj}_\ell\,(P)$ and $\text{proj}_\ell\,(Q + \vec{a}) - \text{proj}_\ell\,(Q)$ is $\vec{0}$.

   (b) Suppose that $\text{proj}_\ell\,(P) = \text{proj}_\ell\,(P + \vec{a})$. Then, by the argument in (a), $\text{proj}_\ell\,(Q) = \text{proj}_\ell\,(Q + \vec{a})$. So, $\text{proj}_\ell\,(P + \vec{a}) - \text{proj}_\ell\,(P) = \vec{0} = \text{proj}_\ell\,(Q + \vec{a}) - \text{proj}_\ell\,(Q)$.

   (c) $\text{proj}_\ell\,(Q) \neq \text{proj}_\ell\,(Q + \vec{a})$, by Theorem I on page 4;

   $$\begin{aligned}[\text{proj}_\ell\,(P + \vec{a}) - \text{proj}_\ell\,(P)] &: [\text{proj}_\ell\,(Q + \vec{a}) - \text{proj}_\ell\,(Q)] \\ &= \big((P + \vec{a}) - P\big) : \big((Q + \vec{a}) - Q\big) \\ &= \vec{a} : \vec{a} \\ &= 1\end{aligned}$$

4. This is a direct consequence of the result in 3(b) and 3(c).

5. They are points of m;
   their ratio is 1 for, by Theorem I,

   $$\begin{aligned}&[\text{proj}_\ell\,(P + \vec{a}) - \text{proj}_\ell\,(P)] : [\text{proj}_m\,(P + \vec{a}) - \text{proj}_m\,(P)] \\ &= \big((P + \vec{a}) - P\big) : \big((P + \vec{a}) - P\big) \\ &= \vec{a} : \vec{a} \\ &= 1.\end{aligned}$$

   [To make an appropriate instance of Theorem I, let $\ell = \overleftrightarrow{P[\vec{a}]}$, $\ell_1 = \ell$, $\ell_2 = m$, $\pi_1 = \sigma_1$ and $\pi_2 = \sigma_2$.]
6. This is a direct consequence of the answer to Exercise 5.
7. (a) If $m = \ell$ then the consequent of (**) reduces to a sentence of the form '$\vec{p} = \vec{p}$'.

   (b) If $\vec{a} = \vec{0}$, and $m \neq \ell$, then the consequent reduces to '$\vec{0} = \vec{0}$'.

   (c) If $\vec{a} \in [\ell]^{\perp}$, then the consequent again reduces to '$\vec{0} = \vec{0}$'.

11.02 Orthogonal Projections of Translations

In the preceding section we listed some intuitive notions concerning the perpendicularity of planes and lines and were led to the notion of the orthogonal projection of a point P on a line l. The point $\mathrm{proj}_l\,(P)$ is the point of intersection of l with the plane which contains P and is perpendicular to l.

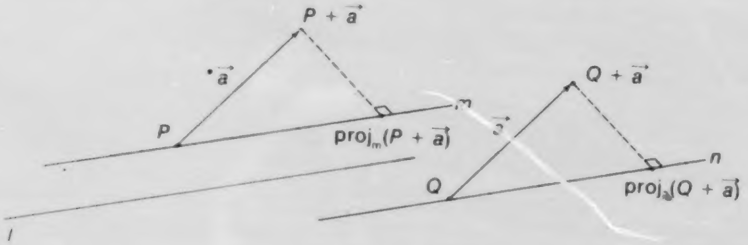
Using our notions concerning perpendicularity, Theorem I and its corollary on pages 4 and 5, the exercises just completed give us two important facts about projections of points on lines. These are:

$$(*) \qquad \mathrm{proj}_l\,(Q + \vec{a}) - \mathrm{proj}_l\,(Q) = \mathrm{proj}_l\,(P + \vec{a}) - \mathrm{proj}_l\,(P)$$

and:

$$(**)\ m \parallel l \Longrightarrow \mathrm{proj}_m\,(P + \vec{a}) - \mathrm{proj}_m\,(P) = \mathrm{proj}_l\,(P + \vec{a}) - \mathrm{proj}_l\,(P)$$

Suppose, now, that we are given a line l and a translation $\vec{a}$. For any point P, let m be the line $\overleftrightarrow{P[l]}$ [which contains P and is parallel to l]. Similarly, let $n = \overleftrightarrow{Q[l]}$.



$P + \vec{a}$
$\vec{a}$
P
$\mathrm{proj}_m(P + \vec{a})$
m
$Q + \vec{a}$
$\vec{a}$
Q
n
$\mathrm{proj}_n(Q + \vec{a})$
l

Fig. 11–6

Let's compare the translations $\mathrm{proj}_n\,(Q + \vec{a}) - Q$ and $\mathrm{proj}_m\,(P + \vec{a}) - P$. Since $Q \in n$ it follows that

$$\mathrm{proj}_n\,(Q + \vec{a}) - Q = \mathrm{proj}_n\,(Q + \vec{a}) - \mathrm{proj}_n\,(Q).$$

Also, by (*),

$$\mathrm{proj}_n\,(Q + \vec{a}) - \mathrm{proj}_n\,(Q) = \mathrm{proj}_n\,(P + \vec{a}) - \mathrm{proj}_n\,(P).$$

Since $n \parallel m$ it follows by (**) that

$$\mathrm{proj}_n\,(P + \vec{a}) - \mathrm{proj}_n\,(P) = \mathrm{proj}_m\,(P + \vec{a}) - \mathrm{proj}_m\,(P).$$

Finally, since $P \in m$,

$$\mathrm{proj}_m\,(P + \vec{a}) - \mathrm{proj}_m\,(P) = \mathrm{proj}_m\,(P + \vec{a}) - P.$$

Combining these results, we see that, with $m = \overleftrightarrow{P[l]}$ and $n = \overleftrightarrow{Q[l]}$,

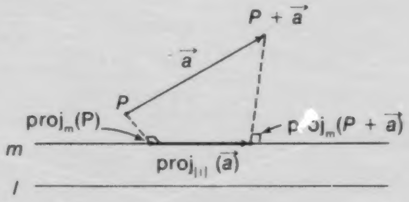
$$\mathrm{proj}_n\,(Q + \vec{a}) - Q = \mathrm{proj}_m\,(P + \vec{a}) - P.$$

It follows that, given l and $\vec{a}$, there is a single translation which, when applied to any point, has the effect of applying $\vec{a}$ to that point and then projecting that image on the line parallel to l and containing the given point. This translation is called *the orthogonal projection of $\vec{a}$ in the direction of l*—for short:

$$\mathrm{proj}_{[l]}\,(\vec{a})$$

For any point P and any line m parallel to l,

$$(1) \qquad \mathrm{proj}_{[l]}\,(\vec{a}) = \mathrm{proj}_m\,(P + \vec{a}) - \mathrm{proj}_m\,(P).$$



$P + \vec{a}$
$\vec{a}$
P
$\mathrm{proj}_m(P)$
$\mathrm{proj}_m(P + \vec{a})$
m
$\mathrm{proj}_{[l]}\,(\vec{a})$
l

Fig. 11–7

Corresponding with (1) in Section 11.01, we have:

$$(2) \qquad \begin{cases} \text{(a)}\ \mathrm{proj}_{[l]}\,(\vec{a}) \in [l] \\ \text{(b)}\ \mathrm{proj}_{[l]}\,(\vec{a}) = \vec{a} \Longrightarrow \vec{a} \in [l] \end{cases}$$

The first part of (2) follows at once from (1) and the fact that projections on l belong to l. To establish the second, let $m = \overleftrightarrow{P[l]}$. Since $P \in m$ and $m \parallel l$ it follows that

$$\mathrm{proj}_{[l]}\,(\vec{a}) = \mathrm{proj}_m\,(P + \vec{a}) - P.$$

So, $\mathrm{proj}_{[l]}\,(\vec{a}) = \vec{a}$ if and only if $\mathrm{proj}_m\,(P + \vec{a}) = P + \vec{a}$. By (1) of Section 11.01, the latter is the case if and only if $P + \vec{a} \in m$. But, since $P \in m$, $P + \vec{a} \in m$ if and only if $\vec{a} \in [m]$. Since $[m] = [l]$ it follows that $\mathrm{proj}_{[l]}\,(\vec{a}) = \vec{a}$ if and only if $\vec{a} \in [l]$.

Exercises

1. On your paper, draw a line l.
   (a) Draw a translation $\overrightarrow{a}$ such that $\overrightarrow{a} \neq \overrightarrow{0}$ and $\mathrm{proj}_{[l]}(\overrightarrow{a}) = \overrightarrow{0}$.
   (b) Draw a translation $\overrightarrow{b}$ such that $\overrightarrow{b} \neq \overrightarrow{0}$ and $\mathrm{proj}_{[l]}(\overrightarrow{b}) = \overrightarrow{b}$.
   (c) Draw a translation $\overrightarrow{c}$ such that $\overrightarrow{c} \neq \overrightarrow{0}$ and $\mathrm{proj}_{[l]}(\overrightarrow{c})$ moves points half as far as does $\overrightarrow{c}$.
   (d) Draw $\mathrm{proj}_{[l]}(-\overrightarrow{c})$. Compare this with the translation $\mathrm{proj}_{[l]}(\overrightarrow{c})$.
2. Make use of sentence (1), above, to show the following.
   (a) $\mathrm{proj}_{[l]}(\overrightarrow{0}) = \overrightarrow{0}$
   (b) $\mathrm{proj}_{[l]}(-\overrightarrow{a}) = -\mathrm{proj}_{[l]}(\overrightarrow{a})$
3. Draw a line l and two translations $\overrightarrow{d}$ and $\overrightarrow{e}$. Picture each of the following translations.
   (a) $\mathrm{proj}_{[l]}(\overrightarrow{d})$
   (b) $\mathrm{proj}_{[l]}(\overrightarrow{e})$
   (c) $\mathrm{proj}_{[l]}(\overrightarrow{d} + \overrightarrow{e})$
   (d) $\mathrm{proj}_{[l]}(\overrightarrow{d}) + \mathrm{proj}_{[l]}(\overrightarrow{e})$
4. Compare the translations drawn in 3(c) and 3(d). What does this suggest about the function $\mathrm{proj}_{[l]}$?
5. Draw a line l and translation $\overrightarrow{a}$. Picture each of the following translations.
   (a) $\mathrm{proj}_{[l]}(\overrightarrow{a}2)$
   (b) $(\mathrm{proj}_{[l]}(\overrightarrow{a}))2$
   (c) $\mathrm{proj}_{[l]}(\overrightarrow{a} \cdot -2)$
   (d) $\mathrm{proj}_{[l]}(\overrightarrow{a}) \cdot -2$
6. Compare each pair of the translations drawn in Exercise 5.
7. (a) Suppose that $\overrightarrow{a}$ is a translation whose orthogonal projection in the direction of a line l is $\overrightarrow{0}$. What can you say about the direction of $\overrightarrow{a}$ in relation to the direction of l?
   (b) Suppose that $\overrightarrow{a}$ is in the orthogonal complement of the direction of l. [That is, suppose that $\overrightarrow{a} \in [l]^\perp$.]. What can you say about $\mathrm{proj}_{[l]}(\overrightarrow{a})$?

Your answers to Exercise 7 may have suggested:

$$\mathrm{proj}_{[l]}(\overrightarrow{a}) = \overrightarrow{0} \Longleftrightarrow \overrightarrow{a} \in [l]^\perp \tag{3}$$

This can be established in much the same way as was the second part of (2) [page 18]. Again let $m = \overleftrightarrow{P[l]}$, and let $\pi = \overrightarrow{P[l]^\perp}$. [Then, m is the line through P parallel to l and π is the plane through P perpendicular to l.] Since $m \parallel l$ and $\pi \perp l$ it follows that $\pi \perp m$. Since $P \in m$, $\mathrm{proj}_{[l]}(\overrightarrow{a}) = \mathrm{proj}_m(P + \overrightarrow{a}) - P$ and, so, $\mathrm{proj}_{[l]}(\overrightarrow{a}) = \overrightarrow{0}$ if and only if $\mathrm{proj}_m(P + \overrightarrow{a}) = P$. Since $\pi \perp m$ and m intersects π at P, the latter is the case if and only if $P + \overrightarrow{a} \in \pi$. But, since $P \in \pi$, $P + \overrightarrow{a} \in \pi$ if and only if $\overrightarrow{a} \in [\pi]$. And, $[\pi] = [l]^\perp$. Hence, (3).

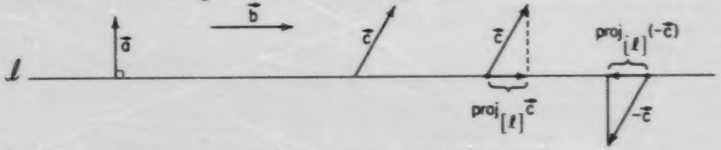
The exercises just completed suggest two other properties of projections:

$$\mathrm{proj}_{[l]}(\overrightarrow{a} + \overrightarrow{b}) = \mathrm{proj}_{[l]}(\overrightarrow{a}) + \mathrm{proj}_{[l]}(\overrightarrow{b}) \tag{4}$$

$$\mathrm{proj}_{[l]}(\overrightarrow{a}a) = \mathrm{proj}_{[l]}(\overrightarrow{a})a \tag{5}$$

Answers for Exercises

1. Here is an appropriate picture for parts (a) - (d). [An arrow for $\vec{a}$ should be drawn so that a line in the direction $[\vec{a}]$ is perpendicular to ℓ; an arrow for $\vec{b}$ should be drawn so that a line in the direction $[\vec{b}]$ is parallel to ℓ; an arrow for $\vec{c}$ should be drawn to "make" an angle of 60° with ℓ.]


ℓ
$\vec{a}$
$\vec{b}$
$\vec{c}$
$\vec{c}$
$\mathrm{proj}_{[\ell]}\vec{c}$
$\mathrm{proj}_{[\ell]}(-\vec{c})$
$-\vec{c}$

   (d) Note that $\mathrm{proj}_{[\ell]}(-\vec{c}) = -\mathrm{proj}_{[\ell]}(\vec{c})$.

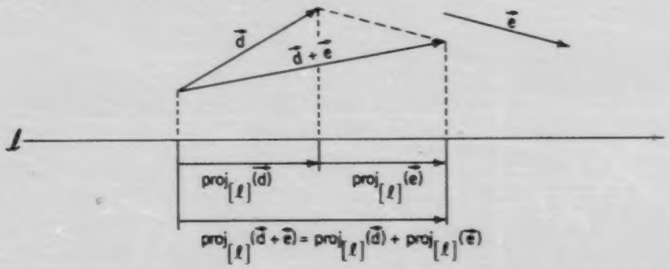
2. (a) By (1) we know that, for any point P,

$$\begin{aligned}\mathrm{proj}_{[\ell]}(\vec{0}) &= \mathrm{proj}_\ell(P + \vec{0}) - \mathrm{proj}_\ell(P)\\ &= \mathrm{proj}_\ell(P) - \mathrm{proj}_\ell(P) = \vec{0}.\end{aligned}$$

   (b) By (1) we know that $\mathrm{proj}_{[\ell]}(-\vec{a}) = \mathrm{proj}_\ell((P + \vec{a}) + -\vec{a}) - \mathrm{proj}_\ell(P + \vec{a})$. So, by (1) and the algebra of points and translations, we have that

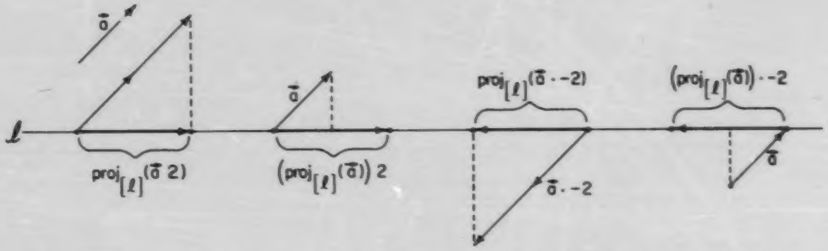
$$\begin{aligned}\mathrm{proj}_{[\ell]}(-\vec{a}) &= \mathrm{proj}_\ell\big((P + \vec{a}) + -\vec{a}\big) - \mathrm{proj}_\ell(P + \vec{a})\\ &= \mathrm{proj}_\ell(P) - \mathrm{proj}_\ell(P + \vec{a})\\ &= -\big(\mathrm{proj}_\ell(P + \vec{a}) - \mathrm{proj}_\ell(P)\big) = -\mathrm{proj}_{[\ell]}(\vec{a}).\end{aligned}$$

3. Here is an appropriate picture for parts (a) - (d):


$\vec{d}$
$\vec{e}$
$\vec{d} + \vec{e}$
ℓ
$\mathrm{proj}_{[\ell]}(\vec{d})$
$\mathrm{proj}_{[\ell]}(\vec{e})$
$\mathrm{proj}_{[\ell]}(\vec{d} + \vec{e}) = \mathrm{proj}_{[\ell]}(\vec{d}) + \mathrm{proj}_{[\ell]}(\vec{e})$

4. They are equal; that $\mathrm{proj}_{[\ell]}(\vec{d} + \vec{e}) = \mathrm{proj}_{[\ell]}(\vec{d}) + \mathrm{proj}_{[\ell]}(\vec{e})$, for any $\vec{d}$ and $\vec{e}$.

5. Here is an appropriate picture for parts (a) - (d):


$\vec{a}$
ℓ
$\mathrm{proj}_{[\ell]}(\vec{a}\,2)$
$\vec{a}$
$(\mathrm{proj}_{[\ell]}(\vec{a}))\,2$
$\mathrm{proj}_{[\ell]}(\vec{a} \cdot -2)$
$\vec{a} \cdot -2$
$(\mathrm{proj}_{[\ell]}(\vec{a})) \cdot -2$
$\vec{a}$

The first of these is easy to establish by using (1) [page 18]:

$$\begin{aligned}
\text{proj}_{[l]}\,(\overrightarrow{a} + \overrightarrow{b}) &= \text{proj}_l\,(P + (\overrightarrow{a} + \overrightarrow{b})) - \text{proj}_l\,(P) && \text{[Why?]}\\
&= \text{proj}_l\,((P + \overrightarrow{a}) + \overrightarrow{b}) - \text{proj}_l\,(P) && \text{[Why?]}\\
&= (\text{proj}_l\,((P + \overrightarrow{a}) + \overrightarrow{b}) - \text{proj}_l\,(P + \overrightarrow{a}))\\
&\qquad + (\text{proj}_l\,(P + \overrightarrow{a}) - \text{proj}_l\,(P)) && \text{[Why?]}\\
&= \text{proj}_{[l]}\,(\overrightarrow{b}) + \text{proj}_{[l]}\,(\overrightarrow{a}) && \text{[Why?]}\\
&= \text{proj}_{[l]}\,(\overrightarrow{a}) + \text{proj}_{[l]}\,(\overrightarrow{b}) && \text{[Why?]}
\end{aligned}$$

In establishing (5) we make use of Theorem I [page 4]. Let π_1, π_2, σ_1, and σ_2 be the planes perpendicular to l which contain $P, P + \overrightarrow{a}, P$, and $P + \overrightarrow{a}a$, respectively. [Then, $\pi_1 = \sigma_1$.] In case $\pi_1 \neq \pi_2$ and $\sigma_1 \neq \sigma_2$ it follows, just as in the proof of (*) [page 17], that

$$\begin{aligned}
(\text{proj}_l\,(P + \overrightarrow{a}a) - \text{proj}_l\,(P)) : (\text{proj}_l\,(P + \overrightarrow{a}) - \text{proj}_l\,(P))
&= ((P + \overrightarrow{a}a) - P) : ((P + \overrightarrow{a}) - P)\\
& && \text{[Why?]}\\
&= (\overrightarrow{a}a) : \overrightarrow{a} = a. && \text{[Why?]}
\end{aligned}$$

So, by (1), (5) holds in case $\pi_1 \neq \pi_2$ and $\sigma_1 \neq \sigma_2$. In case $\pi_1 = \pi_2$ it follows, again as in the proof of (*), that $\overrightarrow{a} \in [\sigma_1]$. So, in this case $\overrightarrow{a}a \in [\sigma_1]$ and $\sigma_2 = \sigma_1 + \overrightarrow{a}a = \sigma_1$. So, in case $\pi_1 = \pi_2$ it follows that

$$\text{proj}_{[l]}\,(\overrightarrow{a}a) = \overrightarrow{0} = \overrightarrow{0}a = \text{proj}_{[l]}\,(\overrightarrow{a})a.$$

A similar argument shows that (5) holds in case $\sigma_1 = \sigma_2$ and $a \neq 0$. Finally, in case $a = 0$,

$$\text{proj}_{[l]}\,(\overrightarrow{a}a) = \text{proj}_{[l]}\,(\overrightarrow{0}) = \overrightarrow{0} = \text{proj}_{[l]}\,(\overrightarrow{a})a.$$

11.03 Orthogonal Translations

In Section 11.01 we considered four notions concerning the perpendicularity of planes to lines. These notions were reasonable ones to accept on intuitive grounds, and might well have been taken as additional postulates for our development of geometry. However, these notions are concerned with lines and planes and our aim is, as it has been, to choose as postulates statements about points, translations, and real numbers. Our hope is that the notions we have about perpendicularity will suggest statements which will be acceptable as postulates.

As a first step toward finding such statements, recall that our notions about perpendicularity led us to the concept of the orthogonal complement, $[l]^{\perp}$, of a proper direction $[l]$. We described $[l]^{\perp}$ as the bi-

Answers for Exercises [cont.]

6. (a) and (b) are equal; (c) and (d) are equal; (a) and (c) are opposites.
7. (a) $[\vec{a}] \subseteq [\ell]^{\perp}$; that is, $[\vec{a}]$ is in the orthogonal complement of $[\ell]$.

   (b) It equals $\vec{0}$.

TC 20

Answers to the 'Why's:

Instance of (1), replacing '$\vec{a}$' by '$\vec{a} + \vec{b}$' and 'm' by 'ℓ'.

Algebra of points and translations

Postulate 3

Consequence of [two applications of] sentence (1)

Postulate 4

Theorem I

Postulate 4 and Theorem 7-24

*

Sample Quiz

1. Suppose that $\pi \perp \ell$. What can you say about a line m such that

   (a) $m \parallel \ell$? (b) $\pi \perp m$? (c) $\pi \not\perp m$?

2. Complete each of the given sentences with one of the following:

   (1) a point (2) a translation (3) a direction

   (4) a bidirection (5) perpendicular (6) parallel

   (a) $\text{proj}_{[\ell]}(\vec{a})$ is ___ (b) $\text{proj}_{\ell}(P)$ is ___

   (c) $[\ell]^{\perp}$ is ___ (d) $[\ell]$ is ___

   (e) $[\pi]^{\perp}$ is ___ (f) $[\pi]$ is ___

   (g) If $[\ell]^{\perp} = [\pi]$, ℓ and π are ___

   (h) If $[\ell]^{\perp} = [m]^{\perp}$, ℓ and m are ___

   (i) If $[\pi]^{\perp} = [\sigma]^{\perp}$, π and σ are ___

   (j) If $[\ell] \subseteq [m]^{\perp}$, ℓ and m are ___

Answers for Sample Quiz

1. (a) $m \perp \pi$ (b) $m \parallel \ell$ (c) $m \not\parallel \ell$
2. (a) (2) (b) (1) (c) (4) (d) (3) (e) (3)

   (f) (4) (g) (5) (h) (6) (i) (6) (j) (5)

direction of the planes which are perpendicular to the lines whose direction is $[l]$. This concept of "orthogonal complementing" may be described as follows:

$$\pi \perp l \longleftrightarrow [\pi] = [l]^{\perp} \tag{1}$$

Orthogonal complementing is an operation which can be applied to any proper direction to yield a proper bidirection. [Make a conjecture as to what the orthogonal complement of $[\vec{0}]$ might be. How about the orthogonal complement of $\mathcal{T}$?] Our notions concerning perpendicularity suggested the following two properties of this operation:

$$(*) \qquad [l]^{\perp} = [m]^{\perp} \longrightarrow [l] = [m]$$
$$[m] \subseteq [l]^{\perp} \longrightarrow [l] \subseteq [m]^{\perp}$$

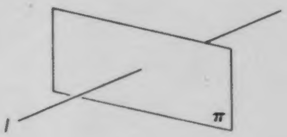
[These are statements (3) and (4) on pages 14 and 15.] As a matter of fact, we noticed that these properties and (1) could be used to deduce three of our notions concerning perpendicularity. Notion 1—that if $\pi \perp l$ then $l \nparallel \pi$—amounts, in view of (1), to saying that

$$(**) \qquad [l] \not\subseteq [l]^{\perp}.$$

Exercises

Part A

1. Here is a picture of a line l and a plane π whose bidirection is the orthogonal complement of $[l]$.



   (a) What can you say about π and l?
   (b) Each of $[l]$ and $[\pi]$ is a set of translations. Is their intersection empty? Are there non-$\vec{0}$ translations in each of them? Is there a non-$\vec{0}$ translation in their intersection? Give "geometric" arguments to support your answers.
   (c) Let $\vec{a}$ and $\vec{b}$ be proper translations in $[l]$ and $[\pi]$, respectively. What can you say about $(\vec{a}, \vec{b})$? Give an argument to support your answer.
2. Assume that $\vec{a} \neq \vec{0}$.
   (a) Given a point—say, P—draw a picture of the line $\overleftrightarrow{P(P + \vec{a})}$. Now, draw a picture of $\overline{P[\vec{a}]}^{\perp}$.
   (b) Describe $\overline{P[\vec{a}]}^{\perp}$. What can you say about $\overline{Q[\vec{a}]}^{\perp}$, for any point Q?
   (c) Does $\vec{a}$ belong to $[\vec{a}]^{\perp}$? Give a "geometric" argument to support your answer.

The exercises on page 21 will help to illustrate the ideas presented in this section. A suggested plan is to use the exercise sets on pages 21 and 24 as class exercises, and to use the exercises on pages 25-26 as homework.

Answers for Part A

1. (a) π is perpendicular to l
   (b) No. [l intersects π in a single point — say, P — and, so, P - P is in both $[l]$ and $[\pi]$. So, $[l] \cap [\pi] \neq \emptyset$.]

   Yes. [Each of l and π is a nondegenerate set of points. So, each contains at least two points. Given that P and Q are two points of l [or, of π] then $Q - P \neq \vec{0}$ and $Q - P \in [l]$ [or, $Q - P \in [\pi]$.]

   No. [For, if $[l] \cap [\pi]$ contains a non-$\vec{0}$ translation, then there are two points — say, P and Q — in both l and π. This implies that l is a subset of π, so that $l \parallel \pi$. But, since $\pi \perp l$ we know, by Notion 1, that $l \nparallel \pi$.]
   (c) Linearly independent. From (b) we know that no proper translation is contained in both $[l]$ and $[\pi]$. Since $\vec{a} \in [l]$, and $\vec{a} \neq \vec{0}$, $\vec{a} \notin [\pi]$. Now, $\vec{b} \in [\pi]$, so that $\vec{a}$ is not a linear combination of $\vec{b}$. So, since $\vec{a}$ and $\vec{b}$ are proper translations, the conclusion follows.

   [Here is an alternate argument: Suppose that $(\vec{a}, \vec{b})$ is linearly dependent. Then, since $\vec{b} \neq \vec{0}$, $\vec{a}$ is a multiple of $\vec{b}$ so that $\vec{a} \in [\pi]$. Thus, since $\vec{a} \neq \vec{0}$, $\vec{a}$ is a proper translation which is contained in both $[l]$ and $[\pi]$. Since there is no proper translation in $[l] \cap [\pi]$, it follows that $(\vec{a}, \vec{b})$ is linearly independent.]
2. (a) [Students should have pictures like the one drawn in Exercise 1.]
   (b) $\overline{P[\vec{a}]}^{\perp}$ is the plane through P and perpendicular to $\overleftrightarrow{P(P + \vec{a})}$. $\overline{Q[\vec{a}]}^{\perp}$ is parallel to $\overline{P[\vec{a}]}^{\perp}$, and is perpendicular to $\overleftrightarrow{P(P + \vec{a})}$.
   (c) No. [The argument is essentially that given for the third question in 1(b).]

3. (a) Since $[\vec{a}]^{\perp}$ is the bidirection of a plane — namely, one which is perpendicular to a line with direction $[\vec{a}]$ — it must contain two linearly independent members.
   (b) Linearly independent; linearly independent; linearly independent.
4. (a) When $(\vec{b}, \vec{c})$ is linearly dependent.
   (b) Linearly independent, for $\vec{a} \notin [\vec{a}]^{\perp} = [\vec{b}, \vec{c}]$.
   (c) When $\vec{c} \in [\pi_1]$.

3. (a) Given that $\vec{a} \neq \vec{0}$ it should be clear that $[\vec{a}]^\perp$ contains two linearly independent members – say $\vec{b}$ and $\vec{c}$. Explain.

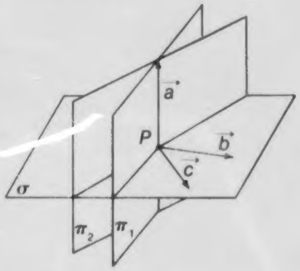
(b) Assuming that $\vec{a} \neq \vec{0}$ and that $\vec{b}$ and $\vec{c}$ are members of $[\vec{a}]^\perp$ such that $(\vec{b}, \vec{c})$ is linearly independent, what can be said of $(\vec{a}, \vec{b})$? Of $(\vec{a}, \vec{c})$? Of $(\vec{a}, \vec{b}, \vec{c})$?

4. Here is a picture in which $\sigma = \overleftrightarrow{P[\vec{a}]^\perp}$, $\pi_1 = \overleftrightarrow{P[\vec{b}]^\perp}$, $\pi_2 = \overleftrightarrow{P[\vec{c}]^\perp}$, and $\vec{b}$ and $\vec{c}$ are members of $[\sigma]$.

(a) Under what conditions will $\pi_1 = \pi_2$?

(b) Given that $(\vec{b}, \vec{c})$ is linearly independent, what can be said of $(\vec{a}, \vec{b}, \vec{c})$? Explain.

(c) Give conditions on $\vec{c}$ and π_1 under which $\vec{b} \in [\vec{c}]^\perp$.



The direction of a line l, $[l]$, is a set of translations. For any non-$\vec{0}$ translations $\vec{a}$ in $[l]$, $[l] = [\vec{a}]$. So, the properties of orthogonal complementing can be expressed in terms of directions of non-$\vec{0}$ translations as well as in terms of directions of lines. For example, instead of saying that, for any line l, the orthogonal complement $[l]^\perp$ is a proper bidirection, we can say:

(2) $\vec{a} \neq \vec{0} \Longrightarrow [\vec{a}]^\perp$ is a proper bidirection

Similarly, instead of (**) on page 21, we might say:

$$\vec{a} \neq \vec{0} \Longrightarrow [\vec{a}] \not\subseteq [\vec{a}]^\perp$$

However, $[\vec{a}]$ is a subset of a proper bidirection if and only if $\vec{a}$ belongs to that bidirection. So, we can also account for (**) by saying:

$$(3) \qquad \vec{a} \neq \vec{0} \Longrightarrow \vec{a} \notin [\vec{a}]^\perp.$$

In a similar fashion, the two parts of (*) on page 21 can be replaced by:

$$(4) \qquad [\vec{a}]^\perp = [\vec{b}]^\perp \Longrightarrow [\vec{a}] = [\vec{b}] \qquad [\vec{a} \neq \vec{0} \neq \vec{b}]$$

and:

$$(5) \qquad \vec{b} \in [\vec{a}]^\perp \Longrightarrow \vec{a} \in [\vec{b}]^\perp \qquad [\vec{a} \neq \vec{0} \neq \vec{b}]$$

Statements (2)–(5) are entirely in terms of translations, and, so, of the kind which we are looking for as postulates. And, if we adopted them as postulates, and adopted (1) as a definition, we would be able to establish as theorems Notions 1–4 concerning perpendicularity. However, it turns out that orthogonal complementing does not easily give us all that we need. As we shall see, there is another operation on translations which has very simple and familiar properties and which will give us all that we need. In order to be prepared to understand this operation, it will be helpful to learn a bit more about orthogonal complements.

To begin with, we note that

$$(6) \qquad \vec{b} \in [\vec{a}]^\perp \Longrightarrow (\vec{a}, \vec{b}) \text{ is linearly independent} \qquad [\vec{a} \neq \vec{0} \neq \vec{b}].$$

For, assuming that $\vec{a} \neq \vec{0}$, if $\vec{b} \in [\vec{a}]^\perp$ then, by (2), $[\vec{b}] \subseteq [\vec{a}]^\perp$ and, by (3), $\vec{a} \notin [\vec{b}]$. Hence, for $\vec{b} \neq \vec{0}$, $(\vec{a}, \vec{b})$ is linearly independent [Theorem 6–13].

Secondly, note that

$$(\vec{c} \in [\vec{b}]^\perp \text{ and } \vec{a} \in [\vec{c}]^\perp) \Longrightarrow \vec{c} \notin [\vec{a}, \vec{b}] \qquad [\vec{b} \neq \vec{0} \neq \vec{c}].$$

To show this assume, for $\vec{b} \neq \vec{0} \neq \vec{c}$, that $\vec{c} \in [\vec{b}]^\perp$ and $\vec{a} \in [\vec{c}]^\perp$. Then, by (5) we have that $\{\vec{a}, \vec{b}\} \subseteq [\vec{c}]^\perp$. So, by (2), $[\vec{a}, \vec{b}] \subseteq [\vec{c}]^\perp$. Hence, by (3), $\vec{c} \notin [\vec{a}, \vec{b}]$.

We can combine the last two results as follows. Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are proper translations such that $\vec{b} \in [\vec{a}]^\perp$, $\vec{c} \in [\vec{b}]^\perp$, and $\vec{a} \in [\vec{c}]^\perp$. Note that, by the result just proved, $\vec{c} \notin [\vec{a}, \vec{b}]$ and that, by (6), $(\vec{a}, \vec{b})$ is linearly independent. So, it follows that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent. Hence, we have that

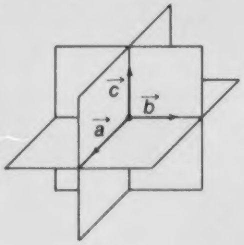


Fig. 11–8

$$(***) \qquad \begin{aligned}&(\vec{b} \in [\vec{a}]^\perp \text{ and } \vec{c} \in [\vec{b}]^\perp \text{ and } \vec{a} \in [\vec{c}]^\perp)\\ &\Longrightarrow (\vec{a}, \vec{b}, \vec{c}) \text{ is linearly independent} \qquad [\vec{a}, \vec{b}, \vec{c} \neq \vec{0}]\end{aligned}$$

When $\vec{b} \in [\vec{a}]^\perp$ it is customary to say that $\vec{b}$ is orthogonal to $\vec{a}$. In these terms, the orthogonal complement of $\vec{a}$, $[\vec{a}]^\perp$, is the set of all translations which are orthogonal to $\vec{a}$. By (5) – at least for non-$\vec{0}$ $\vec{a}$ and $\vec{b}$ – if

$\vec{b}$ is orthogonal to $\vec{a}$ then $\vec{a}$ is orthogonal to $\vec{b}$. So, if $\vec{b}$ is orthogonal to $\vec{a}$ we may say, instead, that $\vec{a}$ and $\vec{b}$ are orthogonal.

We know, since $\mathscr{T}$ is 3-dimensional, that any 3-termed sequence of translations is a basis for $\mathscr{T}$ if and only if it is linearly independent. So, (***) can be restated as follows:

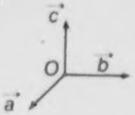
(7) Any 3-termed sequence of non-$\vec{0}$, pair-wise orthogonal translations is a basis for $\mathscr{T}$.

Any such basis is called *an orthogonal basis.*

Part B

Given that $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis for $\mathscr{T}$ and that O is the origin of a coordinate system associated with this basis. Assume that

$$\vec{d_1} = \vec{a}3 + \vec{b}4 + \vec{c}5,$$
$$\vec{d_2} = \vec{a} \cdot -3 + \vec{b}3 + \vec{c} \cdot -2,$$
$$\text{and } \vec{d_3} = \vec{d_1}2 + \vec{d_2} \cdot -1.$$



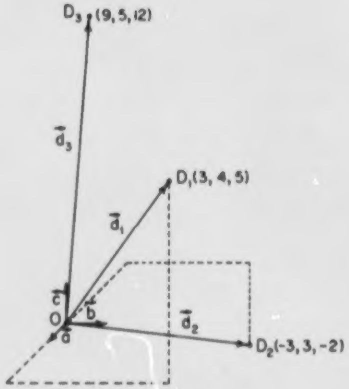
1. On your paper, make a careful sketch of the given coordinate system and mark the points D_1, D_2, and D_3, where $D_1 = O + \vec{d_1}$, $D_2 = O + \vec{d_2}$, and $D_3 = O + \vec{d_3}$. Give the coordinates of D_1, D_2, and D_3.
2. Let π_1 be the plane which contains D_1 and is parallel to the first coordinate plane. [Recall that the first coordinate plane is $\overleftrightarrow{O[\vec{b}, \vec{c}]}$.]
   (a) Give the coordinates of the point of intersection of π_1 and $\overleftrightarrow{O[\vec{a}]}$.
   (b) What can you say about π_1 and $\overleftrightarrow{O[\vec{a}]}$? About the first coordinate plane and $\overleftrightarrow{O[\vec{a}]}$?
   (c) Give the coordinates of the orthogonal projection of D_1 on $\overleftrightarrow{O[\vec{a}]}$.
3. Let π_2 and π_3 be the planes which contain D_1 and are parallel, respectively, to the second and third coordinate planes.
   (a) Give the coordinates of $\pi_2 \cap \overleftrightarrow{O[\vec{b}]}$ and $\pi_3 \cap \overleftrightarrow{O[\vec{c}]}$.
   (b) What can you say about π_2 and $\overleftrightarrow{O[\vec{b}]}$? About the second coordinate plane and $\overleftrightarrow{O[\vec{b}]}$? About π_3 and $\overleftrightarrow{O[\vec{c}]}$? About the third coordinate plane and $\overleftrightarrow{O[\vec{c}]}$?
   (c) Give the coordinates of the orthogonal projections of D_1 on $\overleftrightarrow{O[\vec{b}]}$ and $\overleftrightarrow{O[\vec{c}]}$.
4. Repeat Exercises 2 and 3
   (a) in the case of D_2. (b) in the case of D_3.

*

Given that $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis, any coordinate system associated with this basis is said to be orthogonal. As suggested in the above exercises, an orthogonal cartesian coordinate system has the property that each of its coordinate planes is perpendicular to the cor-

Answers for Part B

1. Here is an appropriate picture for the given information. The coordinates of D_1 are (3, 4, 5), of D_2 are (−3, 3, −2), and of D_3 are (9, 5, 12).



2. (a) (3, 0, 0)
   (b) π_1 is perpendicular to $\overleftrightarrow{O[\vec{a}]}$; perpendicular.
   (c) (3, 0, 0)
3. (a) (0, 4, 0) and (0, 0, 5)
   (b) π_2 is perpendicular to $\overleftrightarrow{O[\vec{b}]}$; perpendicular; π_3 is perpendicular to $\overleftrightarrow{O[\vec{c}]}$; perpendicular
   (c) (0, 4, 0) and (0, 0, 5)

4.	(a)	[2(a)]	(−3, 0, 0)	[3(a)]	(0, 3, 0) and (0, 0, −2)
		[2(b)]	same as 2(b), above	[3(b)]	same as 3(b), above
		[2(c)]	(−3, 0, 0)	[3(c)]	(0, 3, 0) and (0, 0, −2)
	(b)	[2(a)]	(9, 0, 0)	[3(a)]	(0, 5, 0) and (0, 0, 12)
		[2(b)]	same as 2(b), above	[3(b)]	same as 3(b), above
		[2(c)]	(9, 0, 0)	[3(c)]	(0, 5, 0) and (0, 0, 12)

responding coordinate axis. Making use of properties of orthogonal complements, this is not difficult to show. As a matter of fact, it is sufficient to show that

$$(8)\qquad \begin{aligned} &(\overrightarrow{b} \in [\overrightarrow{a}]^{\perp} \text{ and } \overrightarrow{c} \in [\overrightarrow{b}]^{\perp} \text{ and } \overrightarrow{a} \in [\overrightarrow{c}]^{\perp}) \\ &\longrightarrow ([\overrightarrow{b}, \overrightarrow{c}] = [\overrightarrow{a}]^{\perp} \text{ and } [\overrightarrow{c}, \overrightarrow{a}] = [\overrightarrow{b}]^{\perp} \text{ and } [\overrightarrow{a}, \overrightarrow{b}] = [\overrightarrow{c}]^{\perp}) \\ &[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \neq \overrightarrow{0}]. \end{aligned}$$

To show (8), suppose that, for non-$\overrightarrow{0}$ vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$, $\overrightarrow{b} \in [\overrightarrow{a}]^{\perp}$, $\overrightarrow{c} \in [\overrightarrow{b}]^{\perp}$, and $\overrightarrow{a} \in [\overrightarrow{c}]^{\perp}$. Since $\overrightarrow{b} \in [\overrightarrow{a}]^{\perp}$ it follows from (6) that $(\overrightarrow{a}, \overrightarrow{b})$ is linearly independent. Now, $\overrightarrow{c} \in [\overrightarrow{b}]^{\perp}$ and $\overrightarrow{a} \in [\overrightarrow{c}]^{\perp}$ so that, by (5), $\{\overrightarrow{a}, \overrightarrow{b}\} \subseteq [\overrightarrow{c}]^{\perp}$. Hence, by (2), $[\overrightarrow{c}]^{\perp} = [\overrightarrow{a}, \overrightarrow{b}]$. Similarly, $[\overrightarrow{a}]^{\perp} = [\overrightarrow{b}, \overrightarrow{c}]$ and $[\overrightarrow{b}]^{\perp} = [\overrightarrow{c}, \overrightarrow{a}]$.

One advantage of dealing with orthogonal basis is that the components of a vector—and, so, the coordinates of a point—with respect to such a basis are easily found by using the projection operation discussed in Section 11.02. To see this, suppose that $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ is an orthogonal basis and that $\overrightarrow{r} = \overrightarrow{a}a + \overrightarrow{b}b + \overrightarrow{c}c$. Since $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ is orthogonal, it follows that $[\overrightarrow{b}, \overrightarrow{c}] = [\overrightarrow{a}]^{\perp}$. From this, we know that $\overrightarrow{b}b + \overrightarrow{c}c \in [\overrightarrow{a}]^{\perp}$ so that, by (4) on page 19, $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{b}b + \overrightarrow{c}c) = 0$. Since $\overrightarrow{a}a \in [\overrightarrow{a}]$ it follows, by (2) on page 18, that $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{a}a) = \overrightarrow{a}a$. So, since $\overrightarrow{r} = \overrightarrow{a}a + (\overrightarrow{b}b + \overrightarrow{c}c)$, we know that $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{r}) = \overrightarrow{a}a$. Similarly, $\mathrm{proj}_{[\overrightarrow{b}]}(\overrightarrow{r}) = \overrightarrow{b}b$ and $\mathrm{proj}_{[\overrightarrow{c}]}(\overrightarrow{r}) = \overrightarrow{c}c$. That is,

$$(*)\qquad \overrightarrow{r} = \mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{r}) + \mathrm{proj}_{[\overrightarrow{b}]}(\overrightarrow{r}) + \mathrm{proj}_{[\overrightarrow{c}]}\overrightarrow{r}.$$

Part C

Assume that $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ is an orthogonal basis. Let $\overrightarrow{r_1} = \overrightarrow{a}5 + \overrightarrow{b} \cdot -2 + \overrightarrow{c}3$ and $\overrightarrow{r_2} = \overrightarrow{a}4 + \overrightarrow{b} \cdot -3$.

1. Give each of the following in terms of the basis vectors. [You may find it helpful to draw a graph.]
   (a) $\overrightarrow{r_1} + \overrightarrow{r_2}$ (b) $\overrightarrow{r_1} - \overrightarrow{r_2}$
   (c) $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{r_1} + \overrightarrow{r_2})$ (d) $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{r_1} - \overrightarrow{r_2})$
   (e) $\mathrm{proj}_{[\overrightarrow{b}]}(\overrightarrow{r_1} + \overrightarrow{r_2})$ (f) $\mathrm{proj}_{[\overrightarrow{b}]}(\overrightarrow{r_1} - \overrightarrow{r_2})$
   (g) $\mathrm{proj}_{[\overrightarrow{c}]}(\overrightarrow{r_1} + \overrightarrow{r_2})$ (h) $\mathrm{proj}_{[\overrightarrow{c}]}(\overrightarrow{r_1} - \overrightarrow{r_2})$
2. Compute each of the following:
   (a) $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{r_1} + \overrightarrow{r_2}) : \overrightarrow{a}$ [*Hint*: Since $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{r_1} + \overrightarrow{r_2}) = \overrightarrow{a}t$, for some t, and $\overrightarrow{a}t : \overrightarrow{a} = t$, it is enough to compute t in order to do this problem.]
   (b) $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{r_1} - \overrightarrow{r_2}) : \overrightarrow{a}$ (c) $\mathrm{proj}_{[\overrightarrow{b}]}(\overrightarrow{r_1} - \overrightarrow{r_2}) : \overrightarrow{b}$
   (d) $\mathrm{proj}_{[\overrightarrow{c}]}(\overrightarrow{r_1} - \overrightarrow{r_2}) : \overrightarrow{c}$ (e) $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{r_2}) : \overrightarrow{a}$
   (f) $\mathrm{proj}_{[\overrightarrow{b}]}(\overrightarrow{r_2}) : \overrightarrow{b}$ (g) $\mathrm{proj}_{[\overrightarrow{c}]}(\overrightarrow{r_2}) : \overrightarrow{c}$
3. Let $\overrightarrow{t} = \overrightarrow{a}a + \overrightarrow{b}b + \overrightarrow{c}c$, for some a, b, and c.
   (a) Express each of $\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{t})$, $\mathrm{proj}_{[\overrightarrow{b}]}(\overrightarrow{t})$, and $\mathrm{proj}_{[\overrightarrow{c}]}(\overrightarrow{t})$ in terms of the basis vectors.

Some review questions which might be asked at this time are:

1. Are the points D_1, D_2, and D_3 collinear?
2. Show that D_1 is the midpoint of $\overline{D_2D_3}$.
3. Write parametric equations for the line $\overleftrightarrow{D_2D_3}$.
4. In what ratio does D_2 divide the segment from D_1 to D_3? [i.e. What is $(D_2 - D_1):(D_3 - D_2)$?]
5. What are the coordinates of the point which divides the segment from D_2 to D_3 in the ratio $-3/2$?

Answers to review questions

1. Yes. [$D_1 - D_2 = \vec{d}_1 - \vec{d}_2 = \vec{a}6 + \vec{b} + \vec{c}7$ and $D_1 - D_3 = \vec{d}_1 - \vec{d}_3 = \vec{a} \cdot -6 + \vec{b} \cdot -1 + \vec{c} \cdot -7$. So, $(D_1 - D_2, D_1 - D_3)$ is linearly dependent.]
2. From 1, we know that $D_2 - D_1 = D_1 - D_3$. So, by definition, D_1 is the midpoint of $\overline{D_2D_3}$.
3. One set of parametric equations for $\overleftrightarrow{D_2D_3}$ is: $\begin{cases} x_1 = -3 + 12t \\ x_2 = 3 + 2t \\ x_3 = -2 + 14t \end{cases}$

   [There are, of course, others.]
4. $-1/2$
5. (33, 9, 40) [Choose $t = 3$ in the equations given in 3.]

Answers for Part C

1. (a) $\vec{a}9 + \vec{b} \cdot -5 + \vec{c}3$ (b) $\vec{a} + \vec{b} + \vec{c}3$
   (c) $\vec{a}9$ (d) $\vec{a}$
   (e) $\vec{b} \cdot -5$ (f) $\vec{b}$
   (g) $\vec{c}3$ (h) $\vec{c}3$
2. (a) 9 (b) 1
   (c) 1 (d) 3
   (e) 4 (f) -3
   (g) 0
3. (a) $\vec{a}a$, $\vec{b}b$, and $\vec{c}c$

(b) Express each of a, b, and c as ratios in terms of the projection operation.

(c) By definition, the components of $\vec{t}$ with respect to $(\vec{a}, \vec{b}, \vec{c})$ are (a, b, c). Express the components of $\vec{t}$ as ratios in terms of the projection operation.

*

We summarize the results suggested in the preceding discussion and exercises as follows:

(9) If $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis for $\mathscr{T}$ then
$\vec{r} = \text{proj}_{[\vec{a}]}(\vec{r}) + \text{proj}_{[\vec{b}]}(\vec{r}) + \text{proj}_{[\vec{c}]}(\vec{r})$
and the components of $\vec{r}$ with respect to $(\vec{a}, \vec{b}, \vec{c})$ are
$\text{proj}_{[\vec{a}]}(\vec{r}) : \vec{a}$, $\text{proj}_{[\vec{b}]}(\vec{r}) : \vec{b}$, and $\text{proj}_{[\vec{c}]}(\vec{r}) : \vec{c}$.

The results summarized in (9) show that when we are dealing with an orthogonal basis for $\mathscr{T}$ [or with an orthogonal coordinate system for $\mathscr{E}$] each component of a translation [or each coordinate of a point] can be expressed in terms of just one of the basis vectors and the orthogonal projection of the given translation [or of the position vector of the given point] in the direction of this basis vector. This is in contrast to what occurs when we deal with a nonorthogonal basis $(\vec{a}, \vec{b}, \vec{c})$. To find, say, the first component of $\vec{r}$ with respect to such a basis we need to consider not only $\vec{r}$ and $\vec{a}$ but, also, $\vec{b}$ and $\vec{c}$. As a result, when using coordinates to solve a geometric problem it is often much easier to use orthogonal coordinates than it is to use nonorthogonal coordinates. Examples of this will occur in later chapters.

11.04 Three Notions Concerning Distance

In order to arrive at postulates we can use to complete our study of Euclidean geometry we need to consider the notion of distance as well as that of perpendicularity. As it turns out, besides the four notions concerning perpendicularity which we studied in Section 11.01, we need three notions concerning the distance from a first point to a second and one additional notion which concerns both perpendicularity and distance.

The first of the needed notions about distance is certainly intuitively obvious. It is merely that the distance from one point to another is a positive real number. We may state it for the record as:

Notion 5. $d(A, B) \in \mathscr{R}$ and $[B \neq A \Longrightarrow d(A, B) > 0]$

The second and third notions go back to ideas about translations which were already introduced in Volume 1 in order to suggest the operation

Answers for Part C [cont.]

3. (b) $a = \text{proj}_{[\vec{a}]}(\vec{t}):\vec{a}$, $b = \text{proj}_{[\vec{b}]}(\vec{t}):\vec{b}$, $c = \text{proj}_{[\vec{c}]}(\vec{t}):\vec{c}$
(c) $(\text{proj}_{[\vec{a}]}(\vec{t}):\vec{a}$, $\text{proj}_{[\vec{b}]}(\vec{t}):\vec{b}$, $\text{proj}_{[\vec{c}]}(\vec{t}):\vec{c})$

of multiplying a translation by a real number. At that time [and, even earlier, in Chapter 1] we noted that, intuitively, a given translation "moves" any two points the same distance. We state this as:

Notion 6. $d(A, A + \vec{a}) = d(B, B + \vec{a})$

We also noted that the translation $\vec{aa}$ moves each point $|a|$ times as far as $\vec{a}$ does:

Notion 7. $d(A, A + \vec{aa}) = d(A, A + \vec{a}) \cdot |a|$

As we have done in the case of perpendicularity, it will be useful to restate these notions in terms of an operation on vectors. According to Notion 6 a given translation $\vec{a}$ moves all points the same distance. According to Notion 5 this distance is a real number. This number is usually called *the norm of* $\vec{a}$. Because the norm of a vector is analogous to the absolute value of a real number, the customary notation for the norm of $\vec{a}$ is '$\|\vec{a}\|$'. Using this notation, Notion 6 amounts to saying that $d(A, A + \vec{a}) = \|\vec{a}\|$. Equivalently:

(1) $$d(A, B) = \|B - A\|$$

Notions 5 and 7 can then be rewritten as:

(2) $$\|\vec{a}\| \in \mathscr{R} \text{ and } [\vec{a} \neq \vec{0} \longrightarrow \|\vec{a}\| > 0]$$

(3) $$\|\vec{aa}\| = \|\vec{a}\| \cdot |a|$$

Exercises

Part A

1. Make use of (1), (2), and (3), above, to derive the following.
   - **(a)** $\|-\vec{a}\| = \|\vec{a}\|$ **(b)** $\|-\vec{a}a\| = \|\vec{a}a\|$
   - **(c)** $d(B, A) = d(A, B)$ **(d)** $\|\vec{0}\| = 0$
   - **(e)** If $B = A$ then $d(A, B) = 0$ **(f)** If $d(A, B) = 0$ then $B = A$
2. Assume that $\vec{a} \neq \vec{0}$.
   - **(a)** Show that $\vec{a}/\|\vec{a}\|$ belongs to $[\vec{a}]^+$ – that is, belongs to the sense of $\vec{a}$. [*Hint*: What kind of real number is $/\|\vec{a}\|$?]
   - **(b)** Use (3) above to compute the norm of $\vec{a}/\|\vec{a}\|$.
3. Assume that $\vec{b} \in [\vec{a}]$ and that $\|\vec{b}\| = \|\vec{a}\|$. Show that $\vec{b} = \vec{a}$ or $\vec{b} = -\vec{a}$. [*Hint*: In the case $\vec{a} \neq \vec{0}$, make use of (2), (3), and the fact that $\vec{b} \in [\vec{a}]$ only if $\vec{b} = \vec{a}b$, for some b.]
4. A *unit translation* [or: unit vector] is one whose norm is 1.
   - **(a)** Show that each proper sense contains at least one unit vector.
   - **(b)** Show that each proper direction contains exactly two unit vectors.
   - **(c)** Show that each proper sense contains at most one unit vector.

Answers for Part A

1. (a) $||-\vec{a}|| = ||\vec{a} \cdot -1|| = ||\vec{a}||\,|-1| = ||\vec{a}||$
   (b) $||-\vec{a}a|| = ||-(\vec{a}a)|| = ||\vec{a}a||$
   (c) $d(B, A) = ||A - B|| = ||-(B - A)|| = ||B - A|| = d(A, B)$
   (d) $||\vec{0}|| = ||\vec{a}0|| = ||\vec{a}||\,|0| = 0$
   (e) Suppose that $B = A$. Then $d(A, B) = d(A, A) = ||A - A|| = ||\vec{0}|| = 0$. Hence, if $B = A$ then $d(A, B) = 0$.
   (f) Suppose that $d(A, B) = 0$. Then $||B - A|| = 0$ so that by (2) and a rule for contraposition, $B - A = \vec{0}$. So, $B = A$. Hence, if $d(A, B) = 0$ then $B = A$.
2. (a) Since $\vec{a} \neq \vec{0}$, it follows by (2) that $||\vec{a}|| > 0$. So, $/||\vec{a}|| > 0$. Hence, $\vec{a}/||\vec{a}||$ belongs to $[\vec{a}]^+$.
   (b) By (3) and (2), for $\vec{a} \neq \vec{0}$,
   $$\|\vec{a}/\|\vec{a}\|\| = \|\vec{a}\| \cdot |/\|\vec{a}\|| = \|\vec{a}\|/|\|\vec{a}\|| = \|\vec{a}\|/\|\vec{a}\| = 1.$$
3. Suppose that $\vec{b} = \vec{a}b$ and $||\vec{b}|| = ||\vec{a}||$. It follows, by (3) that $||\vec{b}|| = ||\vec{a}|| \cdot |b|$ and, so that $||\vec{a}|| = ||\vec{a}|| \cdot |b|$. In case $\vec{a} \neq \vec{0}$, $||\vec{a}|| \neq 0$ and it follows that $|b| = 1$. So, for $\vec{a} \neq \vec{0}$, $b = 1$ or $b = -1$ and, so, $\vec{b} = \vec{a}$ or $\vec{b} = -\vec{a}$. In case $\vec{a} = \vec{0}$ it follows, since $\vec{b} = \vec{a}b$, that $\vec{b} = \vec{0} = \vec{a}$. So, in any case, if $\vec{b} \in [\vec{a}]$ and $||\vec{b}|| = ||\vec{a}||$ then $\vec{b} = \vec{a}$ or $\vec{b} = -\vec{a}$.
4. (a) Given a proper sense, let $\vec{a}$ be a vector belonging to it. It follows that $\vec{a} \neq \vec{0}$ and that the given sense is $[\vec{a}]^+$. Hence, by Exercise 2, $\vec{a}/||\vec{a}||$ is a unit vector in the given proper sense.
   (b) Given a proper direction, let $\vec{a}$ be a unit vector belonging to it. [That there is such a unit vector follows from part (a) and the fact that the given sense contains a proper translation.] Since $\vec{a} \neq \vec{0}$ it follows that the given direction is $[\vec{a}]$. By Exercise 3, if $\vec{b}$ is any unit vector in $[\vec{a}]$ then $\vec{b} = \vec{a}$ or $\vec{b} = -\vec{a}$. Since $\vec{a} \neq \vec{0}$, $-\vec{a} \neq \vec{a}$ and, by Exercise 1(a), $-\vec{a}$ is a unit vector. Hence, there are exactly two unit vectors in a given proper direction.
   (c) [Like (b) except that, since $\vec{a} \neq \vec{0}$, $-\vec{a} \neq \vec{a}a$ for any $a > 0$.]

From the exercises just completed, we obtain the following property of norms of translations:

(4) $([\vec{b}] = [\vec{a}] \text{ and } \|\vec{b}\| = \|\vec{a}\|) \longrightarrow (\vec{b} = \vec{a} \text{ or } \vec{b} = -\vec{a})$

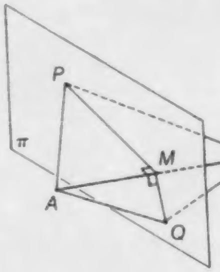
As we saw, (4) together with the notion of a unit vector can be used to show that

(5) $\vec{a}/\|\vec{a}\|$ is *the* unit vector in the sense of $\vec{a}$ $[\vec{a} \neq \vec{0}]$.

[And, this vector and its opposite are the only unit vectors in the direction of $\vec{a}$.]

Part B

1. Suppose that M is the midpoint of the segment $\overline{AB}$ and is also the projection of a point P on $\overleftrightarrow{AB}$.
   (a) Make a conjecture about $d(A,P)$ and $d(B,P)$. [A cardboard and string model will help here.]
   (b) Let Q be any point in the plane π which contains M and which is perpendicular to $\overleftrightarrow{AB}$. What can you say about $d(A, Q)$ and $d(B, Q)$?
   (c) Let R be any point such that $d(A, R) = d(B, R)$. What is the projection of R on $\overleftrightarrow{AB}$? Is R in the plane π?



2. Suppose that $\overline{AB}$ and $\overline{BC}$ are two collinear segments whose midpoints are M and N, respectively.
   (a) Given that π and σ are the planes perpendicular to $\overline{AB}$ and $\overline{BC}$, respectively, at M and N, what can you say about π and σ?
   (b) Let $P \in \pi$ and $Q \in \sigma$, where π and σ are the planes described in (a). What can you say about $d(A, P)$ and $d(B, P)$? About $d(B, Q)$ and $d(C, Q)$?
   (c) Assume that R is a point such that $d(A, R) = d(B, R)$. Is R in π? Is R in σ? Is it possible for R to be in both π and σ?
3. Answer the questions in Exercise 2, given that $\overline{AB}$ and $\overline{BC}$ are two *noncollinear* segments.
4. Consider the case described in Exercise 3. Let S be any point in both π and σ. [Are there such points?]

Answers for Part B

[The purpose of this set of exercises is to develop the notion, at least intuitively, that the plane perpendicular to a segment through the midpoint of that segment is the set of all points equidistant from the end points of the segment.]

1. (a) They are equal.
   (b) They are equal.
   (c) M; Yes.
2. (a) They are parallel.
   (b) They are equal.; They are equal.
   (c) Yes.; No, for π and σ are parallel and, since $M \in \pi$ and $M \notin \sigma$, $\pi \cap \sigma = \emptyset$.; No.
3. (a) They are not parallel. [If they were then $[B - A]^{\perp}$ would be $[C - B]^{\perp}$ and, by (4) on page 22, $[B - A]$ would be $[C - B]$.]
   (b) They are equal.; They are equal.
   (c) Yes.; Only if R is equidistant from B and C.; Yes, if it is on the line of intersection of π and σ.
4. There are points in $\pi \cap \sigma$ because $\pi \not\parallel \sigma$.
   (a) Yes.; M
   (b) Yes.; N
   (c) Yes, for given that $d(A, S) = d(B, S)$ and $d(B, S) = d(C, S)$, it follows by the replacement rule for equations that $d(A, S) = d(C, S)$.
   (d) π' intersects $\overline{AC}$ at its midpoint.

(a) Is S equidistant from A and B – that is, is $d(A, S) = d(B, S)$? What is the projection of S on $\overleftrightarrow{AB}$?

(b) Is S equidistant from B and C? What is the projection of S on $\overleftrightarrow{BC}$?

(c) Given that S is both equidistant from A and B and from B and C, can you conclude that S is equidistant from A and C? Explain.

(d) Let π' be the plane which contains S and is perpendicular to $\overleftrightarrow{AC}$. Make a conjecture about where π' intersects $\overline{AC}$.

11.05 A Notion Concerning Perpendicularity and Distance

The exercises just completed suggest our final notion:

Notion 8. The projection of P on $\overleftrightarrow{AB}$ is the midpoint of $\overline{AB}$

$$\Longleftrightarrow$$

$$d(A, P) = d(B, P) \qquad [A \neq B]$$

We have managed to restate each of Notions 1–7 in a more convenient form of terms of orthogonal complements of directions and norms of vectors. Notions 1–4 were restated in these terms in sentences (1)–(5) of Section 11.03, and Notions 5–7 were restated in sentences (1)–(3) of Section 11.04. Our aim here is to restate Notion 8 in these same terms.

In the first place, the midpoint of $\overline{AB}$ is $A + (B - A)\frac{1}{2}$ and, of course, belongs to $\overleftrightarrow{AB}$. So, to say that

the projection of P on $\overleftrightarrow{AB}$ is the midpoint of $\overline{AB}$
amounts to saying that
$P - (A + (B - A)\frac{1}{2}) \in [B - A]^{\perp}$.
[Explain.]

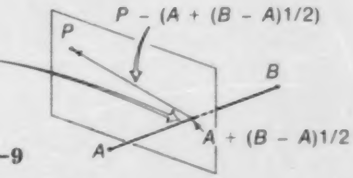


Fig. 11–9

Now, $P - (A + (B - A)\frac{1}{2}) = \dfrac{(P - A) + (P - B)}{2}$ and $B - A = (P - A) - (P - B)$. [Why?] So,

the projection of P on $\overleftrightarrow{AB}$ is the midpoint of $\overline{AB}$
if and only if
$(P - A) + (P - B) \in [(P - A) - (P - B)]^{\perp}$.

The restatement of our eight "Notions" in terms of orthogonal complementing and norming is given on **pp.** 35-36, **sentences** (1) - (10). Our postulates could be completed by adopting these ten sentences as additional parts of Postulate 4. This would introduce both orthogonal complementing and norming into our formal system as "undefined terms". As we shall see in section 11.07 we could, on this basis define a kind of multiplication of vectors, called inner multiplication. This multiplication has simple properties listed at the beginning of section 11.08. These, which we shall actually adopt as postulates, follow by using the sentences (3) - (10) on **pp.** 35-36. Conversely, sentences (3) - (10) follow by using these postulates and the definitions of orthogonal complementing and norming which are given in Definition 11-1 on page 45.

In summary, our eight notions concerning perpendicularity and distance are equivalent, as postulates, to our five postulates on page 42 concerning inner multiplication, together with appropriate definitions of orthogonal complementing, norming, perpendicularity, and distance. The fact that these postulates and definitions are, when added to Postulates 1 - 5′ a sufficient basis for developing Euclidean geometry can be interpreted as highlighting the fundamental character of our eight geometric notions.

Explanation called for in text: The orthogonal projection of a point P on $\overleftrightarrow{AB}$ is a point — say, Q — such that P - Q is in the orthogonal complement of the direction of $\overleftrightarrow{AB}$. In short, if $\text{proj}_{\overleftrightarrow{AB}}(P) = Q$ then $P - Q \in [B - A]^{\perp}$. Thus, to say that $\text{proj}_{\overleftrightarrow{AB}}(P)$ is $A + (B - A)\frac{1}{2}$ amounts to saying that $P - (A + (B - A)\frac{1}{2}) \in [B - A]^{\perp}$.

Answer for 'Why?': Algebra of points and translations.

Using these results, we have the following restatement of Notion 8 in terms of orthogonal complements and norms of vectors:

$$(P - A) + (P - B) \in [(P - A) - (P - B)]^{\perp} \Longleftrightarrow \|P - A\| = \|P - B\| \qquad [A \neq B]$$

An even simpler restatement of Notion 8 is:

$$(1) \qquad \vec{a} + \vec{b} \in [\vec{a} - \vec{b}]^{\perp} \Longleftrightarrow \|\vec{a}\| = \|\vec{b}\| \qquad [\vec{a} \neq \vec{b}]$$

Using Notion 8 we can establish some interesting results concerning triangles. These results will imply the final property of projections [(6) on page 35] which we need in order to arrive at our postulates.

To begin with, Notion 2 tells us that, for a given nondegenerate interval $\overline{AB}$, there is exactly one plane which contains the midpoint M of $\overline{AB}$ and is perpendicular to $\overleftrightarrow{AB}$. This plane is called *the perpendicular bisector* of $\overline{AB}$. Since a point P belongs to this plane if and only if the projection of P on $\overleftrightarrow{AB}$ is M, what Notion 8 says is that, for $A \neq B$,

(2) the perpendicular bisector of $\overline{AB}$ consists of just those points which are equidistant from A and B.

Exercises

Part A

Consider a triangle—say $\triangle ABC$—and let π_1, π_2, and π_3 be the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively.

1. Are π_1 and π_2 parallel? Explain your answer.
2. What kind of set is $\pi_1 \cap \pi_2$?
3. What can you say about the plane $\overleftrightarrow{ABC}$ of $\triangle ABC$ and $\pi_1 \cap \pi_2$? Explain.
4. What can be said about each point of $\pi_1 \cap \pi_2$ and the endpoints of $\overline{AB}$? Of each point of $\pi_1 \cap \pi_2$ and the endpoints of $\overline{BC}$?
5. What follows from Exercise 4 concerning the points of $\pi_1 \cap \pi_2$ and the endpoints of $\overline{AC}$?
6. What can be said about any point which is equidistant from A and C?
7. What conclusion can you draw from Exercises 5 and 6 concerning $\pi_1 \cap \pi_2$ and π_3?
8. What follows from Exercises 2, 4, and 7 concerning $\pi_1 \cap \pi_2 \cap \pi_3$ and $\overleftrightarrow{ABC}$?

Parts A and B provide exercises that can be used for class discussion. With this background, students should be able to do Part C (page 33) for homework.

Answers for Part A

1. No. Suppose that $\pi_1 \parallel \pi_2$. Then, since $\pi_1 \perp \overline{AB}$ and $\pi_2 \perp \overline{BC}$, it follows that $\overleftrightarrow{AB} \parallel \overline{BC}$. So, A, B, and C are collinear. Hence, if $\pi_1 \parallel \pi_2$ then $\{A, B, C\}$ is collinear. Since $\{A, B, C\}$ is noncollinear — for, ABC is a triangle — it follows that $\pi_1 \nparallel \pi_2$.
2. $\pi_1 \cap \pi_2$ is a line.
3. By (5) on page 15, $\overleftrightarrow{ABC} \perp (\pi_1 \cap \pi_2)$.

4. Each point of $\pi_1 \cap \pi_2$ is equidistant from the end points of $\overline{AB}$, for each such point is in π_1. Also, each point of $\pi_1 \cap \pi_2$ is equidistant from the end points of $\overline{BC}$, for each such point is in π_2.
5. Each point in $\pi_1 \cap \pi_2$ is equidistant from the end points of $\overline{AC}$, for each such point is equidistant from A, B, and C.
6. It is in π_3.
7. $(\pi_1 \cap \pi_2) \subset \pi_3$
8. $\pi_1 \cap \pi_2 \cap \pi_3$ is a line each point of which is equidistant from A, B, and C and which is perpendicular to $\overleftrightarrow{ABC}$.

The result obtained in Part A may be stated as follows:

(3) The intersection of the three perpendicular bisectors of the sides of any triangle is a line to which the plane of the triangle is perpendicular.

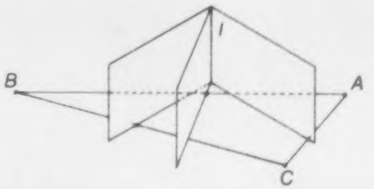


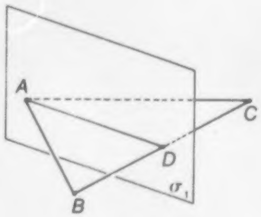
Fig. 11–10

From (3) [and Notion 1] it is easy to see that

(3′) the lines in which the perpendicular bisectors of the sides of a triangle intersect the plane of the triangle are concurrent.

Using this result we shall go on to establish another theorem concerning triangles.

Part B



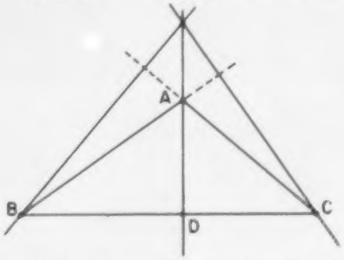
Consider, again, $\triangle ABC$ and let σ_1 be the plane which contains A and is perpendicular to $\overleftrightarrow{BC}$. By Notion 1, the intersection of σ_1 and $\overleftrightarrow{BC}$ consists of a single point – say, D. [Explain why $\sigma_1 \cap \overleftrightarrow{ABC}$ is $\overleftrightarrow{AD}$.] The interval $\overline{AD}$ is called *the altitude of $\triangle ABC$ from A* [or: *to* $\overleftrightarrow{BC}$].

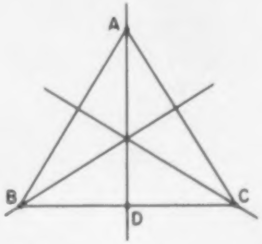
1. Let the surface of your paper represent the plane $\overleftrightarrow{ABC}$ and draw $\triangle ABC$ and the line $\overleftrightarrow{AD}$ which contains the altitude of $\triangle ABC$ from A.
2. Draw the lines which contain the altitudes of $\triangle ABC$ from B and from C, respectively.
3. Repeat Exercises 1 and 2 for each of at least two other triangles. [*Suggestion.* Try to find a triangle such that the altitudes from the respective vertices end at points of the opposite sides, and triangles for which this is not the case. Is there a triangle one of whose altitudes has vertices of the triangle for *both* of its endpoints?]
4. Make a conjecture concerning the lines which contain the altitudes of a triangle.

Answers for Part B

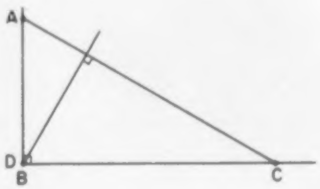
Since A and D are in both σ_1 and $\overleftrightarrow{ABC}$, it follows that $\overleftrightarrow{AD}$ is in both σ_1 and $\overleftrightarrow{ABC}$ and, so, is in their intersection. Since σ_1 and $\overleftrightarrow{ABC}$ are different planes, $\sigma_1 \cap \overleftrightarrow{ABC} = \overleftrightarrow{AD}$.

1-2. Here are two appropriate pictures, one in which $\triangle ABC$ is obtuse and one in which $\triangle ABC$ is acute.





Here is an appropriate picture where $\triangle ABC$ is a right triangle:



3. [See figures drawn for 1 and 2 above.]
4. They are concurrent.

*

The exercises of Part B may have suggested to you that

(4) the lines which contain the altitudes of a triangle are concurrent.

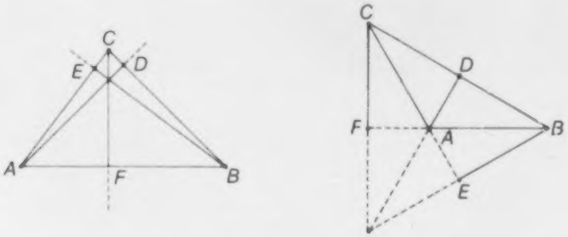
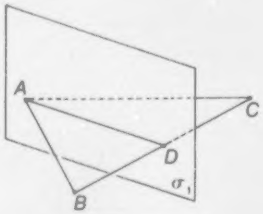


Fig. 11–11

One way of establishing (4) for $\triangle ABC$ is to find a $\triangle A'B'C'$ in the same plane such that the lines containing the altitudes of $\triangle ABC$ are the intersections with $\overline{A\overset{\uparrow}{B}C}$ of the perpendicular bisectors of the sides of $\triangle A'B'C'$. One way to do this is to see what might happen if we had already solved the problem. So, draw a $\triangle P'Q'R'$ and try to find a $\triangle PQR$ whose altitudes are contained in the perpendicular bisectors of the sides of $\triangle P'Q'R'$. If you succeed, consider any $\triangle ABC$. Can you find a $\triangle A'B'C'$ whose sides have perpendicular bisectors which contain the altitudes of $\triangle ABC$? If you manage to draw such a triangle, give an argument to show that it is one. If you don't, the exercises which follow suggest how to locate one such triangle.

Part C

Let's return to the situation described in Part B. Recall that σ_1 is the plane which contains A and is perpendicular to $\overleftrightarrow{BC}$ and that, consequently, $\overleftrightarrow{AD}$ is the line which contains the altitude of $\triangle ABC$ from A. We are looking for a $\triangle A'B'C'$ in the plane $\overline{A\overset{\uparrow}{B}C}$ such that σ_1 is the perpendicular bisector of one of its sides—say, the side $\overline{B'C'}$. [Of course, we want the perpendicular bisectors of the other sides of $\triangle A'B'C'$ to contain the other altitudes of $\triangle ABC$, but it will be easier to concentrate on a side at a time.] We wish, then, to choose $\overleftrightarrow{B'C'}$ so that σ_1 is perpendicular to $\overleftrightarrow{B'C'}$ at the midpoint of $\overline{B'C'}$ and so that $\overleftrightarrow{B'C'} \subseteq \overline{A\overset{\uparrow}{B}C}$.



Sample Quiz

1. Given that $(\vec{i}, \vec{j}, \vec{k})$ is an orthogonal basis, assume that

$$\vec{a} = \vec{i}2 - \vec{j} + \vec{k}3 \quad\text{and}\quad \vec{b} = -\vec{i} + \vec{j}2 + \vec{k}2.$$

Find the following.

(a) $\text{proj}_{[\vec{i}]}(\vec{a}2 - \vec{b}3)$ (b) $\text{proj}_{[\vec{j}]}(\vec{a}2 - \vec{b}3):\vec{j}$

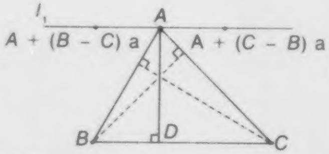
(c) $\text{proj}_{[\vec{k}]}(\vec{b}3 - \vec{a}2)$ (d) $\text{proj}_{[\vec{j}]}(\vec{b}3 - \vec{a}2):\vec{j}$

2. Suppose that P and Q are two points of ℓ, that $R \notin \ell$, and that $\text{proj}_{\ell}(R) = P$.
   (a) Find S such that R is equidistant from Q and S.
   (b) Find T such that Q is equidistant from R and T.
   (c) Is S equidistant from R and T? Explain.
3. Suppose that the median, $\overline{PS}$, of $\triangle PQR$ is its altitude from P.
   (a) What is $\text{proj}_{\overleftrightarrow{QR}}(P)$?
   (b) Give an argument in support of or against the claim that P is equidistant from Q and R.

Answers for Sample Quiz

1. [Note that $\vec{a}2 - \vec{b}3 = \vec{i}7 - \vec{j}8$ and, so, that $\vec{b}3 - \vec{a}2 = \vec{i} \cdot -7 + \vec{j}8$.]
   (a) $\vec{i}7$ (b) -8 (c) $\vec{0}$ (d) 8
2. (a) $S = P + (P - Q)$. That is, S is such that P is the midpoint of $\overleftrightarrow{QS}$.
   (b) $T = P + (P - R)$. That is, T is such that P is the midpoint of $\overleftrightarrow{RT}$.
   (c) Yes, for $P = \text{proj}_{\overleftrightarrow{RT}}(S)$ and P is the midpoint of $\overleftrightarrow{RT}$.
3. (a) S
   (b) $S = \text{proj}_{\overleftrightarrow{QR}}(P)$ and S is the midpoint of $\overleftrightarrow{QR}$. Thus, by Notion 8, P is equidistant from Q and R.

1. If σ_1 is to be perpendicular to $\overleftrightarrow{B'C'}$, to what line must $\overleftrightarrow{B'C'}$ be parallel?
2. If σ_1 is to contain the midpoint of $\overline{B'C'}$, and $\overleftrightarrow{B'C'}$ is to be in $\overleftrightarrow{ABC}$, to what line must the midpoint of $\overline{B'C'}$ belong?
3. Describe all lines $\overleftrightarrow{B'C'}$ which satisfy the conditions referred to in Exercises 1 and 2.
4. Since we are trying to relate a $\triangle A'B'C'$ to a given $\triangle ABC$, it would be to our advantage to try to describe B' and C' in terms of 'A', 'B', and 'C'. Consider, then, the line – say, l_1 – which contains A and is parallel to $\overleftrightarrow{BC}$.
   - (a) What can you say about l_1 and σ_1?
   - (b) Show that, for any a, A is the midpoint of the segment whose endpoints are $A + (B - C)a$ and $A + (C - B)a$.
   - (c) What can you say about σ_1 and any interval whose endpoints are $A + (B - C)a$ and $A + (C - B)a$, for $a \neq 0$?
   - (d) Show that there are points – say, B' and C' – on l_1 such that A is the midpoint of $\overline{B'C'}$. What is σ_1 in relation to any such segment?
5. The picture at the right is a "plane" picture of what we have so far. Draw a picture like this one on your paper, and add to it as you do the rest of these exercises.



   - (a) Consider, now, the plane σ_2 which contains B and is perpendicular to $\overleftrightarrow{AC}$. Let l_2 be the line containing B and perpendicular to σ_2. To what line [or, lines] is l_2 parallel?
   - (b) Show that, for any b, B is the midpoint of the segment whose endpoints are $B + (A - C)b$ and $B + (C - A)b$.
   - (c) What can you say about σ_2 and any interval whose endpoints are $B + (A - C)b$ and $B + (C - A)b$, for $b \neq 0$?
6. The lines l_1 and l_2 described in Exercises 4 and 5 are coplanar lines which are not parallel. [Explain.] Give the point of intersection of l_1 and l_2 in terms of 'A', 'B', and 'C'. [*Hint*: Compute values of 'a' and 'b' for which $A + (B - C)a = B + (A - C)b$.]
7. (a) Consider, now, the plane σ_3 which contains C and is perpendicular to $\overleftrightarrow{AB}$. Let l_3 be the line containing C and perpendicular to σ_3. To what line [or, lines] is l_3 parallel?
   - (b) Show that, for any c, C is the midpoint of the segment whose endpoints are $C + (B - A)c$ and $C + (A - B)c$.
   - (c) What can you say about σ_3 and any interval whose endpoints are $C + (B - A)c$ and $C + (A - B)c$, for $c \neq 0$?
   - (d) What point is common to l_1 and l_3? To l_2 and l_3?
8. Give the vertices of a $\triangle A'B'C'$ which is such that the perpendicular bisectors of its sides contain the altitudes of $\triangle ABC$.
9. Give an argument, now, to show that the lines containing the altitudes of a triangle are concurrent.

Answers for Part C

1. $\overleftrightarrow{B'C'}$ must be parallel to $\overleftrightarrow{BC}$.
2. The midpoint of $\overline{B'C'}$ must belong to $\overleftrightarrow{AD}$.
3. They are the lines which are parallel to $\overleftrightarrow{BC}$ [or. to which σ_1 is perpendicular] and contain points of $\overleftrightarrow{AD}$.
4. (a) $\sigma_1 \perp l_1$
   - (b) $A - (A + (B - C)a) = (C - B)a = (A + (C - B)a) - A$
   - (c) σ_1 is the perpendicular bisector of any such interval.
   - (d) l_1 is the line $\overleftrightarrow{A[C - B]}$. So, given any b and choosing $B' = A + (C - B)b$ and $C' = A + (C - B) \cdot -b$, A is the midpoint of $\overline{B'C'}$. For $b \neq 0$, σ_1 is the perpendicular bisector of $\overline{B'C'}$.
5. (a) l_2 must be parallel to $\overleftrightarrow{CA}$.
   - (b) [See answer for Exercise 4(b).]
   - (c) [See answer for Exercise 4(c).]
6. [$l_1 \nparallel l_2$ because $l_1 \parallel \overleftrightarrow{BC}$, $l_2 \parallel \overleftrightarrow{CA}$, and $\overleftrightarrow{BC} \nparallel \overleftrightarrow{CA}$.]
   $l_1 \cap l_2 = \{A + (B - C)\} = \{B + (A - C)\}$
7. (a) $l_3 \parallel \overleftrightarrow{AB}$
   - (b) [See answer for Exercise 4(b).]
   - (c) [See answer for Exercise 4(c).]
   - (d) $C + (A - B)$ [or: $A + (C - B)$]; $C + (B - A)$ [or: $B + (C - A)$]
8. $A' = B + (C - A)$, $B' = C + (A - B)$, $C' = A + (B - C)$
9. Given any $\triangle ABC$, consider $\triangle A'B'C'$ described in the answer for Exercise 7. Since $A' - B' = (B + (C - A)) - (C + (A - B))$ $= (C + (B - A)) - (C + (A - B)) = (B - A)2$ it follows that $\overleftrightarrow{A'B'} \parallel \overleftrightarrow{AB}$ and, so, the plane through C perpendicular to $\overleftrightarrow{A'B'}$ is, also, perpendicular to $\overleftrightarrow{AB}$. So, by definition, the intersection of this plane with $\overleftrightarrow{ABC}$ contains the altitude of $\triangle ABC$ from C. Since $C - A' = C - (B + (C - A)) = A - B = B' - C$ it follows that the plane through C perpendicular to $\overleftrightarrow{A'B'}$ is the perpendicular bisector of $\overline{A'B'}$. Repeating the preceding argument we see that the altitudes of $\triangle ABC$ are contained in the intersections with $\overleftrightarrow{ABC}$ of the perpendicular bisectors of $\triangle A'B'C'$. Since the lines in which the perpendicular bisectors of the sides of $\triangle A'B'C'$ intersect the plane $\overleftrightarrow{A'B'C'}$ are concurrent [and since $\overleftrightarrow{A'B'C'} = \overleftrightarrow{ABC}$] it follows that the lines which contain the altitudes of $\triangle ABC$ are concurrent.

*

In the next section we shall make use of a special case of (4). To arrive at this special case suppose that, in $\triangle ABC$, A and B are equidistant from C. It follows by Notion 8 that the line containing the altitude from C passes through the midpoint of $\overline{AB}$. Thus, the altitude of $\triangle ABC$ from C is the median of $\triangle ABC$ from C. Hence,

(5) if, in $\triangle ABC$, A and B are equidistant from C then the lines containing the altitudes from A and B intersect on the line containing the median from C.

Now, by Theorem 8-10(b) it follows, in case $d(A, C) = d(B, C)$, that the points in which the altitudes from A and B intersect $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$, respectively, divide the intervals from C to A and from C to B in the same ratio. This is illustrated in Fig. 11-12 where

$$(N - C) : (A - C) = (M - C) : (B - C).$$

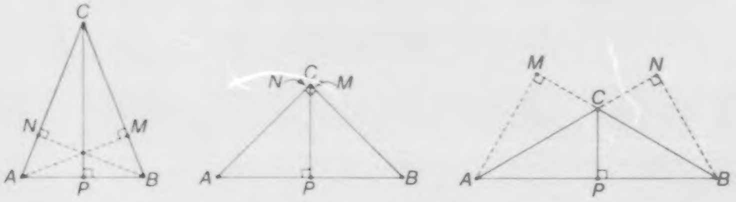


Fig. 11-12

Since $N - C = \text{proj}_{[A - C]}(B - C)$ and $M - C = \text{proj}_{[B - C]}(A - C)$, we see that

$$\text{proj}_{[A - C]}(B - C) : (A - C) = \text{proj}_{[B - C]}(A - C) : (B - C).$$

This last result can be expressed conveniently in terms of vectors:

$$(*) \qquad \begin{gathered} ((\vec{a}, \vec{b}) \text{ is linearly independent and } \|\vec{a}\| = \|\vec{b}\|) \\ \Longrightarrow \\ \text{proj}_{[\vec{a}]}(\vec{b}) : \vec{a} = \text{proj}_{[\vec{b}]}(\vec{a}) : \vec{b} \end{gathered}$$

In the previous section, we found that if $(\vec{a}, \vec{b})$ is linearly dependent and $\|\vec{a}\| = \|\vec{b}\|$ then $\vec{a} = \vec{b}$ or $\vec{a} = -\vec{b}$. Note, in case $\vec{a} = \vec{b}$, that the consequent in (*) is trivially true. Note further, in case $\vec{a} = -\vec{b}$, that

$$\text{proj}_{[\vec{a}]}(\vec{b}) = \text{proj}_{[\vec{a}]}(-\vec{a}) = -\text{proj}_{[\vec{a}]}(\vec{a}) = -\vec{a}$$

By this time, students should feel quite comfortable with the notions of perpendicularity and distance, and should be able to speak of these notions both in terms of planes, lines, and measures of segments as well as in terms of orthogonal complements, directions, bidirections, and norms. The eleven statements given on pp. 35-36 simply summarize what we have done so far with perpendicularity and distance in terms of translations. We shall be making considerable use of these particular statements in our search for additional postulates which describe what we know about perpendicularity and distance and which, at the same time, are easy to work with. So, it may be worthwhile to have the students either put a bookmark at this page or make a copy of the statements for reference purposes.

There are some who would question the strategy of looking at one's intuitions about space in order to get at some workable formal statements which might be adopted as postulates. Those are the ones who wish to give the postulates and "get on with game". If you will take note of what has been accomplished already, you will see that we have indeed been "getting on with the game". And, when we get to the point where we have developed a working knowledge of the operation on which the rest of our postulates are based, we will have already collected a sizable number of theorems. Thus, all of the "playing around" with our intuitions in order to gain the insights needed for an understanding of the postulates enables us to bring our intuitive notions into our formal system as theorems. It is doubtful whether the strategy of adopting postulates and then seeking the insights necessary to prove one's intuitive ideas from those postulates will result in the same feeling for how the system "works". For, there will always be those students who wonder what motivated the choice of those postulates. And, it is one of our objectives to develop this course in such a way as to enable students to "get involved" with the development of the postulates.

and

$$\mathrm{proj}_{[\overrightarrow{b}]}\,(\overrightarrow{a}) = -\overrightarrow{b},$$

so that $\mathrm{proj}_{[\overrightarrow{a}]}\,(\overrightarrow{b}) : \overrightarrow{a} = -1 = \mathrm{proj}_{[\overrightarrow{b}]}\,(\overrightarrow{a}) : \overrightarrow{b}$. So, the consequent in (*) holds whether or not $(\overrightarrow{a}, \overrightarrow{b})$ is linearly independent. Hence, we have:

$$\|\overrightarrow{a}\| = \|\overrightarrow{b}\| \Longrightarrow \mathrm{proj}_{[\overrightarrow{a}]}\,(\overrightarrow{b}) : \overrightarrow{a} = \mathrm{proj}_{[\overrightarrow{b}]}\,(\overrightarrow{a}) : \overrightarrow{b} \tag{6}$$

In the next two sections, we shall get at a way of multiplying translations, describing this multiplication in terms of our intuitive notions of orthogonal complements and norms, and at properties of this multiplication. We shall then return to our formal development by adopting as postulates some of the properties of this multiplication and by defining '$[\overrightarrow{a}]^{\perp}$' and '$\|\overrightarrow{a}\|$' in terms of this multiplication. Doing so, we shall see that all of our notions about perpendicularity and distance which were obtained in the preceding sections will be theorems. So, we shall not only have completed our postulates concerning $\mathscr{T}$ but shall also have made considerable progress beyond Volume 1 in developing the geometry of Euclidean space.

11.06 Orthogonal Components

We have been investigating our intuitive notions concerning perpendicularity of planes to lines and distance for the purpose of formulating new postulates. Before we continue, let us review the results obtained so far.

Study of eight of our notions about perpendicularity and distance led us to the concepts of the orthogonal complement $[\overrightarrow{a}]^{\perp}$ of a direction and the norm $\|\overrightarrow{a}\|$ of a vector $\overrightarrow{a}$. In our investigations, we saw that perpendicularity of planes to lines and distance could be defined by:

(1) $\pi \perp l \Longleftrightarrow [\pi] = [l]^{\perp}$ [(1), p. 21]

and:

(2) $d(A, B) = \|B - A\|$ [(1), p. 27]

We saw, further, that with (1) and (2) as definitions, our Notions 1–8 are equivalent [on the basis of the postulates we have already adopted] to statements about orthogonal complements and norms. For convenience, we list these statements:

(3) $\overrightarrow{a} \neq \overrightarrow{0} \Longrightarrow [\overrightarrow{a}]^{\perp}$ is a proper bidirection [(2), p. 22]

(4) $\overrightarrow{a} \neq \overrightarrow{0} \Longrightarrow \overrightarrow{a} \notin [\overrightarrow{a}]^{\perp}$ [(3), p. 22]

(5) $[\overrightarrow{a}]^{\perp} = [\overrightarrow{b}]^{\perp} \Longrightarrow [\overrightarrow{a}] = [\overrightarrow{b}]$ $[\overrightarrow{a} \neq \overrightarrow{0} \neq \overrightarrow{b}]$ [(4), p. 22]

(6) $\overrightarrow{b} \in [\overrightarrow{a}]^{\perp} \Longrightarrow \overrightarrow{a} \in [\overrightarrow{b}]^{\perp}$ $[\overrightarrow{a} \neq \overrightarrow{0} \neq \overrightarrow{b}]$ [(5), p. 22]

(7) $\|\overrightarrow{a}\| \in \mathscr{R}$ } [(2), p. 27]

(8) $\overrightarrow{a} \neq \overrightarrow{0} \Longrightarrow \|\overrightarrow{a}\| > 0$ } [(2), p. 27]

(9) $\|\overrightarrow{a}a\| = \|\overrightarrow{a}\| \cdot |a|$ [(3), p. 27]

(10) $\overrightarrow{a} + \overrightarrow{b} \in [\overrightarrow{a} - \overrightarrow{b}]^{\perp} \Longleftrightarrow \|\overrightarrow{a}\| = \|\overrightarrow{b}\|$ $[\overrightarrow{a} \neq \overrightarrow{b}]$ [(1), p. 30]

For future reference, recall that from (8), (9), and theorems about multiplying translations by real numbers, it follows that

(11) $\overrightarrow{a}_1 = \overrightarrow{a}/\|\overrightarrow{a}\| \Longrightarrow (\|\overrightarrow{a}_1\| = 1$ and $\overrightarrow{a} = \overrightarrow{a}_1\|\overrightarrow{a}\|)$ $[\overrightarrow{a} \neq \overrightarrow{0}]$.

In the next section we shall discover a way of multiplying a first translation by a second. This multiplication can be described in terms of our intuitive notions of orthogonal complements and norms and, because of (3)–(10) turns out to have interesting properties. In fact, we shall see that orthogonal complements and norms can be defined in terms of this multiplication and that (3)–(10) can be derived from these definitions and a few statements of some of the properties of this operation. These latter statements are the postulates we have been seeking.

In seeking a way to multiply translations it will be helpful to use the notions of orthogonal projections which were developed in Sections 11.01 and 11.02. We continue our review with a brief discussion of these notions.

By (1) and (3), the plane $\overline{P[l]^{\perp}}$ is the unique plane which contains P and is perpendicular to l. By (4), l is a transversal of this plane and, so, intersects this plane in a single point. In Section 11.01 we made use of this fact to define the orthogonal projection of P on l as follows:

$$\mathrm{proj}_l\,(P) = \text{the point of intersection of } l \text{ and } \overline{P[l]^{\perp}}$$

In terms of this we defined, for any $\overrightarrow{a} \neq \overrightarrow{0}$ and any $\overrightarrow{b}$, a mapping $\mathrm{proj}_{[\overrightarrow{a}]}\,(\overrightarrow{b})$ of $\mathscr{E}$ into itself. The definition we used was:

$$P + \mathrm{proj}_{[\overrightarrow{a}]}\,(\overrightarrow{b}) = \mathrm{proj}_{\overleftrightarrow{P[\overrightarrow{a}]}}\,(P + \overrightarrow{b})$$

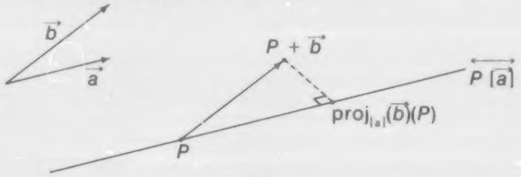

$\overrightarrow{b}$
$\overrightarrow{a}$
$P + \overrightarrow{b}$
$\overleftrightarrow{P[\overrightarrow{a}]}$
$\mathrm{proj}_{[\overrightarrow{a}]}(\overrightarrow{b})(P)$
P

Fig. 11–13

This amounts to saying:

$$P + \text{proj}_{[\vec a]}(\vec b) = Q \Longleftrightarrow Q \in \overleftrightarrow{P[\vec a]} \cap \overline{(P + \vec b)[\vec a]^{\perp}}$$

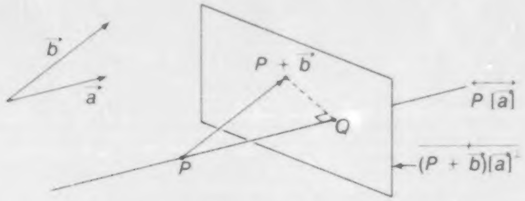


Fig. 11–14

and, so, to saying that

$$(12)\quad P + \text{proj}_{[\vec a]}(\vec b) = Q \Longleftrightarrow (Q - P \in [\vec a] \text{ and } (P + \vec b) - Q \in [\vec a]^{\perp}).$$

Using properties of transversals of parallel planes we were able to show that, for any $\vec a \neq \vec 0$ and any $\vec b$, the mapping $\text{proj}_{[\vec a]}(\vec b)$ is a translation. As a matter of fact we showed that

$$(13)\quad [l] = [\vec a] \Longrightarrow \text{proj}_{[\vec a]}(\vec b) = \text{proj}_l(P + \vec b) - \text{proj}_l(P).$$

We also established a number of basic properties of orthogonal projections, which we list as follows: For $\vec a \neq \vec 0$,

$$(14)\quad \text{proj}_{[\vec a]}(\vec b) \in [\vec a] \text{ and } [\vec b \in [\vec a] \Longrightarrow \text{proj}_{[\vec a]}(\vec b) = \vec b],$$
$$(15)\quad \text{proj}_{[\vec a]}(\vec 0) = \vec 0 \text{ and } [\text{proj}_{[\vec a]}(\vec b) = \vec 0 \Longleftrightarrow \vec b \in [\vec a]^{\perp}],$$
$$(16)\quad \text{proj}_{[\vec a]}(\vec b + \vec c) = \text{proj}_{[\vec a]}(\vec b) + \text{proj}_{[\vec a]}(\vec c), \text{ and}$$
$$(17)\quad \text{proj}_{[\vec a]}(\vec b b) = \left(\text{proj}_{[\vec a]}(\vec b)\right)b.$$

In discussing orthogonal bases in Section 11.03, we noted another important property of orthogonal projections – that

$$(18)\quad (\vec a, \vec b, \vec c) \text{ is orthogonal} \Longrightarrow \vec r = \text{proj}_{[\vec a]}(\vec r) + \text{proj}_{[\vec b]}(\vec r) + \text{proj}_{[\vec c]}(\vec r).$$

Finally, in Section 11.05 we found that

$$(19)\quad \text{proj}_{[\vec a]}(\vec b) : \vec a = \text{proj}_{[\vec b]}(\vec a) : \vec b \qquad [\|\vec a\| = \|\vec b\| \neq 0].$$

Recall that the notion of the components of a vector with respect to a given basis was introduced in Chapter 10. In brief, we said that if $(\vec a, \vec b, \vec c)$ is a basis for $\mathscr{T}$ and $\vec r = a\vec a + b\vec b + c\vec c$, then a is the component

Answers for Exercises

1. By (20), we know that $\text{proj}_{[\vec a]}(\vec r) = \vec a \,\text{comp}_{\vec a}(\vec r)$, for $\vec a \neq \vec 0$. Given that $(\vec a, \vec b, \vec c)$ is orthogonal, it follows that $\vec a$, $\vec b$, and $\vec c$ are non-$\vec 0$. So, by (20), we have that

$$\begin{aligned}\vec r &= \text{proj}_{[\vec a]}(\vec r) + \text{proj}_{[\vec b]}(\vec r) + \text{proj}_{[\vec c]}(\vec r)\\ &= \vec a\,\text{comp}_{\vec a}(\vec r) + \vec b\,\text{comp}_{\vec b}(\vec r) + \vec c\,\text{comp}_{\vec c}(\vec r).\end{aligned}$$

Hence, (18′).

2. To justify (14′): We know that $\text{proj}_{[\vec b]}(\vec a) : \vec b \in \mathcal{R}$. So, by (20), $\text{comp}_{\vec b}(\vec a) \in \mathcal{R}$.

Suppose that $\vec a \in [\vec b]$ and that $\vec b \neq \vec 0$. Then, by (14) $\text{proj}_{[\vec b]}(\vec a) = \vec a$ so that, by (20), $\vec b\,\text{comp}_{\vec b}(\vec a) = \vec a$. Thus, $\text{comp}_{\vec b}(\vec a) = \vec a : \vec b$.

Hence for $\vec b \neq \vec 0$, if $\vec a \in [\vec b]$ then $\text{comp}_{\vec b}(\vec a) = \vec a : \vec b$.

To justify (15′): $\text{comp}_{\vec b}(\vec 0) = \text{proj}_{[\vec b]}(\vec 0) : \vec b = \vec 0 : \vec b = 0$.

For $\vec b \neq \vec 0$, $\text{comp}_{\vec b}(\vec a) = \text{proj}_{[\vec b]}(\vec a) : \vec b$. So, $\text{comp}_{\vec b}(\vec a) = 0$ if and only if $\text{proj}_{[\vec b]}(\vec a) : \vec b = 0$.

The latter is the case, by (15), if and only if $\vec a \in [\vec b]^{\perp}$. Hence, for $\vec b \neq \vec 0$, $\text{comp}_{\vec b}(\vec a) = 0$ if and only if $\vec a \in [\vec b]^{\perp}$.

To justify (16′): For $\vec c \neq \vec 0$,

$$\begin{aligned}\text{comp}_{\vec c}(\vec a + \vec b) &= \text{proj}_{[\vec c]}(\vec a + \vec b) : \vec c\\ &= (\text{proj}_{[\vec c]}(\vec a) + \text{proj}_{[\vec c]}(\vec b)) : \vec c && [(16)]\\ &= \text{proj}_{[\vec c]}(\vec a) : \vec c + \text{proj}_{[\vec c]}(\vec b) : \vec c\\ &= \text{comp}_{\vec c}(\vec a) + \text{comp}_{\vec c}(\vec b) && [(20)]\end{aligned}$$

To justify (17′): For $\vec c \neq \vec 0$,

$$\begin{aligned}\text{comp}_{\vec c}(\vec a b) &= \text{proj}_{[\vec c]}(\vec a b) : \vec c\\ &= (\text{proj}_{[\vec c]}(\vec a))b : \vec c && [(17)]\\ &= (\text{proj}_{[\vec c]}(\vec a) : \vec c)b\\ &= \text{comp}_{\vec c}(\vec a)b && [(20)]\end{aligned}$$

To justify (19′): For $\|\vec a\| = \|\vec b\| \neq 0$,

$$\begin{aligned}\text{comp}_{\vec a}(\vec b) &= \text{proj}_{[\vec a]}(\vec b) : \vec a && [(20)]\\ &= \text{proj}_{[\vec b]}(\vec a) : \vec b && [(19)]\\ &= \text{comp}_{\vec b}(\vec a). && [(20)]\end{aligned}$$

[Note to the teacher: These exercises contain results which ought to be reasonable on intuitive grounds and whose justifications can serve as review of the concept of ratios of translations as well as practice in using the newly-defined concept of orthogonal components. Rather than have each student work on each of these justifications, it is probably wise to split up the work among several groups of students and to give one from each such group an opportunity to present his justification to the entire class.]

of $\vec{r}$ with respect to $\vec{a}$, b is the component of $\vec{r}$ with respect to $\vec{b}$ and c is the component of $\vec{r}$ with respect to $\vec{c}$. For orthogonal bases, it is reasonable to speak of the numbers a, b, and c as *orthogonal components* of $\vec{r}$. This we shall do. In view of this and the first part of sentence (14)—that $\text{proj}_{[\vec{a}]}(\vec{b}) \in [\vec{a}]$—sentence (18) suggests the following definition:

$$\text{comp}_{\vec{a}}(\vec{r}) = \text{proj}_{[\vec{a}]}(\vec{r}) : \vec{a} \qquad [\vec{a} \neq \vec{0}] \tag{20}$$

[Read '$\text{comp}_{\vec{a}}(\vec{r})$' as 'the orthogonal component of $\vec{r}$ with respect to $\vec{a}$'.] Note that, as we have already done implicitly in (9) on page 36, we shall extend our notion of ratio to allow $\vec{0} : \vec{a}$ to be 0 for any $\vec{a} \neq \vec{0}$. Since, for $b \neq 0$, $b : a \neq 0$ it follows that $b : a = 0$ if and only if $b = 0$. Since, for $\vec{b} \neq \vec{0}$, $\vec{b} : \vec{a} \neq 0$ it follows that $\vec{b} : \vec{a} = 0$ if and only if $\vec{b} = \vec{0}$.

Exercises

1. Making use of (20) as a definition, sentence (18) can be rewritten, in terms of orthogonal components, as:

$$(18') \quad (\vec{a}, \vec{b}, \vec{c}) \text{ is orthogonal} \longrightarrow \vec{r} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{r}) + \vec{b}\,\text{comp}_{\vec{b}}(\vec{r}) + \vec{c}\,\text{comp}_{\vec{c}}(\vec{r})$$

Explain.

2. Similarly, each of the properties of projections expressed in (14)–(17) and (19) is, in view of (20), closely related to a property of components. For example, from (14) and (20) we obtain:

$$(14') \quad \text{comp}_{\vec{b}}(\vec{a}) \in \mathscr{R} \text{ and } \left[\vec{a} \in [\vec{b}] \longrightarrow \text{comp}_{\vec{b}}(\vec{a}) = \vec{a} : \vec{b}\right] \qquad [\vec{b} \neq \vec{0}]$$

[Explain.] Justify each of the following:

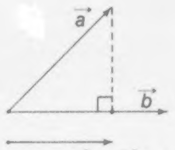
$$(15') \quad \text{comp}_{\vec{b}}(\vec{0}) = 0 \text{ and } \left[\text{comp}_{\vec{b}}(\vec{a}) = 0 \longleftrightarrow \vec{a} \in [\vec{b}]^{\perp}\right] \quad [\vec{b} \neq \vec{0}]$$

$$(16') \quad \text{comp}_{\vec{c}}(\vec{a} + \vec{b}) = \text{comp}_{\vec{c}}(\vec{a}) + \text{comp}_{\vec{c}}(\vec{b}) \qquad [\vec{c} \neq \vec{0}]$$

$$(17') \quad \text{comp}_{\vec{c}}(\vec{a}b) = (\text{comp}_{\vec{c}}(\vec{a}))b \qquad [\vec{c} \neq \vec{0}]$$

$$(19') \quad \text{comp}_{\vec{a}}(\vec{b}) = \text{comp}_{\vec{b}}(\vec{a}) \qquad [\|\vec{a}\| = \|\vec{b}\| \neq 0]$$

3. Draw a picture of vectors $\vec{a}$ and $\vec{b}$, something like the one at the right, in which $\text{proj}_{[\vec{b}]}(\vec{a}) = \vec{b}\frac{2}{3}$. Compute each of the following.
   (a) $\text{comp}_{\vec{b}}(\vec{a})$
   (b) $\text{comp}_{\vec{b}2}(\vec{a})$
   (c) $\text{comp}_{\vec{b}3}(\vec{a})$
   (d) $\text{comp}_{\vec{b}5}(\vec{a})$



$\text{proj}_{[\vec{b}]}(\vec{a}) = \vec{b}\,2/3$

Answers for Exercises [cont.]

3. (a) 2/3
   (b) 1/3 [$\text{proj}_{[\vec{b}]}(\vec{a}) = (\vec{b}2)/3$, so that $\text{comp}_{\vec{b}2}(\vec{a}) = \text{proj}_{[\vec{b}2]}(\vec{a}) : (\vec{b}2) = 1/3$.]
   (c) 2/9 [$\text{proj}_{[\vec{b}3]}(\vec{a}) = (\vec{b}3) \cdot 2/9$]
   (d) 2/15 [$\text{proj}_{[\vec{b}5]}(\vec{a}) = (\vec{b}5) \cdot 2/15$]

   (e) −2/9 [$\text{proj}_{[\vec{b} \cdot -3]}(\vec{a}) = (\vec{b} \cdot -3) \cdot -2/9$]
   (f) 4/3 [$\text{proj}_{[\vec{b}]}(\vec{a}) = (\vec{b}\tfrac{1}{2}) \cdot 4/3$]
4. 1/2; 1/3; 1/5; −1/3; 2
5. Suppose that $\vec{b} \neq \vec{0}$ and $c \neq 0$. Then,

$$\begin{aligned}\text{comp}_{\vec{b}c}(\vec{a}) &= \text{proj}_{[\vec{b}c]}(\vec{a}) : \vec{b}c && [(20)]\\ &= (\text{proj}_{[\vec{b}c]}(\vec{a}) : \vec{b})/c\\ &= (\text{proj}_{[\vec{b}]}(\vec{a}) : \vec{b})/c && [\text{since } [\vec{b}c] = [\vec{b}]]\\ &= \text{comp}_{\vec{b}}(\vec{a})/c.\end{aligned}$$

  (e) $\text{comp}_{\vec{b}\,-3}(\vec{a})$
  (f) $\text{comp}_{\vec{b}\,1.2}(\vec{a})$

4. Consider the numbers $\text{comp}_{\vec{b}}(\vec{a})$ and $\text{comp}_{\vec{b}2}(\vec{a})$ computed in Exercise 3. What must $\text{comp}_{\vec{b}}(\vec{a})$ be multiplied by to obtain $\text{comp}_{\vec{b}2}(\vec{a})$? Answer similar questions with respect to $\text{comp}_{\vec{b}}(\vec{a})$ and the other numbers computed in Exercise 3.
5. The results in Exercises 3 and 4 suggest that

$$(20') \qquad \text{comp}_{\vec{b}c}(\vec{a}) = \text{comp}_{\vec{b}}(\vec{a})/c \qquad [\vec{b} \neq \vec{0},\ c \neq 0].$$

Make use of (20) to show that this is so. [*Hint*: You will have to make use of the facts that $\text{proj}_{[\vec{b}]}(\vec{a}) \in [\vec{b}]$ and that, for $\vec{c} \in [\vec{d}] \neq \{\vec{0}\}$ and $e \neq 0$, $\vec{c} : (\vec{d}e) = (\vec{c} : \vec{d})/e$.]

*

According to the second part of (14′), the notion of the orthogonal component of $\vec{a}$ with respect to a non-$\vec{0}$ translation $\vec{b}$ can be thought of as an extension of the notion of the ratio of $\vec{a}$ with respect to a non-$\vec{0}$ translation $\vec{b}$ in the direction of $\vec{a}$. This suggests that "componenting" is analogous to dividing. This analogy is also suggested by (16′), (17′), and (20′) which are analogous to the real number theorems:

$$(a + b) \div c = a \div c + b \div c,$$
$$(ab) \div c = (a \div c)b,$$
$$\text{and: } a \div (bc) = (a \div b)/c$$

Statement (19′) is analogous to the less familiar theorem according to which $a \div b = b \div a$ if a and b are nonzero real numbers with the same absolute value. Moreover, if we recall that $\mathscr{R}$ can itself be thought of as a vector space in which all vectors have the same direction, we see by (14′) that in this 1-dimensional vector space "componenting" coincides with "ratioing". And, as we have seen in Chapter 7, a ratio of nonzero members of *this* vector space is their quotient.

In the next section, we shall make use of the properties of orthogonal components we have found together with the properties of orthogonal complements and norms which we have just reviewed to introduce a multiplication operation on vectors.

11.07 A Kind of Multiplication

We have just seen that orthogonal componenting of vectors is analogous to division of real numbers and even coincides with division in the case of the vector space $\mathscr{R}$. This suggests that there may be an operation on vectors which is analogous to multiplication of real numbers and which coincides with multiplication when the vector space is $\mathscr{R}$. How to describe such an operation is suggested by the real number theorems:

$$\begin{cases} a \cdot b = (a \div b) \cdot |b|^2 & [b \neq 0] \\ a \cdot 0 = 0 \end{cases}$$

In view of the discussion in the preceding section, these theorems suggest introducing a multiplication operation on translations by:

$$(*) \quad \begin{cases} \vec{a} \cdot \vec{b} = \text{comp}_{\vec{b}}(\vec{a})\, \|\vec{b}\|^2 & [\vec{b} \neq \vec{0}] \\ \vec{a} \cdot \vec{0} = 0 \end{cases}$$

[We shall call this *dot multiplication* and shall never omit the '·'.]

Up to now, our notions concerning orthogonal components and norms of translations – and, so, concerning dot multiplication – are based on intuitive notions concerning perpendicularity and distance. In this section we shall continue to operate on this intuitive basis in order to discover properties of dot multiplication. We shall then be in a position to return to our formal development by taking statements of some of these properties as postulates and definitions. This we shall do in the next section.

To begin with, we have already noted that if we apply (*) to the vector space $\mathscr{R}$ and identify the norm of a real number with its absolute value, dot multiplication of real numbers is just the usual multiplication of real numbers. Consequently, it is reasonable to hope that dot multiplication of translations will have at least some of the properties of multiplication of real numbers. These properties will, of course, result from (*) and properties of norms and orthogonal components summarized in Section 11.06 [and properties of multiplication of real numbers].

For example, it follows from (7), that is, from:

$$\|\vec{a}\| \in \mathscr{R}$$

and the first part of (14′) in Section 11.06:

$$\text{comp}_{\vec{b}}(\vec{a}) \in \mathscr{R}$$

that

$$(1) \qquad \vec{a} \cdot \vec{b} \in \mathscr{R}$$

in case $\vec{b} \neq \vec{0}$. [That (1) also holds in case $\vec{b} = \vec{0}$ is left as an exercise.]

Exercises

1. Show that $\vec{a} \cdot \vec{b} \in \mathscr{R}$ in case $\vec{b} = \vec{0}$.
2. For $\vec{a} \neq \vec{0}$, $\mathrm{comp}_{\vec{a}}(\vec{a}) = 1$ [This is so because $\mathrm{comp}_{\vec{a}}(\vec{a}) = \mathrm{proj}_{[\vec{a}]}(\vec{a}) : \vec{a} = \vec{a} : \vec{a} = 1$.] So for $\vec{a} \neq \vec{0}$, $\vec{a} \cdot \vec{a} = \mathrm{comp}_{\vec{a}}(\vec{a})\,\|\vec{a}\|^2 = \|\vec{a}\|^2$.
   - **(a)** Compute $\mathrm{comp}_{\vec{a}}(-\vec{a})$, $\mathrm{comp}_{-\vec{a}}(\vec{a})$ and $\mathrm{comp}_{-\vec{a}}(-\vec{a})$, for $\vec{a} \neq \vec{0}$.
   - **(b)** Express each of $-\vec{a} \cdot \vec{a}$, $\vec{a} \cdot -\vec{a}$, and $-\vec{a} \cdot -\vec{a}$ in terms of $\|\vec{a}\|$, the norm of $\vec{a}$.
   - **(c)** Compute $\vec{0} \cdot \vec{0}$. Express $\vec{0} \cdot \vec{0}$ in terms of the norm of $\vec{0}$.
   - **(d)** Show that $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$.
3. We know that $\vec{a} \neq \vec{0} \Longrightarrow \|\vec{a}\| > 0$. [See (8) on page 36.] In view of this, show:
   - **(a)** $\vec{a} \neq \vec{0} \Longrightarrow \vec{a} \cdot \vec{a} > 0$ **(b)** $\vec{a} \cdot \vec{a} \geq 0$
4. **(a)** By (15′) on page 38, we know that, for $\vec{b} \neq \vec{0}$, $\mathrm{comp}_{\vec{b}}(\vec{a}) = 0 \Longleftrightarrow \vec{a} \in [\vec{b}]^{\perp}$. Use this to help you to show that

$$\vec{a} \cdot \vec{b} = 0 \Longleftrightarrow \vec{a} \in [\vec{b}]^{\perp} \qquad [\vec{b} \neq \vec{0}].$$

   (b) Give a "geometric" interpretation for the result in (a).
5. From Section 11.06, we have that, for $\vec{c} \neq \vec{0}$, $\mathrm{comp}_{\vec{c}}(\vec{a} + \vec{b}) = \mathrm{comp}_{\vec{c}}(\vec{a}) + \mathrm{comp}_{\vec{c}}(\vec{b})$ and $\mathrm{comp}_{\vec{c}}(\vec{a}b) = \mathrm{comp}_{\vec{c}}(\vec{a})b$. Using these notions, show:
   - **(a)** $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$
   - **(b)** $(\vec{a}b) \cdot \vec{c} = (\vec{a} \cdot \vec{c})b$

   [In both (a) and (b), special arguments are needed for the case in which $\vec{c} = \vec{0}$.]
6. From the second part of (*) on page 40, we know that $\vec{a} \cdot \vec{0} = 0$. Make use of the result in 5(b) to show that $\vec{0} \cdot \vec{a} = 0$. [*Hint*: $\vec{0} = \vec{a}0$.]
7. Show that $\vec{a} \cdot (\vec{b}c) = (\vec{a} \cdot \vec{b})c$ [*Hint*: Handle the cases where both $\vec{b} \neq \vec{0}$ and $c \neq 0$ and where either $\vec{b} = \vec{0}$ or $c = 0$ separately. Sentence (20′) on page 39 and sentence (9) on page 36 will be helpful in the first case.]
8. Show that $(\vec{a}a) \cdot (\vec{b}b) = (\vec{a} \cdot \vec{b})(ab)$.
9. Assume that $\vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}$. Show that $(\vec{b}b) \cdot (\vec{a}a) = (\vec{a}a) \cdot (\vec{b}b)$. [*Hint*: See Exercise 8.]
10. Show that $\vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}$. [*Hint*: The case in which $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ is easy. In what other case do we know that $\vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}$? Make use of Exercise 9 and sentence (11) on page 36 to reduce the case in which $\vec{a} \neq \vec{0} \neq \vec{b}$ to that in which $\|\vec{a}\| = \|\vec{b}\| \neq 0$. One way to do this is to let $\vec{a}_1 = \vec{a}/\|\vec{a}\|$ and $\vec{b}_1 = \vec{b}/\|\vec{b}\|$, and compute $\vec{a} \cdot \vec{b}$ in terms of $\vec{a}_1$ and $\vec{b}_1$.]

*

Answers for Exercises

1. By the second part of (*), $\vec{a} \cdot \vec{0} = 0$; and $0 \in \mathscr{R}$.
2. (a) $\mathrm{comp}_{\vec{a}}(-\vec{a}) = \mathrm{comp}_{\vec{a}}(\vec{a} \cdot -1) = (\mathrm{comp}_{\vec{a}}(\vec{a})) \cdot -1 = 1 \cdot -1 = -1$

   $\mathrm{comp}_{-\vec{a}}(\vec{a}) = \mathrm{comp}_{-\vec{a}}(-(-\vec{a})) = -1$

   $\mathrm{comp}_{-\vec{a}}(-\vec{a}) = 1$

   (b) For $\vec{a} \neq \vec{0}$, $-\vec{a} \cdot \vec{a} = \mathrm{comp}_{\vec{a}}(-\vec{a})\|\vec{a}\|^2 = -1 \cdot \|\vec{a}\|^2 = -\|\vec{a}\|^2$. For $\vec{a} = \vec{0}$, $-\vec{a} \cdot \vec{a} = -\vec{a} \cdot \vec{0} = 0$ and, so, since $\|\vec{0}\| = 0 = -0$ we have that $-\vec{a} \cdot \vec{a} = -\|\vec{a}\|^2$. So, in any case $-\vec{a} \cdot \vec{a} = -\|\vec{a}\|^2$.

   For $\vec{a} \neq \vec{0}$, $\vec{a} \cdot -\vec{a} = \mathrm{comp}_{-\vec{a}}(\vec{a})\|-\vec{a}\|^2 = -\|-\vec{a}\|^2$. Since $\|-\vec{a}\| = \|\vec{a}\|$, we have that, for $\vec{a} \neq \vec{0}$, $\vec{a} \cdot -\vec{a} = -\|\vec{a}\|^2$. For $\vec{a} = \vec{0}$, we have that $-\vec{a} = \vec{0}$ so that $\vec{a} \cdot -\vec{a} = \vec{a} \cdot \vec{0} = 0 = -\|\vec{0}\|^2 = -\|\vec{a}\|^2$. So, in any case, $\vec{a} \cdot -\vec{a} = -\|\vec{a}\|^2$.

   For $\vec{a} \neq \vec{0}$, $-\vec{a} \cdot -\vec{a} = \mathrm{comp}_{-\vec{a}}(-\vec{a})\|-\vec{a}\|^2 = \|-\vec{a}\|^2 = \|\vec{a}\|^2$. For $\vec{a} = \vec{0}$, $-\vec{a} = \vec{0}$, so that $-\vec{a} \cdot -\vec{a} = -\vec{a} \cdot \vec{0} = 0 = \|\vec{0}\|^2 = \|\vec{a}\|^2$. So, in any case, $-\vec{a} \cdot -\vec{a} = \|\vec{a}\|^2$.

   (c) By the second part of (*), $\vec{0} \cdot \vec{0} = 0$. Since $\|\vec{0}\| = 0 = \|\vec{0}\|^2$, $\vec{0} \cdot \vec{0} = \|\vec{0}\|^2$ [or, $\vec{0} \cdot \vec{0} = \|\vec{0}\|$].

   (d) For $\vec{a} \neq \vec{0}$, $\vec{a} \cdot \vec{a} = \mathrm{comp}_{\vec{a}}(\vec{a})\,\|\vec{a}\|^2 = \|\vec{a}\|^2$. For $\vec{a} = \vec{0}$, $\vec{a} \cdot \vec{a} = \vec{0} \cdot \vec{0} = \|\vec{0}\|^2 = \|\vec{a}\|^2$. So, in any case, $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$.
3. (a) Suppose that $\vec{a} \neq \vec{0}$. Then, since we have that $\vec{a} \neq \vec{0} \Longrightarrow \|\vec{a}\| > 0$, it follows that $\|\vec{a}\| > 0$. So, $\|\vec{a}\|^2 > 0$ and, since $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$, we have that $\vec{a} \cdot \vec{a} > 0$. Hence, if $\vec{a} \neq \vec{0}$ then $\vec{a} \cdot \vec{a} > 0$.

   (b) For $\vec{a} \neq \vec{0}$, it follows by part (a) that $\vec{a} \cdot \vec{a} > 0$. So, $\vec{a} \cdot \vec{a} \geq 0$. For $\vec{a} = \vec{0}$, it follows by Exercise 2(c) that $\vec{a} \cdot \vec{a} = 0$. So, $\vec{a} \cdot \vec{a} \geq 0$. Hence, in any case, $\vec{a} \cdot \vec{a} \geq 0$.
4. (a) For $\vec{b} \neq \vec{0}$, $\vec{a} \cdot \vec{b} = \mathrm{comp}_{\vec{b}}(\vec{a})\|\vec{b}\|^2$ and, by Exercise 3(a), $\|\vec{b}\|^2 > 0$. So, $\vec{a} \cdot \vec{b} = 0$ if and only if $\mathrm{comp}_{\vec{b}}(\vec{a}) = 0$. By (15′) and the replacement rule for biconditions, $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a} \in [\vec{b}]^{\perp}$.

   (b) For $\vec{b} \neq \vec{0}$, $[\vec{b}]^{\perp}$ is a proper bidirection. Let π be any plane whose bidirection is $[\vec{b}]^{\perp}$. By part (a), the only vectors whose dot product with $\vec{b}$ is 0 are those in $[\vec{b}]^{\perp}$. the orthogonal complement of $\vec{b}$. Given a line ℓ whose direction is $[\vec{b}]$, we know that $\pi \perp \ell$. So, a "geometric" interpretation of the result in part (a) is that a plane is perpendicular to a line if and only if the dot product of any vector in the bidirection of the plane by any non-$\vec{0}$ vector in the direction of the line is 0.

Answers for Exercises [cont.]

5. (a) For $\vec{c} \neq \vec{0}$,
$$\begin{aligned}(\vec{a} + \vec{b})\cdot\vec{c} &= \mathrm{comp}_{\vec{c}}(\vec{a} + \vec{b})\|\vec{c}\|^2\\ &= \left(\mathrm{comp}_{\vec{c}}(\vec{a}) + \mathrm{comp}_{\vec{c}}(\vec{b})\right)\|\vec{c}\|^2\\ &= \left(\mathrm{comp}_{\vec{c}}(\vec{a})\right)\|\vec{c}\|^2 + \left(\mathrm{comp}_{\vec{c}}(\vec{b})\right)\|\vec{c}\|^2\\ &= \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}.\end{aligned}$$

For $\vec{c} = \vec{0}$, $(\vec{a} + \vec{b})\cdot\vec{c} = (\vec{a} + \vec{b})\cdot\vec{0} = 0 = \vec{a}\cdot\vec{0} + \vec{b}\cdot\vec{0} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$. So, in any case, $(\vec{a} + \vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$.

(b) For $\vec{c} \neq \vec{0}$,
$$\begin{aligned}(\vec{a}b)\cdot\vec{c} &= \mathrm{comp}_{\vec{c}}(\vec{a}b)\|\vec{c}\|^2\\ &= \left(\mathrm{comp}_{\vec{c}}(\vec{a})\right)b\|\vec{c}\|^2\\ &= \left((\mathrm{comp}_{\vec{c}}(\vec{a}))\|\vec{c}\|^2\right)b\\ &= (\vec{a}\cdot\vec{c})b.\end{aligned}$$

For $\vec{c} = \vec{0}$, $(\vec{a}b)\cdot\vec{c} = (\vec{a}b)\cdot\vec{0} = 0 = 0b = (\vec{a}\cdot\vec{0})b = (\vec{a}\cdot\vec{c})b$. So, in any case, $(\vec{a}b)\cdot\vec{c} = (\vec{a}\cdot\vec{c})b$.

6. $\vec{0}\cdot\vec{a} = (\vec{b}0)\cdot\vec{a} = (\vec{b}\cdot\vec{a})0 = 0$.

7. Suppose that $\vec{b} \neq \vec{0}$ and $c \neq 0$. Then,
$$\begin{aligned}\vec{a}\cdot(\vec{b}c) &= \mathrm{comp}_{\vec{b}c}(\vec{a})\|\vec{b}c\|^2\\ &= \left(\mathrm{comp}_{\vec{b}}(\vec{a})\right)\frac{1}{c}\|\vec{b}c\|^2 && [(20')]\\ &= \left(\mathrm{comp}_{\vec{b}}(\vec{a})\right)\frac{1}{c}\|\vec{b}\|^2c^2 && [(9)]\\ &= \mathrm{comp}_{\vec{b}}(\vec{a})\|\vec{b}\|^2c\\ &= (\vec{a}\cdot\vec{b})c.\end{aligned}$$

Suppose, next, that either $\vec{b} = \vec{0}$ or $c = 0$. For $\vec{b} = \vec{0}$, $\vec{a}\cdot(\vec{b}c) = \vec{a}\cdot\vec{0} = 0 = \vec{0}c = (\vec{a}\cdot\vec{0})c = (\vec{a}\cdot\vec{b})c$. For $c = 0$, $\vec{a}\cdot(\vec{b}c) = \vec{a}\cdot(\vec{b}0) = \vec{a}\cdot\vec{0} = 0 = (\vec{a}\cdot\vec{b})0 = (\vec{a}\cdot\vec{b})c$.

Hence, in any case, $\vec{a}\cdot(\vec{b}c) = (\vec{a}\cdot\vec{b})c$.

8.
$$\begin{aligned}(\vec{a}a)\cdot(\vec{b}b) &= (\vec{a}\cdot\vec{b}b)a && [\text{Ex. } 5(b)]\\ &= (\vec{a}\cdot\vec{b})ba && [\text{Ex. } 7]\\ &= (\vec{a}\cdot\vec{b})(ab). && [\text{Post. } 5]\end{aligned}$$

9. Suppose that $\vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b}$. Then,
$$\begin{aligned}(\vec{b}b)\cdot(\vec{a}a) &= (\vec{b}\cdot\vec{a})(ba) && [\text{Ex. } 8]\\ &= (\vec{a}\cdot\vec{b})(ba) && [\text{Assumption}]\\ &= (\vec{a}\cdot\vec{b})(ab) && [\text{Post. } 5]\\ &= (\vec{a}a)\cdot(\vec{b}b). && [\text{Ex. } 8]\end{aligned}$$

Answers for Exercises [cont.]

10. Suppose, first, that $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$. Given that $\vec{a} = \vec{0}$, $\vec{b}\cdot\vec{a} = \vec{b}\cdot\vec{0} = 0 = \vec{0}\cdot\vec{b} = \vec{a}\cdot\vec{b}$. Given that $\vec{b} = \vec{0}$, $\vec{b}\cdot\vec{a} = \vec{0}\cdot\vec{a} = 0 = \vec{a}\cdot\vec{0} = \vec{a}\cdot\vec{b}$.

Suppose, next, that $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$. Let $\vec{a}_1 = \vec{a}/\|\vec{a}\|$ and $\vec{b}_1 = \vec{b}/\|\vec{b}\|$. Then, $\|\vec{a}_1\| = 1$ and $\|\vec{b}_1\| = 1$ so that $\|\vec{a}_1\| = \|\vec{b}_1\| \neq 0$. So,
$$\begin{aligned}\vec{b}\cdot\vec{a} &= (\vec{b}_1\|\vec{b}\|)\cdot(\vec{a}_1\|\vec{a}\|)\\ &= (\vec{b}_1\cdot\vec{a}_1)(\|\vec{b}\|\,\|\vec{a}\|) && [\text{Ex. } 8]\\ &= (\vec{a}_1\cdot\vec{b}_1)(\|\vec{b}\|\,\|\vec{a}\|) && [(19') \text{ and } (*)]\\ &= (\vec{a}_1\cdot\vec{b}_1)(\|\vec{a}\|\,\|\vec{b}\|) && [\text{Post. } 5]\\ &= (\vec{a}_1\|\vec{a}\|)\cdot(\vec{b}_1\|\vec{b}\|) && [\text{Ex. } 8]\\ &= \vec{a}\cdot\vec{b}\end{aligned}$$

*

Sample Quiz

1. Suppose that $\mathrm{comp}_b(\vec{a}) = t$, for some $t \neq 0$. Find the following in terms of 't'.

(a) $\mathrm{comp}_{\vec{b}}(\vec{a}3)$ (b) $\mathrm{comp}_{\vec{b}2}(\vec{a}4)$ (c) $\mathrm{comp}_{\vec{b}4}(\vec{a}2)$

(d) $\mathrm{comp}_{\vec{b}}(-\vec{a}3)$ (e) $\mathrm{comp}_{-\vec{b}}(-\vec{a}3)$ (f) $\mathrm{comp}_{\vec{b}5}(-\vec{a})$

2. We have that, for $\vec{b} \neq \vec{0}$, $\vec{a}\cdot\vec{b} = \mathrm{comp}_{\vec{b}}(\vec{a})\|\vec{b}\|^2$. Express each of the following in terms of '$\|\vec{a}\|$', where $\vec{a}$ is any non-$\vec{0}$ vector.

(a) $\vec{a}2\cdot\vec{a}$ (b) $-\vec{a}2\cdot\vec{a}$ (c) $(\vec{a}4)\cdot(\vec{a}4)$

Answers for Sample Quiz

1. (a) $3t$ $[\mathrm{comp}_{\vec{b}}(\vec{a}3) = [\mathrm{comp}_{\vec{b}}(\vec{a})]3 = t\cdot 3]$

(b) $2t$ $[\mathrm{comp}_{\vec{b}2}(\vec{a}4) = [\mathrm{comp}_{\vec{b}}(\vec{a}4)]/2 = [\mathrm{comp}_{\vec{b}}(\vec{a})]4/2 = t\cdot 2]$

(c) $t/2$ (d) $-3t$

(e) $3t$ (f) $-t/5$

2. (a) $2\|\vec{a}\|^2[\vec{a}2\cdot\vec{a} = \mathrm{comp}_{\vec{a}}(\vec{a}2)\|\vec{a}\|^2 = 2\|\vec{a}\|^2]$

(b) $-2\|\vec{a}\|^2[-\vec{a}2\cdot\vec{a} = \mathrm{comp}_{\vec{a}}(-\vec{a}2) = -2\|\vec{a}\|^2]$

(c) $16\|\vec{a}\|^2[(\vec{a}4)\cdot(\vec{a}4) = \mathrm{comp}_{\vec{a}4}(\vec{a}4)\|\vec{a}4\|^2 = 1\cdot\|\vec{a}\|^2\cdot 16]$

The exercises just completed and the discussion preceding them suggest several properties of dot multiplication, which we summarize here:

(1) $\vec{a} \cdot \vec{b} = \mathscr{R}$

(2) $\vec{a} \cdot \vec{a} = \|a\|^2$

(3) $\vec{a} \cdot \vec{a} > 0 \qquad [\vec{a} \neq \vec{0}]$

(4) $\vec{a} \cdot \vec{b} = 0 \Longleftrightarrow \vec{a} \in [\vec{b}]^{\perp} \qquad [\vec{b} \neq \vec{0}]$

(5) $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

(6) $(\vec{a}b) \cdot \vec{c} = (\vec{a} \cdot \vec{c})b$

(7) $\vec{a} \cdot (\vec{b}c) = (\vec{a} \cdot \vec{b})c$

(8) $\vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}$

11.08 Postulates For Dot Multiplication

Motivated by the considerations in the preceding sections, we introduce dot multiplication into our formal system by adopting as postulates the following statements:

Postulate 4_0 (e) $\vec{a} \cdot \vec{b} \in \mathscr{R}$
Postulate 4_{11} $\vec{a} \cdot \vec{a} > 0 \qquad [\vec{a} \neq \vec{0}]$
Postulate 4_{12} $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$
Postulate 4_{13} $(\vec{a}a) \cdot \vec{b} = (\vec{a} \cdot \vec{b})a$
Postulate 4_{14} $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

In Chapters 3–5 of Volume 1 we saw that the $\mathscr{T}$ of translations of $\mathscr{E}$ is a vector space over the ordered field $\mathscr{R}$ of real numbers. What this means is expressed by Postulates 4_0(a)–(d) and 4_1–4_8. In Chapter 10 we saw that $\mathscr{T}$ is 3-dimensional [Postulates 4_9 and 4_{10}]. All of this is summarized in:

Postulate 4′ $\mathscr{T}$ is a 3-dimensional vector space over $\mathscr{R}$.

Now, a vector space over $\mathscr{R}$ on which a [dot] multiplication satisfying Postulates 4_0(e) and 4_{11}–4_{14} is defined is called *an inner product space over* $\mathscr{R}$. So, the content of Postulates 4_0–4_{14} may be summarized in:

Postulate 4 $\mathscr{T}$ is a 3-dimensional inner product space over $\mathscr{R}$.

As you have probably learned by now, the only way to gain a thorough understanding of a new operation is to use it. You can begin [in Part B of the following exercises] to gain a better understanding of dot multiplication by proving the various parts of the following theorem. Your proofs should, of course, be based on nothing but postulates [including the new ones], definitions, and previously proved theorems. To begin with, the only theorems you can use are those proved in Volume 1. But, you can use parts of Theorem 11–1, below, in proving later parts.

Theorem 11–1
(a) $\vec{a} \cdot (\vec{b}b) = (\vec{a} \cdot \vec{b})b$ (b) $\vec{0} \cdot \vec{a} = 0 = \vec{a} \cdot \vec{0}$
(c) $\vec{a} = \vec{0} \Longleftrightarrow \vec{a} \cdot \vec{a} = 0$ (d) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
(e) $\vec{a} \cdot -\vec{b} = -(\vec{a} \cdot \vec{b}) = -\vec{a} \cdot \vec{b}$ (f) $-\vec{a} \cdot -\vec{b} = \vec{a} \cdot \vec{b}$
(g) $(\vec{a} - \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c}$ (h) $\vec{a} \cdot (\vec{b} - \vec{c}) = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$
(i) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{b}$
(j) $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{c} \cdot \vec{a} + \vec{b} \cdot \vec{b} + 2(\vec{a} \cdot \vec{b})$
(k) $(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2(\vec{a} \cdot \vec{b})$

Exercises

Part A

1. In each of the following, tell whether the given expression refers to points, to translations, to real numbers, or is nonsense.
   (a) $-\vec{a} \cdot (\vec{b} - \vec{c})$ (b) $(\vec{b}3 + \vec{c}5) \cdot (\vec{b}2 - \vec{c}3)$
   (c) $\vec{b}(\vec{c} \cdot \vec{b}) + \vec{c}(\vec{b} \cdot \vec{c})$ (d) $(P - O)[(P - O) \cdot \vec{r}]$
   (e) $P + \vec{r} \cdot (P - O)$ (f) $P + \vec{r}[(P - O) \cdot \vec{r}]$
   (g) $(\vec{b} \cdot \vec{c}) \cdot \vec{d}$ (h) $\vec{b} \cdot (\vec{c} \cdot \vec{d})$
   (i) $[(P + \vec{r}) - (Q + \vec{r})] \cdot (P - Q)$ (j) $[P + (Q - P)] \cdot (Q - P)$
2. In Theorem 11–1(d) there are two occurrences of '+'. Notice that the one on the left side refers to addition of translations, while the one on the right refers to addition of real numbers. Find other parts of Theorem 11–1 where two occurrences of the *same* operator refer to *different* operations.

Part B

Prove the specified parts of Theorem 11–1.

1. Part (a) [Note that this is (7) on page 42. There it was derived from intuitive notions concerning orthogonal components etc., and was used in showing that, on the basis of these notions, dot multiplication is commutative. Now that we are starting fresh and postulating commutativity of dot multiplication, you can use 4_{14} in giving a formal proof of (a).]
2. Part (b) [*Hint*: Use 4_{13} and the theorems '$\vec{a}0 = \vec{0}$' and '$a0 = 0$'.]
3. Part (c)
4. Part (d)
5. Part (e) [*Hint*: There are two ways to prove '$\vec{a} \cdot -\vec{b} = -(\vec{a} \cdot \vec{b})$'. One depends on part (d), the other on part (a). Try to get both proofs. Having done this, you should be able to think of three ways to prove '$-\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{b})$'.]

Section 11.08 can usually be covered in two class sessions and two homework assignments. One way to achieve this is to use the exercises of Part A and Part B (pp. 43-44) for class discussion and supervised seat work. This will help insure that students begin to apply the properties of dot multiplication properly. One homework assignment consists of Part A, page 45. Part B (pages 45-46) can provide a nice variation between class discussion and supervised work. The second homework assignment can be Part C, pages 46-47.

Answers for Part A

1. (a) real number
   (b) real number
   (c) vector
   (d) vector
   (e) nonsense
   (f) point
   (g) nonsense
   (h) nonsense
   (i) real number
   (j) nonsense

2. In (e): The '−' in '$\vec{a} \cdot -\vec{b}$' and in '$-\vec{a} \cdot \vec{b}$' refers to taking the inverse of a translation, while the '−' in '$-(\vec{a} \cdot \vec{b})$' refers to taking the opposite of a real number.

   In (g): The '-' in '$\vec{a} - \vec{b}$' refers to subtraction of vectors, while '-' in '$\vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c}$' refers to subtraction of real numbers.

   In (h): The '-' in '$\vec{b} - \vec{c}$' refers to subtraction of vectors, while the '-' in '$\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$' refers to subtraction of real numbers.

   In (i): The '-' in '$\vec{a} - \vec{b}$' refers to subtraction of vectors, while the '-' in '$\vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{b}$' refers to subtraction of real numbers.

   In (j): Both of the '+'s on the left refer to addition of vectors, while both of the '+'s on the right refer to addition of real numbers.

   In (k): Both of the '-'s on the left refer to subtraction of vectors, while the '-' on the right refers to subtraction of real numbers.

Answers for Part B

1. $$\begin{aligned} \vec{a} \cdot (\vec{b}b) &= (\vec{b}b) \cdot \vec{a} && [4_{14}] \\ &= (\vec{b} \cdot \vec{a})b && [4_{13}] \\ &= (\vec{a} \cdot \vec{b})b && [4_{14}] \end{aligned}$$

2. $$\begin{aligned} \vec{0} \cdot \vec{a} &= (\vec{b}0) \cdot \vec{a} && [\vec{0} = \vec{b}0] \\ &= (\vec{b} \cdot \vec{a})0 && [4_{13}] \\ &= 0 && [5] \end{aligned}$$

   Also, $\vec{a} \cdot \vec{0} = \vec{0} \cdot \vec{a} = 0$.

3. Suppose, first, that $\vec{a} = \vec{0}$. Then, $\vec{a} \cdot \vec{a} = \vec{a} \cdot \vec{0} = 0$. Hence, $\vec{a} = \vec{0} \Longrightarrow \vec{a} \cdot \vec{a} = 0$.

   Next, suppose that $\vec{a} \cdot \vec{a} = 0$. Then, $\vec{a} \cdot \vec{a} \not> 0$. So, by Postulate 4_{11} and modus tollens, $\vec{a} = \vec{0}$. Hence, $\vec{a} \cdot \vec{a} = 0 \Longrightarrow \vec{a} = \vec{0}$.

   Since we have $\vec{a} = \vec{0} \Longrightarrow \vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{a} = 0 \Longrightarrow \vec{a} = \vec{0}$, (c) follows.

4. $$\begin{aligned} \vec{a} \cdot (\vec{b} + \vec{c}) &= (\vec{b} + \vec{c}) \cdot \vec{a} && [4_{14}] \\ &= \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a} && [4_{12}] \\ &= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} && [4_{14}] \end{aligned}$$

5. [One proof of '$\vec{a} \cdot -\vec{b} = -(\vec{a} \cdot \vec{b})$']

   $$\begin{aligned} \vec{a} \cdot -\vec{b} &= -\vec{b} \cdot \vec{a} && [4_{14}] \\ &= (\vec{b} \cdot -1) \cdot \vec{a} && [-\vec{b} = \vec{b} \cdot -1] \\ &= (\vec{b} \cdot \vec{a}) \cdot -1 && [4_{13}] \\ &= (\vec{a} \cdot \vec{b}) \cdot -1 && [4_{14}] \\ &= -(\vec{a} \cdot \vec{b}) && [5] \end{aligned}$$

   So, $\vec{a} \cdot -\vec{b} = -(\vec{a} \cdot \vec{b})$.

   [Alternate proof. This makes use of the real number theorem '$a + b = 0 \Longrightarrow b = -a$'.]

   $$\begin{aligned} \vec{a} \cdot \vec{b} + \vec{a} \cdot -\vec{b} &= \vec{a} \cdot (\vec{b} + -\vec{b}) \\ &= \vec{a} \cdot \vec{0} \\ &= 0 \end{aligned}$$

   Since $\vec{a} \cdot \vec{b} + \vec{a} \cdot -\vec{b} = 0$, it follows that $\vec{a} \cdot -\vec{b} = -(\vec{a} \cdot \vec{b})$.

   Now, here are three proofs of '$-\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{b})$':

   (i) $$\begin{aligned} -\vec{a} \cdot \vec{b} &= (\vec{a} \cdot -1) \cdot \vec{b} && [-\vec{a} = \vec{a} \cdot -1] \\ &= (\vec{a} \cdot \vec{b}) \cdot -1 && [4_{13}] \\ &= -(\vec{a} \cdot \vec{b}) && [5] \end{aligned}$$

   So, $-\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{b})$.

   (ii) $$\begin{aligned} -\vec{a} \cdot \vec{b} &= \vec{b} \cdot -\vec{a} && [4_{14}] \\ &= -(\vec{b} \cdot \vec{a}) && [\text{Ex. } 5] \\ &= -(\vec{a} \cdot \vec{b}) && [4_{14}] \end{aligned}$$

   (iii) $$\begin{aligned} \vec{a} \cdot \vec{b} + -\vec{a} \cdot \vec{b} &= (\vec{a} + -\vec{a}) \cdot \vec{b} \\ &= \vec{0} \cdot \vec{b} \\ &= 0. \end{aligned}$$

   Since $\vec{a} \cdot \vec{b} + -\vec{a} \cdot \vec{b} = 0$, it follows that $-\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{b})$.

6. Part (f) **7.** Part (g) **8.** Part (h)
9. Part (i) **10.** Part (j) **11.** Part (k)

*

Of the numbered statements in Section 11.07 we have, so far, adopted (1), (3), (5), (6), and (8) as postulates. Using these, both (7) and the second part of our intuitive description (*) of dot multiplication turned out to be theorems. This leaves us with (2):

$$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2,$$

with (4):

$$\vec{a} \cdot \vec{b} = 0 \longleftrightarrow \vec{a} \in [\vec{b}]^{\perp} \qquad [\vec{b} \neq \vec{0}],$$

and with the first part of (*):

$$\vec{a} \cdot \vec{b} = \text{comp}_{\vec{b}}(\vec{a})\, \|\vec{b}\|^2 \qquad [\vec{b} \neq \vec{0}]$$

In our formal development we shall take these as the bases of definitions. With this in mind, we turn our attention to each of these statements.

By Postulate 4_{11} and Theorem 11–1(b), $\vec{a} \cdot \vec{a} \geq 0$. Since it is evident, intuitively, that $\|\vec{a}\| \geq 0$ and $\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$, we are justified in adopting '$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$' as a definition.

Since, intuitively, $\vec{b} \in [\vec{a}]^{\perp}$ if and only if $\vec{b} \cdot \vec{a} = 0$ [at least, for $\vec{a} \neq \vec{0}$], we might think of adopting the following:

$$(1) \qquad [\vec{a}]^{\perp} = \{\vec{x}: \vec{x} \cdot \vec{a} = 0\}$$

as a definition of the orthogonal complement of $[\vec{a}]$. To justify doing so, however, it would be necessary to prove that

$$[\vec{b}] = [\vec{a}] \longrightarrow [\vec{c} \cdot \vec{b} = 0 \longleftrightarrow \vec{c} \cdot \vec{a} = 0].$$

[Why would this be necessary?] This is not difficult to derive from our present postulates, but a simpler procedure is to take as our definition:

$$[\vec{a}]^{\perp} = \{\vec{x}: \forall_{\vec{y} \in [\vec{a}]}\, \vec{x} \cdot \vec{y} = 0\}$$

Doing so, it is easy to show that (1) is a theorem. This justifies sentence (4) of Section 11.07. [Note that there is no need to restrict the definition to non-$\vec{0}$ values of '$\vec{a}$'. What, then, is $[\vec{0}]^{\perp}$?]

Finally, the first part of (*) together with sentence (2) suggest the definition:

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Answers for Part B [cont.]

6. $-\vec{a} \cdot -\vec{b} = -\,-\vec{a} \cdot \vec{b}$ [Th. 11-1(e)]
$\quad = \vec{a} \cdot \vec{b}$ [$--\vec{a} = \vec{a}$]

7. $(\vec{a} - \vec{b}) \cdot \vec{c} = (\vec{a} + -\vec{b}) \cdot \vec{c}$
$\quad = \vec{a} \cdot \vec{c} + -\vec{b} \cdot \vec{c}$
$\quad = \vec{a} \cdot \vec{c} + -(\vec{b} \cdot \vec{c})$
$\quad = \vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c}$

8. $\vec{a} \cdot (\vec{b} - \vec{c}) = (\vec{b} - \vec{c}) \cdot \vec{a}$
$\quad = \vec{b} \cdot \vec{a} - \vec{c} \cdot \vec{a}$
$\quad = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$

9. $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = (\vec{a} + \vec{b}) \cdot \vec{a} - (\vec{a} + \vec{b}) \cdot \vec{b}$
$\quad = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} - (\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b})$
$\quad = \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{b}$
$\quad = \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{b}$

10. $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (\vec{a} + \vec{b}) \cdot \vec{a} + (\vec{a} + \vec{b}) \cdot \vec{b}$
$\quad = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + (\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b})$
$\quad = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + 2(\vec{a} \cdot \vec{b})$

11. By 11-1(j) and 11-1(e) and (f).

Taking (1) as a definition it would be necessary to prove the second displayed sentence in order to be sure that orthogonal complementing is an operation on directions and is independent of our way of specifying them. On adopting Definition 11-1(b) the sentence (1) is a theorem [see Theorem 11-2 on page 46]. The second displayed sentence is proved in Exercise 2 of Part C on page 46.

Since, for any $\vec{a} \in T$, $\vec{a} \cdot \vec{0} = 0$ it follows by Definition 11-1(b) that $[\vec{0}]^{\perp} = T$.

TC 45 (1)

Answers for Part A

1. (a) A direct consequence of Def. 11-1(a) and (3) of Part B on page 7.

(b) $\|\vec{a}\| = 0 \Longleftrightarrow \sqrt{\vec{a} \cdot \vec{a}} = 0$
$\quad \Longleftrightarrow \vec{a} \cdot \vec{a} = 0$
$\quad \Longleftrightarrow \vec{a} = \vec{0}$ [4_{11} and Theorem 11-1(b)]

(c) $\|-\vec{a}\| = \sqrt{-\vec{a} \cdot -\vec{a}}$ [Def. 11-1(a)]
$\quad = \sqrt{\vec{a} \cdot \vec{a}}$ [Th. 11-1(f)]
$\quad = \|\vec{a}\|$ [Def. 11-1(a)]

(d) Since $B - A \in T$, this follows directly from part (a).

(e) $\|A - B\| = \|-(B - A)\|$
$\quad = \|B - A\|$ [part (c)]

(f) $\|B - A\| = 0 \Longleftrightarrow B - A = \vec{0}$ [part (b)]
$\quad \Longleftrightarrow B = A$

$$\text{comp}_{\vec{b}}(\vec{a}) = (\vec{a}\cdot\vec{b})/(\vec{b}\cdot\vec{b})$$

for the orthogonal component of $\vec{a}$ with respect to $\vec{b}$. Because of our treatment of reciprocals in Chapter 4, it is not necessary to restrict this definition even though $\vec{0}\cdot\vec{0} = 0$. This is because, by Postulates 4_0(e), 5_0(d), and 5_0(f), $\text{comp}_{\vec{b}}(\vec{a})$, as defined above, is a real number in any case. In fact, since $\vec{a}\cdot\vec{0} = 0$ and since the product of 0 by any real number [in this case, /0] is 0, it follows that, for any vector $\vec{a}$, $\text{comp}_{\vec{0}}(\vec{a}) = 0$.

Summarizing the preceding discussion, we have:

> **Definition 11-1**
> (a) $\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}}$
> (b) $[\vec{a}]^{\perp} = \{\vec{x}: \forall_{\vec{y}\in[\vec{a}]}\ \vec{x}\cdot\vec{y} = 0\}$
> (c) $\text{comp}_{\vec{b}}(\vec{a}) = (\vec{a}\cdot\vec{b})/(\vec{b}\cdot\vec{b})$

Exercises

Part A

1. Prove each of the following statements.
   (a) $\|\vec{a}\| \geq 0$ and $\|\vec{a}\|^2 = \vec{a}\cdot\vec{a}$ [*Hint*: See (3) in Part B on page 7.]
   (b) $\|\vec{a}\| = 0 \Longleftrightarrow \vec{a} = \vec{0}$
   (c) $\|-\vec{a}\| = \|\vec{a}\|$
   (d) $\|B - A\| \geq 0$
   (e) $\|A - B\| = \|B - A\|$
   (f) $\|B - A\| = 0 \Longleftrightarrow B = A$
2. Our intuitive notions concerning the norm of a translation suggest that $\|\vec{a}\|$ is the distance from any point to its image under $\vec{a}$. In particular, $d(A, B) = \|B - A\|$. [Explain.] Tell what parts (d), (e), and (f) of Exercise 1 say about the distance between points.
3. Tell which of the following should be theorems. If a given sentence does not hold for all translations, give a counterexample and find a restriction under which the sentence should be a theorem.
   (a) $\vec{a} = \vec{b} \longrightarrow \|\vec{a}\| = \|\vec{b}\|$
   (b) $\|\vec{a}\| = \|\vec{b}\| \longrightarrow \vec{a} = \vec{b}$
   (c) $\|\vec{a} + \vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$
   (d) $\|\vec{a}b\| = \|\vec{a}\|b$
   (e) $B - A = C - D \longrightarrow \|D - A\| = \|C - B\|$
   (f) $\|B - A\| = \|C - D\| \longrightarrow \|D - A\| = \|C - B\|$

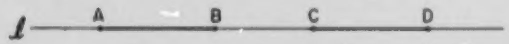
Part B

Suppose that $A \neq B \neq C \neq D$ and that $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$.

1. Suppose that $(B - A)3 = (D - C)\cdot -2$.
   (a) What is $(B - A) : (D - C)$?
   (b) What is $\|B - A\| : \|D - C\|$? [*Hint*: $(B - A)\cdot(B - A) = [(D - C)\cdot -\frac{2}{3}]\cdot[(D - C)\cdot -\frac{2}{3}] = \ldots$]
   (c) If $\|B - A\| = 5$, what is $\|D - C\|$?

Answers for Part A [cont.]

2. (d) says that the distance between any two points is nonnegative.
   (e) says that the distance between A and B is the same as the distance between B and A.
   (f) says that the distance between points A and B is 0 if and only if A and B are the same point.
3. (a) Theorem.
   (b) Not a theorem, for let $\vec{b} = -\vec{a} \neq \vec{0}$. Then $\|\vec{b}\| = \|\vec{a}\|$, but $\vec{b} \neq \vec{a}$.
   [Note, however, that for $\vec{a} \in [\vec{c}]^{+}$ and $\vec{b} \in [\vec{c}]^{+}$, $\|\vec{a}\| = \|\vec{b}\| \Longleftrightarrow \vec{a} = \vec{b}$.]
   (c) Not a theorem, for let $\vec{b} = -\vec{a} \neq \vec{0}$. Then $\|\vec{a} + \vec{b}\| = 0$, but $\|\vec{a}\| + \|\vec{b}\| = \|\vec{a}\|2 \neq 0$.
   [Note, however, that if one of the vectors is a nonnegative multiple of the other, the statement is a theorem.]
   (d) Not a theorem. $\|\vec{a}\cdot -1\| \neq \|\vec{a}\|\cdot -1$ when $\vec{a} \neq \vec{0}$, for $\|\vec{a}\cdot -1\| > 0$ and $\|\vec{a}\|\cdot -1 < 0$.
   [Note, however, that $\|\vec{a}b\| = \|\vec{a}\|b$ if $b \geq 0$.]
   (e) Theorem, for $B - A = C - D \Longleftrightarrow D - A = C - B$ and $D - A = C - B \Longrightarrow \|D - A\| = \|C - B\|$.
   (f) Not a theorem, for let A, B, C, and D be four points of a line such that $\|B - A\| = \|C - D\|$ and $[B - A]^{+} \neq [C - D]^{+}$, as shown in the following picture:

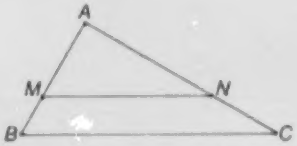


Then, $\|B - A\| = \|C - D\|$ but $\|D - A\| \neq \|C - B\|$. [Note, however, that (f) is a theorem when the senses of B - A and C - D are the same.]

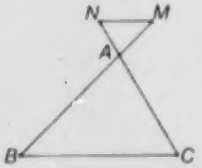
Answers for Part B

1. (a) −2/3
   (b) 2/3 [By the hint, $\|B - A\|^2 = \|D - C\|^2\cdot 4/9$. So, $\|B - A\| = \|D - C\|\cdot 2/3$.]
   (c) 15/2 [From (b), we know that $\|B - A\| = \|D - C\|\cdot 2/3$. So, $5 = \|D - C\|\cdot 2/3$.]

2. Suppose that $C = A + (B - A)\frac{5}{2}$. Compute each of the following:
   (a) $(C - A) : (B - C)$ (b) $(C - A) : (B - A)$
   (c) $(C - B) : (B - A)$ (d) $\|C - A\| : \|B - A\|$
   (e) $\|C - B\| : \|A - B\|$ (f) $\|C - A\| : \|B - C\|$
3. Suppose that $(B - A)b = (D - C)d$, for some nonzero numbers b and d.
   (a) What is $(B - A) : (D - C)$?
   (b) What is $\|B - A\| : \|D - C\|$? [*Hint*: See Exercise 1(b). And, note the properties of square rooting that you need.]
4. Suppose that $C = A + (B - A)t$, for some t. Compute each of the following, giving appropriate restrictions on the values for 't' when necessary.
   (a) $(C - A) : (B - C)$ (b) $(C - A) : (B - A)$
   (c) $(C - B) : (B - A)$ (d) $\|C - A\| : \|B - A\|$
   (e) $\|C - B\| : \|A - B\|$ (f) $\|C - A\| : \|B - C\|$
5. Given $\triangle ABC$, suppose that $\overleftrightarrow{MN} \parallel \overleftrightarrow{BC}$ and $(M - A) : (B - A) = \frac{2}{3}$, as shown in the picture at the right.



   (a) Compute $(N - A) : (C - A)$, $(N - A) : (C - N)$, and $(N - M) : (C - B)$.
   (b) Given that $\|M - A\| = 4$, compute $\|B - A\|$ and $\|B - M\|$.
   (c) Given that $\|C - N\| = 3$, compute $\|N - A\|$ and $\|C - A\|$.
   (d) Given that $\|B - C\| = 12$, compute $\|N - M\|$.
6. Given $\triangle ABC$, suppose that $\overleftrightarrow{MN} \parallel \overleftrightarrow{BC}$ and $(M - A) : (B - A) = -\frac{1}{4}$, as shown in the picture at the right.



   (a) Compute $(N - A) : (C - A)$, $(N - A) : (C - N)$, $(N - M) : (C - B)$, and $(M - B) : (B - A)$.
   (b) Given that $\|N - A\| = 2$ and $\|M - N\| = \|M - A\| = 5$, compute each of the following: $\|C - A\|$, $\|C - B\|$, $\|B - A\|$, $\|M - B\|$, and $\|N - C\|$.

Part C

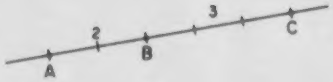
1. Prove:

‖ Theorem 11–2 $\vec{b} \in [\vec{a}]^{\perp} \longleftrightarrow \vec{b} \cdot \vec{a} = 0$

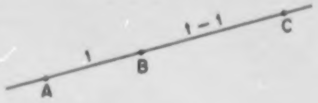
2. Show that $[\vec{b}] = [\vec{a}] \longrightarrow [\vec{c} \cdot \vec{b} = 0 \longleftrightarrow \vec{c} \cdot \vec{a} = 0]$.
3. In the discussion which motivated Definition 11–1, a question was raised concerning $[\vec{0}]^{\perp}$. Prove that $[\vec{0}]^{\perp} = \mathscr{T}$. That is, prove that the orthogonal complement of $\vec{0}$ is the set $\mathscr{T}$ of all translations. [*Hint*: Since $[\vec{0}]^{\perp} \subseteq \mathscr{T}$ [Why?], it is enough to show that $\mathscr{T} \subseteq [\vec{0}]^{\perp}$.]

Answers for Part B [cont.]

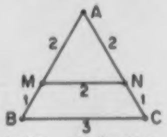
2. From the information given, we know that (C - A) = (B - A)5/2, so that (C - A):(B - A) = 5/2. Making a picture of this situation, we will be able to "read off" the values of the required ratios. Of course, we should be able to compute these values from the information given.



   (a) -5/3 (b) 5/2
   (c) 3/2 (d) 5/2
   (e) 3/2 (f) 5/3
3. (a) d/b
   (b) |d/b| [or, |d|/|b|] We prove this as follows:
   (B - A)·(B - A) = ((D - C)·d/b)·((D - C)·d/b)
   = (D - C)·(D - C)d^2/b^2. So, $||B - A||^2 = ||D - C||^2 \cdot d^2/b^2$
   which means that ||B - A|| = ||D - C||·|d|/|b|.
4. From the given information, (C - A):(B - A) = t. As in Exercise 2, it is helpful to draw a picture. Then, we will be able to "read off" the values of the required ratios.



   (a) t/(1 - t) [t ≠ 1] (b) t
   (c) t - 1 (d) |t| [from part (b)]
   (e) |t - 1| [from part (c)] (f) |t|/|1 - t| [t ≠ 1]
5. It may be helpful to label a picture of the given △ ABC in such a way as to indicate known ratios.



   (a) 2/3; 2; 2/3
   (b) 6, 2 [(B - A):(M - A) = 3/2 and (B - M):(M - A) = 1/2.
   Thus, ||B - A||/||M - A|| = 3/2 and ||B - M||/||M - A|| = 1/2 so that ||B - A|| = ||M - A||·3/2 = 4·3/2 = 6 and ||B - M|| = ||M - A||·1/2 = 4·1/2 = 2.]
   (c) 6, 9 [||N - A||/||C - N|| = 2 and ||C - A||/||C - N|| = 3 so that ||N - A|| = ||C - N||2 = 3·2 = 6 and ||C - A|| = ||C - N||3 = 3·3 = 9.]
   (d) 8 [||N - M||/||B - C|| = 2/3 so that ||N - M|| = ||B - C||·2/3 = 12·2/3 = 8.]
6. (a) (N - A):(C - A) = -1/4 (N - A):(C - N) = -1/5
   (N - M):(C - B) = -1/4 (M - B):(B - A) = -5/4
   (b) 8, 20, 20, 25, and 10

Answers for Part C

1. Only if-part: Suppose that $\vec{b} \in [\vec{a}]^{\perp}$. By Definition 11-1(b), each member of $[\vec{a}]$ is such that its dot product with $\vec{b}$ is 0. Since $\vec{a} \in [\vec{a}]$, it follows that $\vec{b} \cdot \vec{a} = 0$. Hence, $\vec{b} \in [\vec{a}]^{\perp} \Longrightarrow \vec{b} \cdot \vec{a} = 0$.

   If-part: Suppose that $\vec{b} \cdot \vec{a} = 0$ and that $\vec{p} \in [\vec{a}]$. It follows that $\vec{p} = \vec{a}b$ where $b \in \mathcal{R}$ and, so, that $\vec{b} \cdot \vec{p} = \vec{b} \cdot (\vec{a}b) = (\vec{b} \cdot \vec{a})b = 0b = 0$. Hence, for each $\vec{y} \in [\vec{a}]$, $\vec{b} \cdot \vec{y} = 0$ and, by Definition 11-1(b), $\vec{b} \in [\vec{a}]^{\perp}$. Consequently, if $\vec{b} \cdot \vec{a} = 0$ then $\vec{b} \in [\vec{a}]$.

2. Suppose that $[\vec{b}] = [\vec{a}]$. Then, for some $b \neq 0$, $\vec{b} = \vec{a}b$, so that $\vec{c} \cdot \vec{b} = \vec{c} \cdot (\vec{a}b) = (\vec{c} \cdot \vec{a})b$. Thus, $\vec{c} \cdot \vec{b} = 0$ if and only if $(\vec{c} \cdot \vec{a})b = 0$. Since $b \neq 0$, the latter is the case if and only if $\vec{c} \cdot \vec{a} = 0$. So, by the replacement rule for biconditionals, $\vec{c} \cdot \vec{b} = 0$ if and only if $\vec{c} \cdot \vec{a} = 0$. Hence, the theorem.

3. $[\vec{a}]^{\perp} \subseteq \mathcal{T}$, for any $\vec{a}$. So, $[\vec{0}]^{\perp} \subseteq \mathcal{T}$. Suppose that $\vec{a}$ is any translation. Since $[\vec{0}] = \{\vec{0}\}$ and $\vec{a} \cdot \vec{0} = 0$, it follows that, for each $\vec{y}$ in $[\vec{0}]$, $\vec{a} \cdot \vec{y} = 0$. So, by Definition 11-1(b), $\vec{a} \in [\vec{0}]^{\perp}$. Thus, if $\vec{a} \in \mathcal{T}$ then $\vec{a} \in [\vec{0}]^{\perp}$. This means that $\mathcal{T} \subseteq [\vec{0}]^{\perp}$. Since it is also the case that $[\vec{0}]^{\perp} \subseteq \mathcal{T}$, we have that $[\vec{0}]^{\perp} = \mathcal{T}$.

Answers for Part C [cont.]

4. (a) By Definition 11-1(c), $\text{comp}_{\vec{b}}(\vec{a}) = (\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})$. By Postulate 4_0, both $\vec{a} \cdot \vec{b}$ and $/(\vec{b} \cdot \vec{b})$ belong to $\mathcal{R}$ so that their product belongs to $\mathcal{R}$. Hence, $\text{comp}_{\vec{b}}(\vec{a}) \in \mathcal{R}$.

   (b) Suppose that $\vec{b} \in [\vec{a}]$, for $\vec{a} \neq \vec{0}$. Then, by the hint, $\vec{b} = \vec{a}b$, where $b = \vec{b} : \vec{a}$. By Definition 11-1(c), $\text{comp}_{\vec{a}}(\vec{b}) = (\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a}) = (\vec{a}b \cdot \vec{a})/(\vec{a} \cdot \vec{a}) = ((\vec{a} \cdot \vec{a})b)/(\vec{a} \cdot \vec{a}) = b$. Since $b = \vec{b} : \vec{a}$, $\text{comp}_{\vec{a}}(\vec{b}) = \vec{b} : \vec{a}$. Hence, the theorem.

   (c) This holds for $\vec{a} = \vec{0}$ since $[\vec{0}]^{\perp} = \mathcal{T}$ so that $\vec{b} \in [\vec{0}]^{\perp}$, for any $\vec{b}$. For the remainder of the proof, assume that $\vec{a} \neq \vec{0}$.

   Suppose, first, that $\text{comp}_{\vec{a}}(\vec{b}) = 0$. Then, $(\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a}) = 0$ and so, since $\vec{a} \cdot \vec{a} \neq 0$, $\vec{b} \cdot \vec{a} = 0$. By Theorem 11-2, $\vec{b} \in [\vec{a}]^{\perp}$. Hence, if $\text{comp}_{\vec{a}}(\vec{b}) = 0$ then $\vec{b} \in [\vec{a}]^{\perp}$.

   Next, suppose that $\vec{b} \cdot \vec{a} = 0$. Then, $\text{comp}_{\vec{a}}(\vec{b}) = (\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a}) = 0$. Hence, if $\vec{b} \cdot \vec{a} = 0$ then $\text{comp}_{\vec{a}}(\vec{b}) = 0$.

   (d)
$$\begin{aligned} \text{comp}_{\vec{a}}(\vec{b} + \vec{c}) &= (\vec{b} + \vec{c}) \cdot \vec{a}/(\vec{a} \cdot \vec{a}) && \text{[Def. 11-1(c)]} \\ &= (\vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a})/(\vec{a} \cdot \vec{a}) && \text{[Post. } 4_{12}\text{]} \\ &= (\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a}) + (\vec{c} \cdot \vec{a})/(\vec{a} \cdot \vec{a}) && \text{[Post. 5]} \\ &= \text{comp}_{\vec{a}}(\vec{b}) + \text{comp}_{\vec{a}}(\vec{c}) && \text{[Def. 11-1(c)]} \end{aligned}$$

   (e)
$$\begin{aligned} \text{comp}_{\vec{a}}(\vec{b}b) &= (\vec{b}b) \cdot \vec{a}/(\vec{a} \cdot \vec{a}) && \text{[Def. 11-1(c)]} \\ &= (\vec{b} \cdot \vec{a})b/(\vec{a} \cdot \vec{a}) && \text{[Post. } 4_{13}\text{]} \\ &= ((\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a}))b && \text{[Post. 5]} \\ &= \text{comp}_{\vec{a}}(\vec{b})b && \text{[Def. 11-1(c)]} \end{aligned}$$

   (f) Suppose that $a \neq 0$. Then
$$\begin{aligned} \text{comp}_{(\vec{a}a)}(\vec{b}) &= \vec{b} \cdot (\vec{a}a)/(\vec{a}a \cdot \vec{a}a) \\ &= (\vec{b} \cdot \vec{a})a/((\vec{a} \cdot \vec{a})a^2) \\ &= ((\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a}))a/a^2 \\ &= \text{comp}_{\vec{a}}(\vec{b})/a. \end{aligned}$$
   Hence, the theorem.

4. In Section 11.06 we listed some intuitive properties of orthogonal components. Now that the notion of orthogonal components is a part of our formal development, we are in a position to prove those properties. Give proofs for the following:
   (a) $\mathrm{comp}_{\vec b}(\vec a) \in \mathscr{R}$ [*Hint*: Since both $\vec a \cdot \vec b$ and $\vec b \cdot \vec b$ are real numbers [Why?] so is $(\vec a \cdot \vec b)/(\vec b \cdot \vec b)$.]
   (b) $\vec b \in [\vec a] \Longrightarrow \mathrm{comp}_{\vec a}(\vec b) = \vec b : \vec a$ $[\vec a \neq \vec 0]$ [*Hint*: Given that $\vec b \in [\vec a]$, it follows that $\vec b = \vec a b$, for some b. So, $b = \vec b : \vec a$. Now, compute $\mathrm{comp}_{\vec a}(\vec b)$.]
   (c) $\mathrm{comp}_{\vec a}(\vec b) = 0 \Longleftrightarrow \vec b \in [\vec a]^{\perp}$
   (d) $\mathrm{comp}_{\vec a}(\vec b + \vec c) = \mathrm{comp}_{\vec a}(\vec b) + \mathrm{comp}_{\vec a}(\vec c)$
   (e) $\mathrm{comp}_{\vec a}(\vec b b) = \mathrm{comp}_{\vec a}(\vec b)b$
   (f) $\mathrm{comp}_{(\vec a a)}(\vec b) = \mathrm{comp}_{\vec a}(\vec b)/a$ $[a \neq 0]$

11.09 Basic Properties of Norms and Orthogonal Complements

As was remarked earlier, it will be greatly to our advantage to show that sentences (3)–(10) on pages 35–36 are theorems. For, once this is done, it will be a simple matter to garner numerous theorems concerning perpendicularity and distance from the earlier sections of this chapter. Most of the sentences (3)–(10) are easy to prove. Among these, (7)–(10) are especially easy and we list them, together with sentence (11), in:

Theorem 11–3

(a) $\|\vec a\| \in \mathscr{R}$ (b) $\vec a \neq \vec 0 \Longleftrightarrow \|\vec a\| > 0$

(c) $\|\vec a a\| = \|\vec a\|\,|a|$ (d) $\vec a + \vec b \in [\vec a - \vec b]^{\perp} \Longleftrightarrow \|\vec a\| = \|\vec b\|$

(e) $\vec a_1 = \vec a/\|\vec a\| \Longleftrightarrow (\|\vec a_1\| = 1 \text{ and } \vec a = \vec a_1\|\vec a\|)$ $[\vec a \neq \vec 0]$

[Note that proofs using Definition 11–1(a) require theorems about square roots. As indicated in Part B on page 7, such theorems can be derived from Postulate 5 and two special postulates:

$$\sqrt{a} \in \mathscr{R} \quad [a \geq 0] \text{ and: } \sqrt{a} \geq 0 \text{ and } (\sqrt{a})^2 = a \quad [a \geq 0]$$

In proving Theorem 11–3 you may cite as theorems these postulates and any of the results on square roots developed in Part B. You may, of course, also use parts of Theorem 11–1, as well as earlier theorems.]

Exercises

Part A

Prove each of the following:

1. $\vec a \cdot \vec a \geq 0$
2. Theorem 11–3(a)
3. Theorem 11–3(b)
4. Theorem 11–3(c)
5. Theorem 11–3(d)
6. Theorem 11–3(e)

Among the exercises of section 11.09, Parts A, B, and C are best for homework assignments. Parts A and C involve derivations that are not too involved. Part B provides some applications to geometric figures. Because of the sequential nature of the exercises, Parts D, E, and F are recommended for group discussion.

Answers for Part A

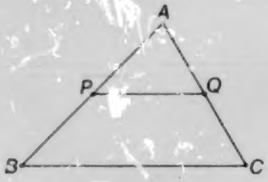
1. For $\vec a = \vec 0$, $\vec a \cdot \vec a = 0 \geq 0$. For $\vec a \neq \vec 0$, $\vec a \cdot \vec a > 0 \geq 0$. So, in any case, $\vec a \cdot \vec a \geq 0$.
2. $\|\vec a\| = \sqrt{\vec a \cdot \vec a} \in \mathscr{R}$, since $\vec a \cdot \vec a \geq 0$.
3. Suppose that $\vec a \neq \vec 0$. Then, $\vec a \cdot \vec a > 0$, so that $\sqrt{\vec a \cdot \vec a} > 0$. Thus, $\|\vec a\| > 0$.

   Suppose, next, that $\|\vec a\| > 0$. Then $\|\vec a\|^2 > 0$ so that $\vec a \cdot \vec a > 0$. Either $\vec a = \vec 0$ or $\vec a \neq \vec 0$. If the former, then $\vec a \cdot \vec a = \vec 0 \cdot \vec 0 = 0 \not> 0$. Thus, $\vec a \neq \vec 0$.
4. $\|\vec a a\| = \sqrt{\vec a a \cdot \vec a a} = \sqrt{(\vec a \cdot \vec a)a^2} = \sqrt{(\vec a \cdot \vec a)}\sqrt{a^2} = \|\vec a\|\,|a|$
5. Suppose that $\vec a + \vec b \in [\vec a - \vec b]^{\perp}$. Then, $(\vec a + \vec b)\cdot(\vec a - \vec b) = 0$. By Theorem 11-1(i), $\vec a \cdot \vec a = \vec b \cdot \vec b$. So, $\|\vec a\|^2 = \|\vec b\|^2$ and, since norms are nonnegative, $\|\vec a\| = \|\vec b\|$.

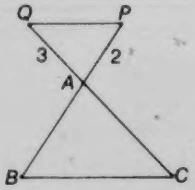
   Suppose, next, that $\|\vec a\| = \|\vec b\|$. Then $\vec a \cdot \vec a = \vec b \cdot \vec b$, so that $(\vec a + \vec b)\cdot(\vec a - \vec b) = 0$. Thus, $\vec a + \vec b \in [\vec a - \vec b]^{\perp}$.
6. Suppose that $\vec a \neq \vec 0$ and that $\vec a_1 = \vec a/\|\vec a\|$. Then, $\|\vec a\| \neq 0$ so that $\vec a = \vec a_1\|\vec a\|$. Also, $\vec a_1 \cdot \vec a_1 = (\vec a/\|\vec a\|)\cdot(\vec a/\|\vec a\|) = (\vec a \cdot \vec a)/\|\vec a\|^2 = (\vec a \cdot \vec a)/(\vec a \cdot \vec a) = 1$. Thus, $\|\vec a_1\| = 1$ and $\vec a = \vec a_1\|\vec a\|$.

Part B

1. Suppose that $\overline{AB}$ and $\overline{CD}$ are nondegenerate parallel intervals. Show that $\overline{AB} : \overline{CD} = \|B - A\|/\|D - C\|$. [Hint: Recall that $\overline{AB} : \overline{CD} = |(B - A) : (D - C)|$ and make use of Theorem 11–3(c).]



2. By Exercise 1, we know that the ratio of two nondegenerate parallel intervals is the ratio of the norms of the translations determined by the endpoints of intervals.
   (a) Given that $\overline{PQ} \parallel \overline{BC}$, as shown in the picture at the right, we know that $(P - A) : (B - A) = (Q - P) : (C - B)$. Show that $\|P - A\|/\|B - A\| = \|Q - P\|/\|C - B\|$.
   (b) Show that $\|P - A\|/\|Q - P\| = \|B - A\|/\|C - B\|$.
3. Given that $\overline{PQ} \parallel \overline{BC}$, $\|P - A\| = 2$ and $\|B - P\| = 3$, as in the picture for Exercise 2.
   (a) Compute the ratios $\|B - P\|/\|B - A\|$, $\|P - Q\|/\|B - C\|$, $\|A - Q\|/\|Q - C\|$, and $\|A - C\|/\|Q - C\|$.
   (b) Do you have enough information to compute $\|A - Q\|$? If so, do it; if not, explain.
   (c) Given that $\|Q - C\| = 4$ and $\|P - Q\| = 4$, compute $\|C - A\|$, $\|Q - A\|$, and $\|B - C\|$.
   (d) Given that $\|A - C\| = 8$ and $\|B - C\| = 10$, compute $\|A - Q\|$, $\|Q - C\|$, and $\|P - Q\|$.



4. Suppose that $\overline{PQ} \parallel \overline{BC}$, $\|P - A\| = 2$, and $\|Q - A\| = 3$, as shown at the right.
   (a) Compute these ratios:
      $\|C - A\|/\|B - A\|$,
      $\|P - Q\|/\|B - C\|$
   (b) Given that $\|B - C\| = 6$ and $\|A - C\| = 5$, compute $\|B - A\|$ and $\|Q - P\|$.
   (c) Given that $\|Q - P\| = 4$ and $\|Q - C\| = 9$, compute $\|A - B\|$ and $\|C - B\|$.
5. Consider a line—say, $\overleftrightarrow{AB}$. Let $C = A + (B - A)c$ and $D = A + (B - A)d$, for some nonzero numbers c and d.
   (a) What is $(C - A) : (D - A)$? What is $\overline{AC} : \overline{AD}$?
   (b) Assume that $\|C - A\| = \|D - A\|$. What can you say about c and d? About C and D?
   (c) Assume that both C and D are points of $\overrightarrow{AB}$ and that $\|C - A\| = \|D - A\|$. What can you say about c and d? About C and D?
6. From Exercise 5(c), we see that if the translations $C - A$ and $D - A$ have the same norm and are in the sense of $\overrightarrow{AB}$, then $C = D$

Answers for Part B

1. From the given information, $B - A = (D - C)b$ for some $b \neq 0$. So, $\|B - A\| = \|D - C\|\,|b|$. Now, $\overline{AB} : \overline{CD} = |(B - A) : (D - C)| = |b| = \|B - A\|/\|D - C\|$.
2. (a) $(P - A) : (B - A) = (Q - P) : (C - B)$
   $\Longrightarrow |(P - A) : (B - A)| = |(Q - P) : (C - B)|$
   $\Longrightarrow \overline{AP} : \overline{AB} = \overline{PQ} : \overline{BC}$
   $\Longrightarrow \|(P - A)\|/\|B - A\| = \|Q - P\|/\|C - B\|$
   (b) $\|P - A\|/\|B - A\| = \|Q - P\|/\|C - B\|$
   $\Longrightarrow \|P - A\| \cdot \|C - B\| = \|Q - P\| \cdot \|B - A\|$
   $\Longrightarrow \|P - A\|/\|Q - P\| = \|B - A\|/\|C - B\|$
3. (a) $\|B - P\|/\|B - A\| = 3/5$ $\quad \|P - Q\|/\|B - C\| = 2/5$
   $\|A - Q\|/\|Q - C\| = 2/3$ $\quad \|A - C\|/\|Q - C\| = 5/3$
   (b) No. There are infinitely many triangles which satisfy the given information. Choose any point C' on $\overleftrightarrow{BC}$ which is distinct from B and $\triangle ABC'$ is such a triangle. Let Q' be the point of intersection of the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AC'}$. It is fairly clear that not all of the segments $\overline{AQ'}$ have the same length.
   (c) 20/3, 8/3, 10
   (d) 16/5, 24/5, 4
4. (a) $\|C - A\|/\|B - A\| = 3/2$ [$\|P - A\|/\|B - A\| = \|Q - A\|/\|C - A\|$ so that $\|C - A\|/\|B - A\| = \|Q - A\|/\|P - A\| = 3/2$]
   $\|P - Q\|/\|B - C\|$ is not determined by data.
   (b) 10/3, 18/5 [$\|B - A\|/\|A - C\| = 2/3$ so that $\|B - A\| = \|A - C\| \cdot 2/3 = 5 \cdot 2/3 = 10/3$; $\|Q - P\|/\|B - C\| = 3/5$ so that $\|Q - P\| = \|B - C\| \cdot 3/5 = 6 \cdot 3/5 = 18/5$.]
   (c) 4, 8 [Since $\|Q - C\| = 9$ and $\|Q - A\| = 3$, $\|A - C\| = 6$. Now, $\|A - B\|/\|C - A\| = 2/3$ so that $\|A - B\| = \|C - A\| \cdot 2/3 = 6 \cdot 2/3 = 4$. Since $\|C - B\|/\|P - Q\| = 2$, $\|C - B\| = \|P - Q\| \cdot 2 = 4 \cdot 2 = 8$.]
5. (a) $(C - A) : (D - A) = (B - A)c : (B - A)d = c/d$;
   $\overline{AC} : \overline{AD} = |c/d| = |c|/|d|$
   (b) $c = d$ or $c = -d$, since $|c| = |d|$. $C = D$ when $c = d$ and A is the midpoint of $\overline{CD}$ when $c = -d$.
   (c) $c = d$, for both c and d are positive and $|c| = |d|$. $C = D$, for $c = d$.

or, more conveniently, $C - A = D - A$. This suggests the following theorem about translations:

$$(\|\vec{a}\| = \|\vec{b}\| \text{ and } [\vec{a}]^+ = [\vec{b}]^+) \Longrightarrow \vec{a} = \vec{b}$$

Prove this theorem. [*Hint*: Treat separately the cases $\vec{b} = \vec{0}$ and and $\vec{b} \neq \vec{0}$.]

*

Statements (7)–(11) on pages 35–36 were listed together in Theorem 11–3. Similarly, we list statements (3)–(6) in:

Theorem 11–4 (a) $[\vec{a}]^\perp$ is a proper bidirection $[\vec{a} \neq \vec{0}]$
(b) $[\vec{a}] \cap [\vec{a}]^\perp = \{\vec{0}\}$
(c) $[\vec{a}]^\perp = [\vec{b}]^\perp \Longleftrightarrow [\vec{a}] = [\vec{b}]$
(d) $\vec{b} \in [\vec{a}]^\perp \Longleftrightarrow \vec{a} \in [\vec{b}]^\perp$

[Note that statement (4)—that, for $\vec{a} \neq \vec{0}$, $\vec{a} \notin [\vec{a}]^\perp$—follows from part (b) and the fact that $\vec{a} \in [\vec{a}]$.] The proofs of two parts of Theorem 11–4 should be very simple. [Which two?] The other two parts are more difficult to prove. To make the task easier, we begin by stating several useful results about orthogonal complements and norms in the following two lemmas:

Lemma 1 (a) $[\vec{0}]^\perp = \mathscr{T}$
(b) $\vec{a} - \vec{b}((\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})) \in [\vec{b}]^\perp$
(c) $\{\vec{b}, \vec{c}\} \subseteq [\vec{a}]^\perp \Longrightarrow [\vec{b}, \vec{c}] \subseteq [\vec{a}]^\perp$

Lemma 2
(a) $\vec{u} \cdot \vec{v} = 1 \Longleftrightarrow \vec{u} = \vec{v}$ $[\|\vec{u}\| = 1 = \|\vec{v}\|]$
(b) $\vec{u} \cdot \vec{v} = -1 \Longleftrightarrow \vec{u} = -\vec{v}$ $[\|\vec{u}\| = 1 = \|\vec{v}\|]$

Part C

1. Prove:
   (a) Theorem 11–4(b) [*Hint*: Make use of Definition 11–1(b), Theorem 11–1(c), and the facts that $\vec{0} \in [\vec{a}]$ and $\vec{0} \cdot \vec{a} = 0$.]
   (b) Theorem 11–4(d).
2. Prove Lemma 1. [*Hint*: In part (b) consider the cases $\vec{b} = \vec{0}$, $\vec{b} \neq \vec{0}$.]
3. Prove Lemma 2. [*Hint*: Note that $\vec{u} = \vec{v}$ if and only if $\vec{u} - \vec{v} = \vec{0}$. Now, use parts (c) and (k) of Theorem 11–1.]

Part D

In order to prove Theorem 11–4(a) we shall, first, find a proper bidirection which is a subset of $[\vec{a}]^\perp$. Then, we shall show that any vector which is not in this bidirection is, also, not in $[\vec{a}]^\perp$. [This will tell us

Answers for Part B [cont.]

6. Assume, first, that $\vec{b} = \vec{0}$. Since $\|\vec{a}\| = \|\vec{b}\|$, $\|\vec{a}\| = 0$. Thus, $\vec{a} = \vec{0}$, so that $\vec{a} = \vec{b}$. Next, assume that $\vec{b} \neq \vec{0}$. Then, since $[\vec{a}]^+ = [\vec{b}]^+$, $\vec{a} = \vec{b}a$, for some $a > 0$. So, $\|\vec{a}\| = \|\vec{b}\|a$ and, since $\|\vec{a}\| = \|\vec{b}\|$, $a = 1$. So, in any case, $\vec{a} = \vec{b}$. Hence, the theorem.

Answers for Part C

1. (a) Clearly, $\vec{0} \in [\vec{a}] \cap [\vec{a}]^\perp$, for $\vec{0} = \vec{a}0$ and $\vec{0} \cdot \vec{a} = 0$, for any $\vec{a}$. So, $\{\vec{0}\} \subseteq [\vec{a}] \cap [\vec{a}]^\perp$. Let $\vec{p} \in [\vec{a}] \cap [\vec{a}]^\perp$. Then, $\vec{p} = \vec{a}p$, for some p, and $\vec{p} \cdot \vec{a} = 0$. Thus, $\vec{p} \cdot \vec{p} = \vec{p} \cdot (\vec{a}p) = (\vec{p} \cdot \vec{a})p = 0p = 0$, so that $\vec{p} = \vec{0}$. Hence, if $\vec{p} \in [\vec{a}] \cap [\vec{a}]^\perp$ then $\vec{p} \in \{\vec{0}\}$. That is, $[\vec{a}] \cap [\vec{a}]^\perp \subseteq \{\vec{0}\}$.

   Since we have established that each of $\{\vec{0}\}$ and $[\vec{a}] \cap [\vec{a}]^\perp$ is a subset of the other, Theorem 11-4(b) is proved.

   (b) $\vec{b} \in [\vec{a}]^\perp \Longleftrightarrow \vec{b} \cdot \vec{a} = 0$ [Th. 11-2]
   $\Longleftrightarrow \vec{a} \cdot \vec{b} = 0$ [Post. 4_{14}]
   $\Longleftrightarrow \vec{a} \in [\vec{b}]^\perp$ [Th. 11-2]

2. Proof of Lemma 1.

   (a) [This was proved in Exercise 3 of Part C on page 46.]

   (b) $$\{\vec{a} - \vec{b}((\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b}))\} \cdot \vec{b} = \vec{a} \cdot \vec{b} - \vec{b}((\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})) \cdot \vec{b}$$
   $$= \vec{a} \cdot \vec{b} - (\vec{b} \cdot \vec{b})((\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b}))$$
   $$= \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} = 0 \quad [\text{for } \vec{b} \neq \vec{0}]$$
   $$\{\vec{a} - \vec{0}\,(\vec{a} \cdot \vec{0})/(\vec{0} \cdot \vec{0})\} \cdot \vec{0} = 0$$
   So, $\vec{a} - \vec{b}((\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})) \in [\vec{b}]^\perp$.

   (c) Suppose that both $\vec{b}$ and $\vec{c}$ belong to $[\vec{a}]^\perp$. Then, $\vec{b} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{a} = 0$. Let $\vec{d} \in [\vec{b}, \vec{c}]$. Then, $\vec{d} = \vec{b}b + \vec{c}c$, for some b and c. Since $\vec{d} \cdot \vec{a} = (\vec{b}b + \vec{c}c) \cdot \vec{a} = (\vec{b}b) \cdot \vec{a} + (\vec{c}c) \cdot \vec{a} = (\vec{b} \cdot \vec{a})b + (\vec{c} \cdot \vec{a})c = 0b + 0c = 0$, $\vec{d} \in [\vec{a}]^\perp$. Thus, $[\vec{b}, \vec{c}] \subseteq [\vec{a}]^\perp$. Hence, the theorem.

3. Proof of Lemma 2.

   (a) Suppose that $\|\vec{u}\| = 1 = \|\vec{v}\|$. Then, $\vec{u} \cdot \vec{u} = \vec{v} \cdot \vec{v} = 1$. Now, $\vec{u} = \vec{v}$ if and only if $\vec{u} - \vec{v} = \vec{0}$ and the latter is the case if and only if $(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$. Since $(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2\vec{u} \cdot \vec{v} = 2 - 2\vec{u} \cdot \vec{v}$ [Remember that $\vec{u} \cdot \vec{u} = \vec{v} \cdot \vec{v} = 1$!], and $2 - 2\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u} \cdot \vec{v} = 1$, it follows that $\vec{u} = \vec{v}$ if and only if $\vec{u} \cdot \vec{v} = 1$.

   (b) Substitute '$-\vec{v}$' for '$\vec{v}$' in (a) and use Theorem 11-1(e) and Exercise 1(c) of Part A on page 45.

that the proper bidirection we found *is equal to* $[\vec a]^\perp$, and will prove the theorem.] Suppose that $\vec a \neq \vec 0$, and choose vectors – say, $\vec b$ and $\vec c$ – such that $(\vec a, \vec b, \vec c)$ is linearly independent.

1. How can you be sure that there are vectors such as $\vec b$ and $\vec c$?
2. Given $\vec b$ and $\vec c$, it is easy to find vectors $\vec b'$ and $\vec c'$ which belong to $[\vec a]^\perp$. Describe such vectors. [*Hint*: See Lemma 1.]
3. Show that $(\vec b', \vec c')$ is linearly independent. What does this tell you about $[\vec b', \vec c']$?
4. Show that $[\vec b', \vec c']$ is a proper bidirection which is a subset of $[\vec a]^\perp$.
5. Suppose that $\vec a'$ is some translation which is not in $[\vec b', \vec c']$. What can you say about $(\vec a', \vec b', \vec c')$? About $[\vec a', \vec b', \vec c']$?
6. From your answers for Exercise 5, it should follow that if $\vec a' \notin [\vec b', \vec c']$ then there are numbers – say, a, b, and c – such that $\vec a = \vec a' a + \vec b' b + \vec c' c$. Recalling that $\vec b' b + \vec c' c \in [\vec a]^\perp$, what is $\vec a \cdot \vec a$?
7. What does your answer for Exercise 6 tell you about $\vec a' \cdot \vec a$? [Remember that $\vec a \neq \vec 0$.] What does this tell you, in turn, about $[\vec a]^\perp$?

Part E

To prove Theorem 11–4(c), it is sufficient to show that if $[\vec a] \neq [\vec b]$ then $[\vec a]^\perp \neq [\vec b]^\perp$. [Why?] We shall consider, first, the case in which $\vec a \neq \vec 0 \neq \vec b$. Suppose, then, that $[\vec a] \neq [\vec b]$ and $\vec a \neq \vec 0 \neq \vec b$.

1. What can you say about $(\vec a, \vec b)$?
2. Let $\vec a_1 = \vec a/\|\vec a\|$ and $\vec b_1 = \vec b/\|\vec b\|$. Explain why $(\vec a_1, \vec b_1)$ is linearly independent.
3. Show that $[\vec a_1]^\perp = [\vec a]^\perp$ and $[\vec b_1]^\perp = [\vec b]^\perp$.
4. We wish to show that $[\vec a]^\perp \neq [\vec b]^\perp$. By Exercise 3, it will be sufficient to show that $[\vec a_1]^\perp \neq [\vec b_1]^\perp$ or, equivalently, to find a vector which belongs to $[\vec b_1]^\perp$ but does not belong to $[\vec a_1]^\perp$. Use Lemma 1(b) to find a vector related to $\vec a_1$ which belongs to $[\vec b_1]^\perp$.
5. Use Lemma 2 to show that if $\vec a_1 - \vec b_1(\vec a_1 \cdot \vec b_1) \in [\vec a_1]^\perp$ then $(\vec a_1, \vec b_1)$ is linearly dependent. What, then, can you conclude about $\vec a_1 - \vec b_1(\vec a_1 \cdot \vec b_1)$?
6. Complete the proof of Theorem 11–4(c) in case $\vec a \neq \vec 0 \neq \vec b$.
7. Suppose that $[\vec a] \neq [\vec b]$ and that $\vec a = \vec 0$. What can you say about $\vec b$? About $[\vec a]^\perp$? About $[\vec b]^\perp$? [*Hint*: See Theorem 11–4(a).]
8. Complete the proof of Theorem 11–4(c) in case $\vec a = \vec 0$ or $\vec b = \vec 0$.

Part F

1. (a) Describe – in terms of $\vec c$ and $\vec d$ – a translation $\vec e$ such that $\vec e \in [\vec c]$ and $\vec d - \vec e \in [\vec c]^\perp$. [*Hint*: Look carefully at Lemma 1(b).]
   (b) Now, describe – in terms of $\vec c$ and $\vec d$ – a translation $\vec f$ such that $\vec f \in [\vec d]$ and $\vec c - \vec f \in [\vec d]^\perp$.
2. Suppose that $\{\vec c_1, \vec c_2\} \subseteq [\vec a]$ and $\{\vec b - \vec c_1, \vec b - \vec c_2\} \subseteq [\vec a]^\perp$. Show that $\vec c_1 = \vec c_2$. [*Hint*: Show that $\vec c_1 - \vec c_2$ belongs to both $[\vec a]$ and $[\vec a]^\perp$. Use Theorem 11–4(b).]

Answers for Part D

1. T is 3-dimensional
2. Let $\vec b' = \vec b - \vec a((\vec b \cdot \vec a)/(\vec a \cdot \vec a))$ and $\vec c' = \vec c - \vec a((\vec c \cdot \vec a)/(\vec a \cdot \vec a))$.
3. Using the vectors $\vec b'$ and $\vec c'$ described in Exercise 2, let b and c be such that $\vec b' b + \vec c' c = \vec 0$. Then we have that
$$\{\vec b - \vec a((\vec b \cdot \vec a)/(\vec a \cdot \vec a))\}b + \{\vec c - \vec a((\vec c \cdot \vec a)/(\vec a \cdot \vec a))\}c = \vec 0$$
so that
$$\vec a[-b((\vec b \cdot \vec a)/(\vec a \cdot \vec a)) - c((\vec c \cdot \vec a)/(\vec a \cdot \vec a))] + \vec b b + \vec c c = \vec 0.$$
Since $(\vec a, \vec b, \vec c)$ is linearly independent, it follows from this that $b = 0$ and $c = 0$. So, $(\vec b', \vec c')$ is linearly independent. This tells us that $[\vec b', \vec c']$ is a proper bidirection.
4. This follows directly from Exercises 2 and 3 and Lemma 1(c).
5. Linearly independent [or, a basis for T]; equals T.
6. $\vec a \cdot \vec a = \vec a \cdot (\vec a' a + \vec b' b + \vec c' c) = (\vec a \cdot \vec a')a$, for $\vec a \cdot \vec b' = 0 = \vec a \cdot \vec c'$.
7. $\vec a' \cdot \vec a = (\vec a \cdot \vec a)/a \neq 0$ [$a \neq 0$ since $\vec a \notin [\vec b', \vec c']$]. This tells us that $\vec a'$ is not in $[\vec a]^\perp$, so that $[\vec a]^\perp \subseteq [\vec b', \vec c']$. Also, since we already know, by Exercise 4, that $[\vec b', \vec c'] \subseteq [\vec a]^\perp$, it follows that $[\vec b', \vec c'] = [\vec a]^\perp$.

Answers for Part E

We already know that $[\vec a] = [\vec b] \Longrightarrow [\vec a]^\perp = [\vec b]^\perp$. So, to prove the biconditional Theorem 11-4(c), it is enough to prove that $[\vec a]^\perp = [\vec b]^\perp \Longrightarrow [\vec a] = [\vec b]$; or, we will accomplish the same result if we prove the contrapositive of this last conditional, namely, that $[\vec a] \neq [\vec b] \Longrightarrow [\vec a]^\perp \neq [\vec b]^\perp$.

1. Linearly independent.
2. If $\vec a_1 a + \vec b_1 b = \vec 0$ then $\vec a(a/\|\vec a\|) + \vec b(b/\|\vec b\|) = \vec 0$. Since $(\vec a, \vec b)$ is linearly independent it follows that so is $(\vec a_1, \vec b_1)$.
3. Since $\vec a_1 \in [\vec a]$ and $\vec a_1 \neq \vec 0$, $[\vec a_1] = [\vec a]$. Since $[\vec a]^\perp = [\vec a]^\perp$ it follows, by the replacement rule for equations, that $[\vec a_1]^\perp = [\vec a]^\perp$. Similarly, $[\vec b_1]^\perp = [\vec b]^\perp$.
4. By Lemma 1(b), $\vec a_1 - \vec b_1((\vec a_1 \cdot \vec b_1)/(\vec b_1 \cdot \vec b_1)) \in [\vec b_1]^\perp$. Since $\vec b_1 \cdot \vec b_1 = 1$, it follows that $\vec a_1 - \vec b_1(\vec a_1 \cdot \vec b_1) \in [\vec b_1]^\perp$.
5. Suppose that $\vec a_1 - \vec b_1(\vec a_1 \cdot \vec b_1) \in [\vec a_1]^\perp$. Then, $(\vec a_1 - \vec b_1(\vec a_1 \cdot \vec b_1)) \cdot \vec a_1 = 0$ so that $\vec a_1 \cdot \vec a_1 - (\vec a_1 \cdot \vec b_1)^2 = 0$. Since $\vec a_1 \cdot \vec a_1 = 1$, $(\vec a_1 \cdot \vec b_1)^2 = 1$ so that either $\vec a_1 \cdot \vec b_1 = 1$ or $\vec a_1 \cdot \vec b_1 = -1$. In either case it follows by Lemma 2 that $(\vec a_1, \vec b_1)$ is linearly dependent.

Since $(\vec a_1, \vec b_1)$ is linearly independent, it follows that $\vec a_1 - \vec b_1(\vec a_1 \cdot \vec b_1) \notin [\vec a_1]^\perp$.

TC 50 (7)

Answers for Part E [cont.]

6. What we have shown in Exercises 4 and 5 is that $[\vec{b}_1]^\perp$ contains a vector, namely $\vec{a}_1 - \vec{b}_1(\vec{a}_1 \cdot \vec{b}_1)$, which is not in $[\vec{a}_1]^\perp$. So, $[\vec{a}_1]^\perp \ne [\vec{b}_1]^\perp$. Hence, in the case $\vec{a} \ne \vec{0} \ne \vec{b}$, if $[\vec{a}] \ne [\vec{b}]$ then $[\vec{a}]^\perp \ne [\vec{b}]^\perp$.
7. $\vec{b} \ne \vec{0}$; $[\vec{a}]^\perp = [\vec{0}]^\perp = \mathcal{T}$; $[\vec{b}]^\perp$ is a proper bidirection and, so, is not $\mathcal{T}$.
8. By Exercise 7, it is clear that in case $\vec{a} = \vec{0}$, $[\vec{a}]^\perp \ne [\vec{b}]^\perp$. Similarly, this same result holds in case $\vec{b} = \vec{0}$. Hence, Theorem 11-4(c).

Answers for Part F

1. (a) Let $\vec{e} = \vec{c}((\vec{d} \cdot \vec{c})/(\vec{c} \cdot \vec{c}))$
   (b) Let $\vec{f} = \vec{d}((\vec{c} \cdot \vec{d})/(\vec{d} \cdot \vec{d}))$
2. Since $\{\vec{c}_1, \vec{c}_2\} \subset [\vec{a}]$, $\vec{c}_1 - \vec{c}_2 \in [\vec{a}]$. Since $\{\vec{b} - \vec{c}_1, \vec{b} - \vec{c}_2\} \subset [\vec{a}]^\perp$ it follows by Theorem 11-4(a) [if $\vec{a} \ne \vec{0}$] or by Lemma 1(a) [if $\vec{a} = \vec{0}$] that $(\vec{b} - \vec{c}_2) - (\vec{b} - \vec{c}_1) \in [\vec{a}]^\perp$ — that is, that $\vec{c}_1 - \vec{c}_2 \in [\vec{a}]^\perp$. It follows by Theorem 11-4(b) that $\vec{c}_1 - \vec{c}_2 = \vec{0}$ and, so, that $\vec{c}_1 = \vec{c}_2$.

TC 51 (1)

Several important properties are introduced in the exercises of this section. It is wise to exercise caution relative to exercises assigned for homework. The exercises on page 51 as well as Parts A and B on pages 53-54 lend themselves best to homework. Part C, page 54, should be treated in a group discussion because of the importance of the result in Exercise 1 (d). This result is applied in Exercise 3, page 55, which can be used as a supervised seat exercise. Another important result (Schwarz's Inequality) results from Exercise 5, page 56. Here again it is best to develop the exercises in a group discussion. Part D can be used as homework, but students sometimes have difficulty getting started. You may wish to do the first few exercises as examples.

Answers for Exercises

1. Proof of Theorem 11-5(b). By Definition 11-1(c), $\text{comp}_{\vec{a}}(\vec{b}) = (\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a})$. So, $\text{comp}_{\vec{a}}(\vec{b}) = 0$ if and only if $\vec{b} \cdot \vec{a} = 0$. Now, $\vec{b} \cdot \vec{a} = 0$ if and only if $\vec{b} \in [\vec{a}]^\perp$. Hence, by the replacement rule for biconditionals, $\text{comp}_{\vec{a}}(\vec{b}) = 0$ if and only if $\vec{b} \in [\vec{a}]^\perp$.
2. Proof of Theorem 11-5(c). By Definition 11-1(c), $\text{comp}_{\vec{a}}(\vec{b} + \vec{c}) = ((\vec{b} + \vec{c}) \cdot \vec{a})/(\vec{a} \cdot \vec{a})$. By Postulates 4_{12} and 5, $\text{comp}_{\vec{a}}(\vec{b} + \vec{c}) = (\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a}) + (\vec{c} \cdot \vec{a})/(\vec{a} \cdot \vec{a})$. Hence, by Definition 11-1(c), $\text{comp}_{\vec{a}}(\vec{b} + \vec{c}) = \text{comp}_{\vec{a}}(\vec{b}) + \text{comp}_{\vec{a}}(\vec{c})$.
3. Proof of Theorem 11-5(d). $\text{comp}_{\vec{a}}(\vec{b}b) = ((\vec{b}b) \cdot \vec{a})/(\vec{a} \cdot \vec{a}) = (\vec{b} \cdot \vec{a})b/(\vec{a} \cdot \vec{a}) = [(\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a})]b = \text{comp}_{\vec{a}}(\vec{b})b$.

*

In this section, we have managed to establish that the sentences (5)–(11), about norms and orthogonal complements, are theorems. We have also managed to show, in Exercise 2 of Part F, that there is at most one translation $\vec{x}$ such that $\vec{x} \in [\vec{a}]$ and $\vec{b} - \vec{x} \in [\vec{a}]^\perp$. And, since $\vec{a}((\vec{a} \cdot \vec{b})/(\vec{a} \cdot \vec{a}))$—or, $\vec{a}\,\text{comp}_{\vec{a}}(\vec{b})$—is one such translation, it is the only one. In the next section, we shall make use of this result.

11.10 Components and Projections

Using Definition 11–1(c), the properties of orthogonal components which are summarized on page 38 follow easily from properties of dot multiplication which should, by now, be familiar. For reference, we list some theorems concerning orthogonal components:

Theorem 11–5 For $\vec{a} \ne \vec{0}$,
- (a) $\text{comp}_{\vec{a}}(\vec{b}) \in \mathscr{R}$ and $\vec{b} \in [\vec{a}] \longrightarrow \text{comp}_{\vec{a}}(\vec{b}) = \vec{b} : \vec{a}$,
- (b) $\text{comp}_{\vec{a}}(\vec{b}) = 0 \longleftrightarrow \vec{b} \in [\vec{a}]^\perp$,
- (c) $\text{comp}_{\vec{a}}(\vec{b} + \vec{c}) = \text{comp}_{\vec{a}}(\vec{b}) + \text{comp}_{\vec{a}}(\vec{c})$,
- (d) $\text{comp}_{\vec{a}}(\vec{b}b) = \text{comp}_{\vec{a}}(\vec{b})b$, and
- (e) $\text{comp}_{(\vec{a}a)}(\vec{b}) = \text{comp}_{\vec{a}}(\vec{b})/a \quad [a \ne 0]$.

As a brief review, we sketch proofs for both parts of Theorem 11–5(a).

Suppose that $\vec{a} \ne \vec{0}$. Then, $\text{comp}_{\vec{a}}(\vec{b}) = (\vec{a} \cdot \vec{b})/(\vec{a} \cdot \vec{a})$. Since $\vec{a} \cdot \vec{b}$ and $\vec{a} \cdot \vec{a}$ are real numbers, so is $(\vec{a} \cdot \vec{b})/(\vec{a} \cdot \vec{a})$. So, $\text{comp}_{\vec{a}}(\vec{b}) \in \mathscr{R}$.

Suppose, further, that $\vec{b} \in [\vec{a}]$. Then, $\vec{b} = \vec{a}b$, for some b. So, $\vec{b} : \vec{a} = b$. Now, $\text{comp}_{\vec{a}}(\vec{b}) = (\vec{a} \cdot \vec{b})/(\vec{a} \cdot \vec{a}) = (\vec{a} \cdot \vec{a}b)/(\vec{a} \cdot \vec{a}) = [(\vec{a} \cdot \vec{a})/(\vec{a} \cdot \vec{a})]b = b$. Since $b = \vec{b} : \vec{a}$, $\text{comp}_{\vec{a}}(\vec{b}) = \vec{b} : \vec{a}$. Hence, if $\vec{b} \in [\vec{a}]$ then $\text{comp}_{\vec{a}}(\vec{b}) = \vec{b} : \vec{a}$.

Exercises

Write paragraph proofs for:

1. Theorem 11–5(b)
2. Theorem 11–5(c)
3. Theorem 11–5(d)
4. Theorem 11–5(e)
5. (a) $\vec{b} \in [\vec{a}] \longleftrightarrow \vec{b} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b})$ [*Hint*: Give separate arguments for the cases in which $\vec{a} \ne \vec{0}$ and $\vec{a} = \vec{0}$.]
   (b) $\vec{b} \in [\vec{a}]^+ \longleftrightarrow \text{comp}_{\vec{a}}(\vec{b}) > 0 \quad [\vec{b} \in [\vec{a}] \ne \{\vec{0}\}]$
6. (a) $\vec{b} \in [\vec{a}]^+ \longrightarrow \|\vec{a} + \vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$
   (b) Make a guess about $\|\vec{a} + \vec{b}\|$ and $\|\vec{a}\| + \|\vec{b}\|$ in case $\vec{a}$ and $\vec{b}$ are non-$\vec{0}$ vectors such that $\vec{b} \notin [\vec{a}]^+$.

Answers for Exercises [cont.]

4. Proof of Theorem 11-5(e). Suppose that $a \neq 0$. Then, $\text{comp}_{\vec{a}a}(\vec{b})$ $= (\vec{b}\cdot(\vec{a}a))/(\vec{a}a\cdot\vec{a}a) = ((\vec{b}\cdot\vec{a})a)/((\vec{a}\cdot\vec{a})a^2) = [(\vec{b}\cdot\vec{a})/(\vec{a}\cdot\vec{a})]a/a^2$ $= [(\vec{b}\cdot\vec{a})/(\vec{a}\cdot\vec{a})]/a = \text{comp}_{\vec{a}}(\vec{b})/a$.

5. (a) Assume, first, that $\vec{a} = \vec{0}$. Suppose that $\vec{b} \in [\vec{a}]$. Then, $\vec{b} = \vec{0} = \vec{0}\,\text{comp}_{\vec{a}}(\vec{b}) = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b})$. Thus, $\vec{b} \in [\vec{a}] \Longrightarrow \vec{b} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b})$. Suppose that $\vec{b} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b})$. Then, since $\text{comp}_{\vec{a}}(\vec{b}) = 0 \in \mathcal{R}$, $\vec{b} \in [\vec{a}]$. Thus, $\vec{b} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b}) \Longrightarrow \vec{b} \in [\vec{a}]$. Hence, 5(a) is proved in case $\vec{a} = \vec{0}$.

   Assume, then, that $\vec{a} \neq \vec{0}$. Suppose that $\vec{b} \in [\vec{a}]$. Then, $\vec{b} = \vec{a}b$, for some b. Since $\vec{b}\cdot\vec{a} = (\vec{a}\cdot\vec{a})b$, it follows that $b = (\vec{b}\cdot\vec{a})/(\vec{a}\cdot\vec{a})$. So, $\vec{b} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b})$. Thus, $\vec{b} \in [\vec{a}] \Longrightarrow \vec{b} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b})$. Suppose that $\vec{b} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b})$. Since $\text{comp}_{\vec{a}}(\vec{b}) \in \mathcal{R}$, $\vec{b} \in [\vec{a}]$. Thus, $\vec{b} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b}) \Longrightarrow \vec{b} \in [\vec{a}]$.

   Hence, the theorem holds in any case.

   (b) Suppose that $\vec{b} \in [\vec{a}]^+$. It follows that $\vec{a} \neq \vec{0}$ and that $\vec{b} = \vec{a}b$ for some $b > 0$. By part (a) $\vec{b} = \vec{a}\,\text{comp}_{\vec{a}}(\vec{b})$. So, since $\vec{a} \neq \vec{0}$, $\text{comp}_{\vec{a}}(\vec{b}) = b > 0$. Suppose, on the other hand, that $\text{comp}_{\vec{a}}(\vec{b}) > 0$. It follows at once by part (a) that $\vec{b} \in [\vec{a}]^+$.

6. (a) Suppose that $\vec{b} \in [\vec{a}]^+$. Then $\vec{b} = \vec{a}b$, for some $b > 0$. So, $\|\vec{b}\| = \|\vec{a}\|b$ and

$$\begin{aligned}
\|\vec{a} + \vec{b}\| &= \|\vec{a} + \vec{a}b\| \\
&= \|\vec{a}(1 + b)\| && [\text{Post. } 4] \\
&= \|\vec{a}\|(1 + b) && [1 + b > 0] \\
&= \|\vec{a}\| + \|\vec{a}\|b && [\text{Post. } 5] \\
&= \|\vec{a}\| + \|\vec{b}\|
\end{aligned}$$

   (b) $\|\vec{a} + \vec{b}\| < \|\vec{a}\| + \|\vec{b}\|$

TC 52

Answer to question in text: If there were two such points X, then $\text{proj}_{[\vec{a}]}(\vec{b})$ is not a function. But, we have already seen that, on intuitive grounds, $\text{proj}_{[\vec{a}]}(\vec{b})$ is a translation, and translations are functions.

*

In Section 11.02 we introduced the notion of the orthogonal projection of a vector $\vec{b}$ in the direction of a vector $\vec{a}$. This projection was described as the mapping of $\mathcal{E}$ into itself under which the image of any point P is the projection of $P + \vec{b}$ on the line through P whose direction is $[\vec{a}]$. It turned out that this mapping is, itself, a translation. In Section 11.06 we reworked this description into a form making use of orthogonal complements rather than orthogonal projections of points on lines. Briefly, the image of P under $\text{proj}_{[\vec{a}]}(\vec{b})$ is the point of intersection of the line $\overleftrightarrow{P[\vec{a}]}$ and the plane $\overleftrightarrow{(P + \vec{b})[\vec{a}]^\perp}$. So, our intuitive notion is that

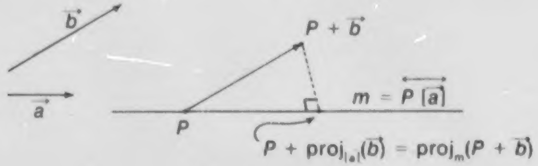

$\vec{b}$
$\vec{a}$
$P + \vec{b}$
$m = \overleftrightarrow{P[\vec{a}]}$
P
$P + \text{proj}_{[\vec{a}]}(\vec{b}) = \text{proj}_m(P + \vec{b})$

Fig. 11–15

$$P + \text{proj}_{[\vec{a}]}(\vec{b}) = Q$$
$$\Longleftrightarrow$$
$$Q - P \in [\vec{a}] \text{ and } (P + \vec{b}) - Q \in [\vec{a}]^\perp.$$

Since $(P + \vec{b}) - Q = \vec{b} - (Q - P)$ it seems reasonable to adopt:

Definition 11–2

$$P + \text{proj}_{[\vec{a}]}(\vec{b}) = Q$$
$$\Longleftrightarrow$$
$$Q - P \in [\vec{a}] \text{ and } \vec{b} - (Q - P) \in [\vec{a}]^\perp$$

But, before we can safely adopt this definition in our formal development, we must make sure that, given $[\vec{a}]$, $\vec{b}$, and P, there is one and only one point X such that

$$X - P \in [\vec{a}] \text{ and } \vec{b} - (X - P) \in [\vec{a}]^\perp.$$

[Why do we need to make certain of this?] Fortunately, we have already seen that there is one and only one translation $\vec{x}$ such that, given $[\vec{a}]$ and $\vec{b}$,

$$(*) \qquad \vec{x} \in [\vec{a}] \text{ and } \vec{b} - \vec{x} \in [\vec{a}]^{\perp},$$

and that this translation is, in fact, $\vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b})$. So, the point X in question is

$$P + \vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b}).$$

It follows that Definition 11 – 2 does define a mapping and that, for any point P,

$$P + \mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = P + \vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b}).$$

So, since each point has the same image under the mapping $\mathrm{proj}_{[\vec{a}]}\,(\vec{b})$ as it does under the translation $\vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b})$ we have:

Theorem 11 – 6 $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b})$

[In view of our intuitive description of components in Section 11.06, this is, of course, what we should have expected. Explain.]

Exercises

Part A

1. (a) We know that $\vec{a}a \in [\vec{a}]$, for any a. What can you say about $\vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b})$? Justify your answer.
   (b) Show that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) \in \mathscr{T}$.
2. (a) Assume that $\vec{c} \in [\vec{a}]$ and $\vec{b} - \vec{c} \in [\vec{a}]^{\perp}$. What can you conclude about $\vec{c}$?
   (b) Give arguments to show that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) \in [\vec{a}]$ and that $\vec{b} - \mathrm{proj}_{[\vec{a}]}\,(\vec{b}) \in [\vec{a}]^{\perp}$.
   (c) Prove: $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{c} \longleftrightarrow (\vec{c} \in [\vec{a}] \text{ and } \vec{b} - \vec{c} \in [\vec{a}]^{\perp})$
3. (a) Describe the translations $\vec{b}$ such that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{0}$. [*Hint*: See Exercise 2(c).]
   (b) Are any of the translations $\vec{b}$ described in part (a) such that $(\vec{a}, \vec{b})$ is linearly independent? Explain.
   (c) Describe the translations $\vec{b}$ such that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{b}$. What can you say about $(\vec{a}, \vec{b})$ in this case?
4. Express $\vec{a} \cdot \mathrm{proj}_{[\vec{a}]}\,(\vec{b})$ in terms of '$\vec{a}$' and '$\vec{b}$'. [*Hint*: Use Definition 11 – 1 and Theorem 11 – 6.]
5. (a) Can you find two nonzero real numbers whose product is zero?
   (b) Describe two non-$\vec{0}$ translations whose dot product is zero.

*

From Theorem 11 – 6 and the fact that $\vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b})$ is the only solution of (*), we obtained two results, which we list as:

Explanation called for in text: In section 11.06, we saw that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) : \vec{a}$ was the orthogonal component of $\vec{b}$ with respect to $\vec{a}$. This gave rise to our intuitive definition '$\mathrm{comp}_{\vec{a}}\,(\vec{b}) = \mathrm{proj}_{[\vec{a}]}\,(\vec{b}) : \vec{a}$' which is equivalent to Theorem 11-6.

Answers for Part A

1. (a) $\vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b}) \in [\vec{a}]$, for $\mathrm{comp}_{\vec{a}}\,(\vec{b}) \in \mathcal{R}$.
   (b) By Theorem 11-6, for $\vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b}) \in \mathcal{T}$.
2. (a) $\vec{c} = \vec{a}\big((\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a})\big) = \vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b})$
   (b) That $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) \in [\vec{a}]$ follows from Theorem 11-6. That $\vec{b} - \mathrm{proj}_{[\vec{a}]}\,(\vec{b}) \in [\vec{a}]^{\perp}$ follows from Exercise 2(a) and Theorem 11-6.
   (c) We have already established that $\vec{c} \in [\vec{a}]$ and $\vec{b} - \vec{c} \in [\vec{a}]^{\perp}$ if and only if $\vec{c} = \vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b})$. Thus, by Theorem 11-6, the theorem in (c) is established.
3. (a) These are all of the translations in $[\vec{a}]^{\perp}$, for by 2(c) we have that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{0} \Longleftrightarrow (\vec{0} \in [\vec{a}] \text{ and } \vec{b} - \vec{0} \in [\vec{a}]^{\perp})$; and since $\vec{0} \in [\vec{a}]$ and $\vec{b} - \vec{0} = \vec{b}$, this is equivalent to saying that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{0} \Longleftrightarrow \vec{b} \in [\vec{a}]^{\perp}$.
   (b) Yes.; All such translations $\vec{b} \neq \vec{0}$, for $\vec{a} \neq \vec{0}$, are such that $(\vec{a}, \vec{b})$ is linearly independent. [Here is a proof of this result: By Exercise 2(c), $\vec{b} \in [\vec{a}]^{\perp}$. Given that $(\vec{a}, \vec{b})$ is linearly dependent, we know that, since $\vec{a} \neq \vec{0}$, $\vec{b} = \vec{a}b$, for some b. Now, $0 = \vec{a} \cdot \vec{b} = \vec{a} \cdot (\vec{a}b) = (\vec{a} \cdot \vec{a})b$, and since $\vec{a} \cdot \vec{a} \neq 0$, $b = 0$. So, $\vec{b} = \vec{a}0 = \vec{0}$. Thus, if $(\vec{a}, \vec{b})$ is linearly dependent, $\vec{b} = \vec{0}$. Since $\vec{b} \neq \vec{0}$, $(\vec{a}, \vec{b})$ is linearly independent.]
   (c) By Exercise 2(c), we have that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{b} \Longleftrightarrow (\vec{b} \in [\vec{a}] \text{ and } \vec{b} - \vec{b} \in [\vec{a}]^{\perp})$. Since $\vec{b} - \vec{b} = \vec{0} \in [\vec{a}]^{\perp}$, for any $\vec{a}$, this is the same as saying that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{b} \Longleftrightarrow \vec{b} \in [\vec{a}]$. So, the translations $\vec{b}$ such that $\mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{b}$ are precisely the translations $\vec{b} \in [\vec{a}]$. In this case, $(\vec{a}, \vec{b})$ is linearly dependent.
4. $\vec{a} \cdot \mathrm{proj}_{[\vec{a}]}\,(\vec{b}) = \vec{a} \cdot \vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b}) = (\vec{a} \cdot \vec{a})\big((\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a})\big) = \vec{b} \cdot \vec{a}$.
5. (a) No, for $ab = 0$ if and only if ($a = 0$ or $b = 0$).
   (b) Given $\vec{a} \neq \vec{0}$, we know that $[\vec{a}]^{\perp}$ is a proper bidirection and that, for any $\vec{b}$, $\vec{b} - \vec{a}\big((\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a})\big) \in [\vec{a}]^{\perp}$. Choosing $\vec{b} \notin [\vec{a}]$, we know that $\vec{a}$ and $\vec{b} - \vec{a}\,\mathrm{comp}_{\vec{a}}\,(\vec{b})$ are non-$\vec{0}$ orthogonal vectors.

Corollary

(a) $\mathrm{proj}_{[\vec{a}]}(\vec{b}) \in \mathcal{T}$

(b) $\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{c} \longleftrightarrow (\vec{c} \in [\vec{a}]$ and $\vec{b} - \vec{c} \in [\vec{a}]^{\perp})$

Part B

Making use of Theorems 11–5 and 11–6, it is easy to prove statements about orthogonal projections which are analogous to the statements about orthogonal components in Theorems 11–5(a)–(d). Prove the following:

1. $\mathrm{proj}_{[\vec{a}]}(\vec{b}) \in [\vec{a}]$
2. $\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{b} \longleftrightarrow \vec{b} \in [\vec{a}]$
3. $\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{0} \longleftrightarrow \vec{b} \in [\vec{a}]^{\perp}$
4. $\mathrm{proj}_{[\vec{a}]}(\vec{b} + \vec{c}) = \mathrm{proj}_{[\vec{a}]}(\vec{b}) + \mathrm{proj}_{[\vec{a}]}(\vec{c})$
5. $\mathrm{proj}_{[\vec{a}]}(\vec{b}b) = \mathrm{proj}_{[\vec{a}]}(\vec{b})b$

*

We collect the results proved in Part B for easy reference in:

Theorem 11–7

(a) $\mathrm{proj}_{[\vec{a}]}(\vec{b}) \in [\vec{a}]$ and $(\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{b} \longleftrightarrow \vec{b} \in [\vec{a}])$

(b) $\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{0} \longleftrightarrow \vec{b} \in [\vec{a}]^{\perp}$

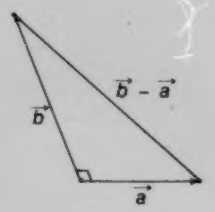
(c) $\mathrm{proj}_{[\vec{a}]}(\vec{b} + \vec{c}) = \mathrm{proj}_{[\vec{a}]}(\vec{b}) + \mathrm{proj}_{[\vec{a}]}(\vec{c})$

(d) $\mathrm{proj}_{[\vec{a}]}(\vec{b}b) = \mathrm{proj}_{[\vec{a}]}(\vec{b})b$

Part C

We are in a position to make use of some of our knowledge of components and projections to learn more about dot products and norms. We begin to do this in these exercises.

1. Suppose that $\vec{b} \in [\vec{a}]^{\perp}$.
    - (a) What does this tell you about $\mathrm{proj}_{[\vec{a}]}(\vec{b})$?
    - (b) What is $\mathrm{proj}_{[\vec{a}]}(\vec{b} - \vec{a})$? What is $\mathrm{proj}_{[\vec{a}]}(\vec{b} + \vec{a})$?
    - (c) Compute the norms of both $\vec{b} - \vec{a}$ and $\vec{b} + \vec{a}$ in terms of $\vec{a}$ and $\vec{b}$. What can you say about these norms?
    - (d) Show that $\|\vec{b} + \vec{a}\|^2 = \|\vec{b}\|^2 + \|\vec{a}\|^2$. What is $\|\vec{b} - \vec{a}\|^2$?
2. Show that $\vec{a} \cdot \vec{b} = 0$ if and only if $\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|$.



Answers for Part B

1. By Theorem 11-6, $\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{a}\,\mathrm{comp}_{\vec{a}}(\vec{b}) \in [\vec{a}]$.
2. By Exercise 2(c) in Part A, $\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{b} \Longleftrightarrow (\vec{b} \in [\vec{a}]$ and $\vec{b} - \vec{b} \in [\vec{a}]^{\perp})$. Since $\vec{b} - \vec{b} = \vec{0} \in [\vec{a}]^{\perp}$, $\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{b} \Longleftrightarrow \vec{b} \in [\vec{a}]$.
3. By Exercise 2(c) in Part A, $\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{0} \Longleftrightarrow (\vec{0} \in [\vec{a}]$ and $\vec{b} - \vec{0} \in [\vec{a}]^{\perp})$. So, $\mathrm{proj}_{[\vec{a}]}(\vec{b}) = \vec{0} \Longleftrightarrow \vec{b} \in [\vec{a}]^{\perp}$.
4. $$\begin{aligned}\mathrm{proj}_{[\vec{a}]}(\vec{b} + \vec{c}) &= \vec{a}\,\mathrm{comp}_{\vec{a}}(\vec{b} + \vec{c}) \\ &= \vec{a}(\mathrm{comp}_{\vec{a}}(\vec{b}) + \mathrm{comp}_{\vec{a}}(\vec{c})) \\ &= \vec{a}\,\mathrm{comp}_{\vec{a}}(\vec{b}) + \vec{a}\,\mathrm{comp}_{\vec{a}}(\vec{c}) \\ &= \mathrm{proj}_{[\vec{a}]}(\vec{b}) + \mathrm{proj}_{[\vec{a}]}(\vec{c})\end{aligned}$$
5. $$\begin{aligned}\mathrm{proj}_{[\vec{a}]}(\vec{b}b) &= \vec{a}\,\mathrm{comp}_{\vec{a}}(\vec{b}b) \\ &= \vec{a}\,\mathrm{comp}_{\vec{a}}(\vec{b})b = \mathrm{proj}_{[\vec{a}]}(\vec{b})b\end{aligned}$$

Answers for Part C

1. (a) $\vec{0}$, by Theorem 11-7(b).
    - (b) $-\vec{a}$, for $\mathrm{proj}_{[\vec{a}]}(\vec{b} - \vec{a}) = \mathrm{proj}_{[\vec{a}]}(\vec{b}) - \mathrm{proj}_{[\vec{a}]}(\vec{a}) = \vec{0} - \vec{a} = -\vec{a}$; $\vec{a}$, for similar reasons.
    - (c) $\|\vec{b} - \vec{a}\| = \sqrt{\vec{b}\cdot\vec{b} + \vec{a}\cdot\vec{a}}$, for $(\vec{b} - \vec{a})\cdot(\vec{b} - \vec{a}) = \vec{b}\cdot\vec{b} + \vec{a}\cdot\vec{a} - 2\vec{a}\cdot\vec{b}$, and $\vec{a}\cdot\vec{b} = 0$; $\|\vec{b} + \vec{a}\| = \sqrt{\vec{b}\cdot\vec{b} + \vec{a}\cdot\vec{a}}$, for similar reasons. Clearly, $\|\vec{b} - \vec{a}\| = \|\vec{b} + \vec{a}\|$.
    - (d) This follows from (c) and the facts that $\vec{b}\cdot\vec{b} = \|\vec{b}\|^2$ and $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$. $\|\vec{b} - \vec{a}\|^2 = \|\vec{b}\|^2 + \|\vec{a}\|^2$.
2. $$\begin{aligned}\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\| &\Longleftrightarrow \|\vec{a} + \vec{b}\|^2 = \|\vec{a} - \vec{b}\|^2 \\ &\Longleftrightarrow \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} + 2\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\vec{a}\cdot\vec{b} \\ &\Longleftrightarrow \vec{a}\cdot\vec{b} = 0\end{aligned}$$

TC 55

3. (a) By the hint, $\|\vec{t}\|^2 = 5^2 + 12^2 = 169$. So, $\|\vec{t}\| = 13$.
    - (b) $\|\vec{t}\|^2 = \|(C + (B - A)) - A\|^2 = \|(C - A) + (B - A)\|^2 = \|C - A\|^2 + \|B - A\|^2 = 5^2 + 4^2 = 41$. So, $\|\vec{t}\| = \sqrt{41}$.
    - (c) $\|\vec{t}\|^2 = \|B - C\|^2 = \|(B - A) + (A - C)\|^2 = \|B - A\|^2 + \|A - C\|^2 = 6^2 + 8^2 = 100$. So, $\|\vec{t}\| = 10$.
    - (d) $\|\vec{t}\|^2 = \|(B - A)2 + (C - A)2\|^2 = \|(B - A)2\|^2 + \|(C - A)2\|^2 = \|B - A\|^2\cdot 4 + \|C - A\|^2\cdot 4 = 6^2\cdot 4 + 3^2\cdot 4 = 45\cdot 4$. So, $\|\vec{t}\| = 6\sqrt{5}$ [or, $\sqrt{180}$].
    - (e) Note that $\vec{t} = (C + (A - B)) - (B + (A - C)) = (A + (C - B)) - (A + (B - C)) = (C - B)2$. Since $C - B = (A - B) + (C - A)$, $\|\vec{t}\|^2 = \|(A - B) + (C - A)\|^2\cdot 4 = (\|A - B\|^2 + \|C - A\|^2)4 = (5^2 + 12^2)4$. So, $\|\vec{t}\| = 26$.

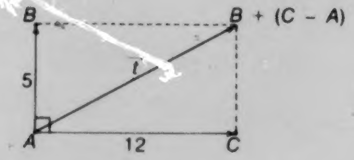
*

Warn students to remember that $\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$ only if $\vec{a}\cdot$

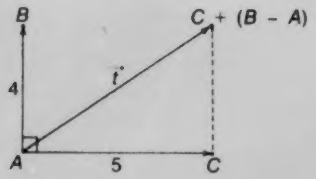
3. In each of the following, you are given the norms of two orthogonal translations, $B - A$ and $C - A$. Compute the norm of $\vec{t}$, which is a translation related to either the sum or the difference of $B - A$ and $C - A$.

[*Hint*: Note that $\vec{t} = (B - A) + (C - A)$. Since $B - A$ and $C - A$ are orthogonal, it follows from 1(d) that $\|\vec{t}\|^2 = \|B - A\|^2 + \|C - A\|^2 = 5^2 + \ldots$]

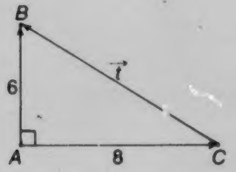
(a)



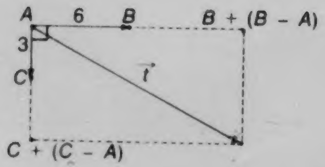
(b)



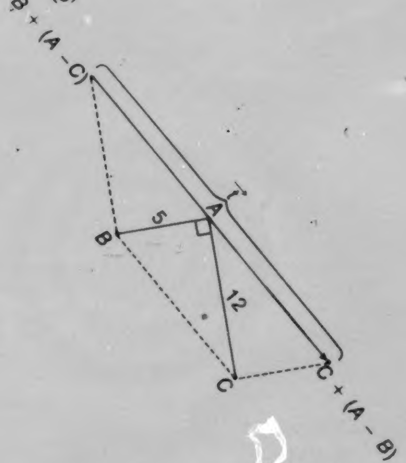
(c)



(d)



(e)



Answers for Part C [cont.]

4. (a) $\|\text{proj}_{[\vec{a}]}(\vec{b})\| = \|\vec{a}((\vec{b}\cdot\vec{a})/(\vec{a}\cdot\vec{a}))\|$

$= \|\vec{a}\|\cdot|(\vec{b}\cdot\vec{a})/(\vec{a}\cdot\vec{a})|$

$= \|\vec{a}\|\cdot|\vec{b}\cdot\vec{a}|\cdot/\|\vec{a}\|^2$

$= |\vec{b}\cdot\vec{a}|/\|\vec{a}\|$

(b) $\|\vec{b} - \text{proj}_{[\vec{a}]}(\vec{b})\|^2 = \vec{b}\cdot\vec{b} + \text{proj}_{[\vec{a}]}(\vec{b})\cdot\text{proj}_{[\vec{a}]}(\vec{b}) - 2\vec{b}\cdot\text{proj}_{[\vec{a}]}(\vec{b})$

$= \vec{b}\cdot\vec{b} + (\vec{b}\cdot\vec{a})^2/(\vec{a}\cdot\vec{a}) - 2(\vec{b}\cdot\vec{a})^2/(\vec{a}\cdot\vec{a})$

$= \vec{b}\cdot\vec{b} - (\vec{b}\cdot\vec{a})^2/(\vec{a}\cdot\vec{a})$

So, $\|\vec{b} - \text{proj}_{[\vec{a}]}(\vec{b})\| = \sqrt{\vec{b}\cdot\vec{b} - (\vec{b}\cdot\vec{a})^2/(\vec{a}\cdot\vec{a})}$.

(c) By (b), and the fact that $\|\vec{a}\|^2 \geq 0$, we have that $\vec{b}\cdot\vec{b} - (\vec{b}\cdot\vec{a})^2/(\vec{a}\cdot\vec{a}) \geq 0$.

(d) Since $\vec{b}\cdot\vec{b} = \|\vec{b}\|^2$ and $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$, it follows from (c) that $\|\vec{b}\|^2 - (\vec{b}\cdot\vec{a})^2/\|\vec{a}\|^2 \geq 0$. Since $\vec{a} \neq \vec{0}$, we have that $\|\vec{a}\|^2\|\vec{b}\|^2 \geq (\vec{b}\cdot\vec{a})^2$. This last result holds in case $\vec{a} = \vec{0}$, for then $\|\vec{a}\| = 0$ and $\vec{b}\cdot\vec{a} = 0$.

5. (a) Using 4(d), we know that $(\|\vec{a}\|\,\|\vec{b}\|)^2 \geq (\vec{b}\cdot\vec{a})^2$. Since both $\|\vec{a}\|$ and $\|\vec{b}\|$ are nonnegative and $(\vec{b}\cdot\vec{a})^2 = |\vec{b}\cdot\vec{a}|^2$, it follows that $\|\vec{a}\|\,\|\vec{b}\| \geq |\vec{b}\cdot\vec{a}|$.

(b) Suppose that $\vec{a} \neq \vec{0}$. Then, by Exercises 4(a) and 5(a), we have that

$$\begin{aligned}\|\text{proj}_{[\vec{a}]}(\vec{b})\| &= |\vec{b}\cdot\vec{a}|/\|\vec{a}\| \\ &\leq (\|\vec{a}\|\,\|\vec{b}\|)/\|\vec{a}\| \\ &= \|\vec{b}\|.\end{aligned}$$

So, for $\vec{a} \neq \vec{0}$, $\|\text{proj}_{[\vec{a}]}(\vec{b})\| \leq \|\vec{b}\|$. [This also holds if $\vec{a} = \vec{0}$.]

(c) $\vec{b} \in [\vec{a}] \Longleftrightarrow \text{proj}_{[\vec{a}]}(\vec{b}) = \vec{b}$ [Theorem 11-7(a)]

$\Longleftrightarrow \|\vec{b} - \text{proj}_{[\vec{a}]}(\vec{b})\| = 0$

$\Longleftrightarrow \|\vec{b}\|^2 = (\vec{b}\cdot\vec{a})^2/\|\vec{a}\|^2$ [Exercise 4(b)]

$\Longleftrightarrow \|\vec{a}\|^2\,\|\vec{b}\|^2 = (\vec{b}\cdot\vec{a})^2$ $[\vec{a} \neq \vec{0}]$

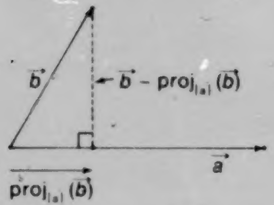
$\Longleftrightarrow \|\vec{a}\|\,\|\vec{b}\| = |\vec{a}\cdot\vec{b}|$

$\vec{b} \in [\vec{a}] \Longleftrightarrow (\vec{a}, \vec{b})$ is linearly dependent $[\vec{a} \neq \vec{0}]$

So, for $\vec{a} \neq \vec{0}$, $(\vec{a}, \vec{b})$ is linearly dependent if and only if $\|\vec{a}\|\,\|\vec{b}\| = |\vec{a}\cdot\vec{b}|$.

For $\vec{a} = \vec{0}$, $\|\vec{a}\|\,\|\vec{b}\| = 0 = |\vec{a}\cdot\vec{b}|$ and, also, $(\vec{a}, \vec{b})$ is linearly dependent. So, for $\vec{a} = \vec{0}$, $(\vec{a}, \vec{b})$ is linearly dependent if and only if $\|\vec{a}\|\,\|\vec{b}\| = |\vec{a}\cdot\vec{b}|$.

4. Suppose that $\vec{a}$ and $\vec{b}$ are translations $[\vec{a} \neq \vec{0}]$, as shown at the right. By Definition 11–1 and Theorem 11–6 we know that $\text{proj}_{[\vec{a}]}(\vec{b}) = \vec{a}[(\vec{a} \cdot \vec{b})/(\vec{a} \cdot \vec{a})]$.


$\vec{b}$
$\leftarrow \vec{b} - \text{proj}_{[\vec{a}]}(\vec{b})$
$\vec{a}$
$\text{proj}_{[\vec{a}]}(\vec{b})$

   (a) Compute the norm of $\text{proj}_{[\vec{a}]}(\vec{b})$ in terms of '$\vec{a}$' and '$\vec{b}$'.
   (b) Compute the norm of $\vec{b} - \text{proj}_{[\vec{a}]}(\vec{b})$ in terms of '$\vec{a}$' and '$\vec{b}$'.
   (c) Use the result in part (b) and the fact that norms of translations are nonnegative to show that
   $$\vec{b} \cdot \vec{b} - (\vec{a} \cdot \vec{b})^2/(\vec{a} \cdot \vec{a}) \geq 0.$$
   (d) Show that $\|\vec{a}\|^2\|\vec{b}\|^2 \geq (\vec{a} \cdot \vec{b})^2$. Does this still hold in case $\vec{a} = \vec{0}$?

5. Prove:
   (a) $\|\vec{a}\|\,\|\vec{b}\| \geq |\vec{a} \cdot \vec{b}|$ [*Hint*: Use the result in Exercise 4(d). Which theorems about square roots are needed?]
   (b) $\|\text{proj}_{[\vec{a}]}(\vec{b})\| \leq \|\vec{b}\|$ $[\vec{a} \neq \vec{0}]$ [*Hint*: Recall Exercise 4(a).]
   (c) $\|\vec{a}\|\,\|\vec{b}\| = |\vec{a} \cdot \vec{b}| \longleftrightarrow (\vec{a}, \vec{b})$ is linearly dependent [*Hint*: $\|\vec{a}\|\,\|\vec{b}\| = |\vec{a} \cdot \vec{b}|$ if and only if $\|\vec{a}\|^2\|\vec{b}\|^2 = (\vec{a} \cdot \vec{b})^2$. Explain. Now, make use of Exercise 4(b) and Theorem 11–7(a).]
   (d)

Theorem 11–8
(a) $\|\vec{a}\|\,\|\vec{b}\| \geq |\vec{a} \cdot \vec{b}|$
(b) $(\vec{a}, \vec{b})$ is linearly dependent if and only if $\|\vec{a}\|\,\|\vec{b}\| = |\vec{a} \cdot \vec{b}|$

[Theorem 11–8 is called *Schwarz's Inequality*. It follows almost at once from earlier parts of this exercise.]

6. Suppose that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent. Since $\mathscr{T}$ is 3-dimensional it follows that any translation is a linear combination of $\vec{a}, \vec{b}$, and $\vec{c}$.
   (a) Suppose that $\vec{p} \in [\vec{a}]$ and that $\vec{p} \cdot \vec{a} = 0$. What can you conclude concerning $\vec{p}$? Justify your answer.
   (b) Suppose that $\vec{p} = \vec{a}a + \vec{b}b$ and that $\vec{p} \cdot \vec{a} = 0$ and $\vec{p} \cdot \vec{b} = 0$. What do you suspect concerning $\vec{p}$? Show that $\vec{p} \cdot \vec{p} = 0$ by simplifying '$\vec{p} \cdot (\vec{a}a + \vec{b}b)$'.
   (c) Suppose that $\vec{p}$ is any translation such that $\vec{p} \cdot \vec{a} = \vec{p} \cdot \vec{b} = \vec{p} \cdot \vec{c} = 0$. Show that $\vec{p} = \vec{0}$.

Part D

Consider the equation:

$$(*) \qquad \vec{r} \cdot \vec{a} = c,$$

where $\vec{a} \neq \vec{0}$ and $c \in \mathscr{R}$.

1. Find a translation $\vec{r} \in [\vec{a}]$ which satisfies (*). Is there more than one such translation?

The rather trivial case of the preceding proof in which $\vec{a} = \vec{0}$ makes use of a rather trivial type of inference:

$$\frac{p \qquad q}{p \Longleftrightarrow q}$$

This can be justified by using two applications of conditionalizing:

$$\cfrac{\cfrac{p}{q \Longrightarrow p} \qquad \cfrac{q}{p \Longrightarrow q}}{p \Longleftrightarrow q}$$

*

Answers for Part C [cont.]

5. (d) The inequality is proved in part (a) and the equality condition is established in part (d).

*

Schwarz's Inequality [also called The Cauchy-Schwarz Inequality] is one of the most basic theorems concerning inner product spaces. We shall make much use of it and shall suggest other proofs for it.

*

6. (a) $\vec{p} = \vec{0}$, for $\vec{p} \in [\vec{a}]$ and $\vec{p} \in [\vec{a}]^\perp$ and $[\vec{a}] \cap [\vec{a}]^\perp = \{\vec{0}\}$.
   (b) $\vec{p} \cdot (\vec{a}a + \vec{b}b) = \vec{p} \cdot \vec{a}a + \vec{p} \cdot \vec{b}b = (\vec{p} \cdot \vec{a})a + (\vec{p} \cdot \vec{b})b = 0a + 0b = 0$.
   (c) Since $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent, $\vec{p} = \vec{a}a + \vec{b}b + \vec{c}c$, for some a, b, and c. So, $\vec{p} \cdot \vec{p} = \vec{p} \cdot (\vec{a}a + \vec{b}b + \vec{c}c) = \vec{p} \cdot (\vec{a}a + \vec{b}b) + \vec{p} \cdot \vec{c}c = 0 + 0c = 0$. Thus, by Theorem 11-1(c), $\vec{p} = \vec{0}$.

Answers for Part D

1. $\vec{a}c/(\vec{a} \cdot \vec{a})$. Suppose that $\vec{a} \neq \vec{0}$, $\vec{r} \in [\vec{a}]$, and that $\vec{r} \cdot \vec{a} = c$. Then $\vec{r} = \vec{a}r$ for some r, so that $(\vec{a}r) \cdot \vec{a} = c$. Since $\vec{a} \neq \vec{0}$, $\vec{a} \cdot \vec{a} \neq 0$ so that $r = c/(\vec{a} \cdot \vec{a})$. So, $\vec{r} = \vec{a}c/(\vec{a} \cdot \vec{a})$. This translation is unique.

TC 57 (1)

2. $\vec{r}_2 - \vec{r}_1 \in [\vec{a}]^\perp$, for $(\vec{r}_2 - \vec{r}_1) \cdot \vec{a} = \vec{r}_2 \cdot \vec{a} - \vec{r}_1 \cdot \vec{a} = c - c = 0$.
3. Suppose that $\vec{p} = (\vec{a}c)/(\vec{a} \cdot \vec{a}) + \vec{t}$, for some $\vec{t} \in [\vec{a}]^\perp$. Then,
$$\begin{aligned} \vec{p} \cdot \vec{a} &= \big((\vec{a}c)/(\vec{a} \cdot \vec{a}) + \vec{t}\big) \cdot \vec{a} \\ &= \big((\vec{a}c)/(\vec{a} \cdot \vec{a})\big) \cdot \vec{a} + \vec{t} \cdot \vec{a} \\ &= (\vec{a} \cdot \vec{a})c/(\vec{a} \cdot \vec{a}) + \vec{t} \cdot \vec{a} \\ &= c + 0 \\ &= c \end{aligned}$$
So, if $\vec{p} = (\vec{a}c)/(\vec{a} \cdot \vec{a}) + \vec{t}$ for some $\vec{t} \in [\vec{a}]^\perp$ then $\vec{p} \cdot \vec{a} = c$ — that is, then $\vec{p}$ is a solution of (*).

Suppose, next, that $\vec{p}$ is a solution of (*) — that is, that $\vec{p} \cdot \vec{a} = c$. Then, for any $\vec{q}$ such that $\vec{q} \cdot \vec{a} = c$, it follows that $\vec{p} - \vec{q} \in [\vec{a}]^\perp$. In particular, for $\vec{q} = (\vec{a}c)/(\vec{a} \cdot \vec{a})$ as in Exercise 1,

2. Suppose that $\vec{r}_1$ and $\vec{r}_2$ both satisfy (*). What can you say about $\vec{r}_2 - \vec{r}_1$? [*Hint*: What is $(\vec{r}_2 - \vec{r}_1) \cdot \vec{a}$?]
3. Show that the solutions of (*) are just the translations

$$\vec{a}c)/(\vec{a} \cdot \vec{a}) + \vec{t}, \text{ where } \vec{t} \in [\vec{a}]^{\perp}.$$

[*Hint*: It should follow from your answers for Exercises 1 and 2 that each solution of (*) is such a translation. And, it's easy to show that each such translation is a solution of (*).]

4. Draw a line $\overleftrightarrow{O[\vec{a}]}$ and mark on it a point P. Assume that $P = O + (\vec{a}c)/(\vec{a} \cdot \vec{a})$. By Exercise 1, $P - O$ satisfies (*). If $R - O$ satisfies (*), what do you know about $R - P$? What does this tell you about the location of R?
5. Describe, in terms of 'P' and '$[\vec{a}]$', the set of all points which satisfy the equation:

$$(**) \qquad (R - O) \cdot \vec{a} = c$$

and picture this set in the figure you drew for Exercise 4.

6. (a) If r, s, and a are real numbers such that $a \neq 0$ and $ra = sa$, does it follow that $r = s$?
   (b) If $\vec{r}$, $\vec{s}$, and $\vec{a}$ are translations such that $\vec{a} \neq \vec{0}$ and $\vec{r} \cdot \vec{a} = \vec{s} \cdot \vec{a}$, does it follow that $\vec{r} = \vec{s}$?
7. In Exercise 5 you should have found that the points, R, which satisfy (**) are just those of the plane through P perpendicular to $\overleftrightarrow{O[\vec{a}]}$. Suppose, now, that $\vec{b}$ is a translation such that $(\vec{a}, \vec{b})$ is linearly independent and that R satisfies both (**) and the equation:

$$(R - O) \cdot \vec{b} = d$$

Describe the location of R.

8. (a) Suppose that $(\vec{a}, \vec{b})$ is linearly independent and that $\vec{r} \cdot \vec{a} = \vec{s} \cdot \vec{a}$ and $\vec{r} \cdot \vec{b} = \vec{s} \cdot \vec{b}$. Does it follow that $\vec{r} = \vec{s}$?
   (b) Suppose, in addition, that $\vec{r} - \vec{s} \in [\vec{a}, \vec{b}]$. Does it now follow that $\vec{r} = \vec{s}$? [*Hint*: See Exercise 6(b) of Part C.]
9. Suppose that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent and that $\vec{r} \cdot \vec{a} = \vec{s} \cdot \vec{a}$, $\vec{r} \cdot \vec{b} = \vec{s} \cdot \vec{b}$, and $\vec{r} \cdot \vec{c} = \vec{s} \cdot \vec{c}$. Does it follow that $\vec{r} = \vec{s}$?

11.11 Bases

You learned in Chapter 10 that a *basis* for a vector space is a linearly independent sequence of vectors which spans the space. You also learned that, because $\mathscr{T}$ is 3-dimensional, any 3-termed linearly independent sequence of translations is a basis for $\mathscr{T}$. Analogously, since $\mathscr{R}$, when viewed as a vector space, is 1-dimensional, any 1-

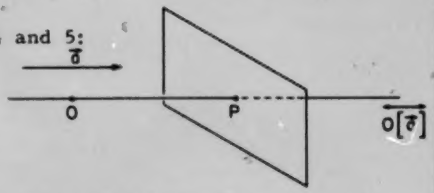
Answers for Part D [cont.]

we have that

$$(**) \qquad \vec{p} - (\vec{a}c)/(\vec{a} \cdot \vec{a}) \in [\vec{a}]^{\perp}$$

so that each solution $\vec{p}$ of (*) must be a solution of (**). Hence, $\vec{p} = (\vec{a}c)/(\vec{a} \cdot \vec{a}) + \vec{t}$, where $\vec{t} \in [\vec{a}]^{\perp}$. So, if $\vec{p}$ is a solution of (*) then $\vec{p} = (\vec{a}c)/(\vec{a} \cdot \vec{a}) + \vec{t}$ for some $\vec{t} \in [\vec{a}]^{\perp}$.

4. Here is a picture for Exercises 4 and 5:



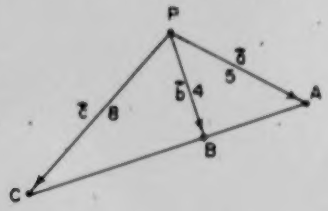
Suppose that R - O satisfies (*). Then $(R - O) \cdot \vec{a} = c$. Since P - O satisfies (*) and (R - O) - (P - O) = R - P, it follows by Exercise 2, that $R - P \in [\vec{a}]^{\perp}$. This tells us that $R \in \overleftrightarrow{P[\vec{a}]^{\perp}}$.

5. The set of all points R which satisfy the equation '$(R - O) \cdot \vec{a} = c$' consists of the points in the plane $\overleftrightarrow{P[\vec{a}]^{\perp}}$, where P and O are the points described in Exercise 4. This is the plane which contains P and is perpendicular to $\overleftrightarrow{O[\vec{a}]}$.
6. (a) Yes, for then $(r - s)a = 0$ in which case either $r - s = 0$ or $a = 0$. Since $a \neq 0$, it is the case that $r - s = 0$, so that $r = s$.
   (b) No. It is the case, however, that $\vec{r} - \vec{s} \in [\vec{a}]^{\perp}$.
7. To satisfy both (**) and '$(R - O) \cdot \vec{b} = d$', R must be on both the plane through P and perpendicular to $\overleftrightarrow{O[\vec{a}]}$ and the plane through T perpendicular to $\overleftrightarrow{O[\vec{b}]}$, where $T = O + (\vec{b}d)/(\vec{b} \cdot \vec{b})$. Since $(\vec{a}, \vec{b})$ is linearly independent, the planes just described intersect in a line and R is on this line of intersection.
8. (a) No. By Exercise 2, $\vec{r} - \vec{s} \in [\vec{a}]^{\perp}$ and $\vec{r} - \vec{s} \in [\vec{b}]^{\perp}$. Since both $[\vec{a}]^{\perp}$ and $[\vec{b}]^{\perp}$ are proper bidirections and $[\vec{a}]^{\perp} \neq [\vec{b}]^{\perp}$, it follows that $[\vec{a}]^{\perp} \cap [\vec{b}]^{\perp}$ is a proper direction. $\vec{r} - \vec{s}$ may be any member of this direction and so, $\vec{r}$ need not be $\vec{s}$.
   (b) Yes. Since $\vec{r} - \vec{s} \in [\vec{a}]^{\perp} \cap [\vec{b}]^{\perp}$, $(\vec{r} - \vec{s}) \cdot \vec{a} = 0$ and $(\vec{r} - \vec{s}) \cdot \vec{b} = 0$. Now, suppose that $\vec{r} - \vec{s} \in [\vec{a}, \vec{b}]$. Then, $\vec{r} - \vec{s} = \vec{a}a + \vec{b}b$, for some a and b. So, $(\vec{r} - \vec{s}) \cdot (\vec{r} - \vec{s}) = (\vec{r} - \vec{s}) \cdot (\vec{a}a + \vec{b}b) = (\vec{r} - \vec{s}) \cdot \vec{a}a + (\vec{r} - \vec{s}) \cdot \vec{b}b = 0$. Thus, by Theorem 11-1(c), $\vec{r} - \vec{s} = \vec{0}$. That is, $\vec{r} = \vec{s}$.
9. Yes, for $\vec{r} - \vec{s} \in [\vec{a}, \vec{b}, \vec{c}]$ so that $\vec{r} - \vec{s} = \vec{a}a + \vec{b}b + \vec{c}c$, for some a, b, and c. Thus, $(\vec{r} - \vec{s}) \cdot (\vec{r} - \vec{s}) = (\vec{r} - \vec{s}) \cdot (\vec{a}a + \vec{b}b + \vec{c}c) = (\vec{r} - \vec{s}) \cdot (\vec{a}a + \vec{b}b) + (\vec{r} - \vec{s}) \cdot \vec{c}c = 0$, so that $\vec{r} - \vec{s} = \vec{0}$. That is, $\vec{r} = \vec{s}$.

Sample Quiz

1. Show that if $||\vec{a}|| = ||\vec{b}||$ then $(\vec{a} + \vec{b} + \vec{c})\cdot(\vec{a} - \vec{b}) = \vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c}$.

2. In the picture at the right, $\vec{a}$, $\vec{b}$, and $\vec{c}$ are the position vectors, with respect to P, of collinear points A, B, and C, respectively, their norms are 5, 4, and 8 [as indicated], and $C - A \in [\vec{b}]^{\perp}$.



   (a) What is $\text{proj}_{[\vec{b}]}(\vec{c})$? What is $\text{proj}_{[\vec{b}]}(\vec{a})$?

   (b) Find $\vec{a}\cdot\vec{b}$ and $\vec{c}\cdot\vec{b}$.

   (c) Express each of A - B and C - B as linear combinations of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

   (d) Find $||A - B||$ and $||C - B||$.

3. Show that, for any non-$\vec{0}$ vector $\vec{b}$, $\vec{a} - \vec{b}\left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right) \in [\vec{b}]^{\perp}$.

Answers for Sample Quiz

1. Suppose that $||\vec{a}|| = ||\vec{b}||$. Then $\vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b}$. So, $(\vec{a} + \vec{b} + \vec{c})\cdot(\vec{a} - \vec{b})$
$= (\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) + \vec{c}\cdot(\vec{a} - \vec{b}) = (\vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{b}) + (\vec{c}\cdot\vec{a} - \vec{c}\cdot\vec{b})$
$= 0 + (\vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c}) = \vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c}$. Hence, if $||\vec{a}|| = ||\vec{b}||$ then
$(\vec{a} + \vec{b} + \vec{c})\cdot(\vec{a} - \vec{b}) = \vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c}$.

2. (a) $\text{proj}_{[\vec{b}]}(\vec{c}) = \vec{b}$; $\text{proj}_{[\vec{b}]}(\vec{a}) = \vec{b}$

   (b) $\vec{a}\cdot\vec{b} = 16$ [for $\text{comp}_{\vec{b}}(\vec{a}) = (\vec{a}\cdot\vec{b})/(\vec{b}\cdot\vec{b})$ and $\text{comp}_{\vec{b}}(\vec{a}) = 1$ and $\vec{b}\cdot\vec{b} = 16$]; $\vec{c}\cdot\vec{b} = 16$ [for $\vec{c}\cdot\vec{b} = (\vec{b}\cdot\vec{b})\,\text{comp}_{\vec{b}}(\vec{c}) = 16\cdot 1 = 16$]

   (c) $A - B = \vec{a} - \vec{b}$; $C - P = \vec{c} - \vec{b}$

   (d) $||A - B|| = 3$ [for $||A - B||^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b}) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\vec{a}\cdot\vec{b} = 25 + 16 - 2\cdot 16$];
   $||C - B|| = 4\sqrt{3}$ [for $||C - B||^2 = (\vec{c} - \vec{b})\cdot(\vec{c} - \vec{b}) = \vec{c}\cdot\vec{c} + \vec{b}\cdot\vec{b} - 2\vec{c}\cdot\vec{b} = 64 + 16 - 2\cdot 16$]

3. Note that $\vec{b}\cdot\left(\vec{a} - \vec{b}\left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\right) = \vec{b}\cdot\vec{a} - (\vec{b}\cdot\vec{b})\left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right) = 0$, for $\vec{b} \neq \vec{0}$. So, for any non-$\vec{0}$ vector $\vec{b}$, $\vec{a} - \vec{b}\left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right) \in [\vec{b}]^{\perp}$.

---

termed linearly independent sequence of real numbers is a basis for $\mathscr{R}$. What this last means is that if a is any nonzero real number then each real number is a multiple of a. And, the linear independence of the 1-termed sequence (a) in the case that $a \neq 0$ implies that

$$(*) \qquad \text{if } ar = as \text{ then } r = s.$$

In $\mathscr{T}$ we have two operations analogous to multiplication of real numbers. The first one we studied is multiplication of a translation by a real number. For this operation we were able to prove a statement very much like (*):

$$\overrightarrow{ar} = \overrightarrow{as} \longrightarrow r = s \qquad [\overrightarrow{a} \neq \overrightarrow{0}]$$

The second analogue of multiplication of real numbers is dot multiplication of a translation by a translation. For this operation the simplest statement analogous to (*) is false. In fact, given $\overrightarrow{a} \neq \overrightarrow{0}$, it is not difficult to find translations $\overrightarrow{r}$ and $\overrightarrow{s}$ such that

$$\overrightarrow{a} \cdot \overrightarrow{r} = \overrightarrow{a} \cdot \overrightarrow{s}, \text{ but } \overrightarrow{r} \neq \overrightarrow{s}.$$

[Find two such translations $\overrightarrow{r}$ and $\overrightarrow{s}$, given that $\overrightarrow{a} \neq \overrightarrow{0}$. You have done this several times in earlier exercises.] In spite of this, it is still possible that there is a theorem about dot multiplication which is analogous to (*). The reason for this possibility is that the restriction '$a \neq 0$' has two analogues. One is '$\overrightarrow{a} \neq \overrightarrow{0}$'. The other is '$(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ is a basis for $\mathscr{T}$' [or, equivalently, since $\mathscr{T}$ is 3-dimensional, '$(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ is linearly independent'].

To get a theorem about $\mathscr{T}$ which is analogous to (*) we first state as a lemma a result which was established in Exercise 6(c) on page 56.

> **Lemma** If $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ is linearly independent then $\overrightarrow{a} \cdot \overrightarrow{r} = \overrightarrow{b} \cdot \overrightarrow{r} = \overrightarrow{c} \cdot \overrightarrow{r} = 0 \longrightarrow \overrightarrow{r} = \overrightarrow{0}$.

For the record, we sketch the proof of this lemma. Given that $(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c})$ is a basis for $\mathscr{T}$, there are numbers—say, a, b, and c—such that $\overrightarrow{r} = \overrightarrow{a}a + \overrightarrow{b}b + \overrightarrow{c}c$. It follows that

$$\begin{aligned}
\overrightarrow{r} \cdot \overrightarrow{r} &= (\overrightarrow{a}a + \overrightarrow{b}b + \overrightarrow{c}c) \cdot \overrightarrow{r} && \text{[Why?]}\\
&= (\overrightarrow{a}a) \cdot \overrightarrow{r} + (\overrightarrow{b}b) \cdot \overrightarrow{r} + (\overrightarrow{c}c) \cdot \overrightarrow{r} && \text{[Why?]}\\
&= (\overrightarrow{a} \cdot \overrightarrow{r})a + (\overrightarrow{b} \cdot \overrightarrow{r})b + (\overrightarrow{c} \cdot \overrightarrow{r})c. && \text{[Why?]}
\end{aligned}$$

So, given that $\overrightarrow{a} \cdot \overrightarrow{r}$, $\overrightarrow{b} \cdot \overrightarrow{r}$, and $\overrightarrow{c} \cdot \overrightarrow{r}$ are all 0, it follows that $\overrightarrow{r} \cdot \overrightarrow{r} = 0$. Thus, $\overrightarrow{r} = \overrightarrow{0}$. [Explain.]

We are now in a position to state the following theorem about $\mathscr{T}$ which is analogous to (*):

> Theorem 11–9 If $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent then $(\vec{a} \cdot \vec{r} = \vec{a} \cdot \vec{s}, \vec{b} \cdot \vec{r} = \vec{b} \cdot \vec{s}$, and $\vec{c} \cdot \vec{r} = \vec{c} \cdot \vec{s}) \longrightarrow \vec{r} = \vec{s}$.

The proof is left as an exercise.

Exercises

1. Answer the questions in the proof of the lemma given above.
2. Prove Theorem 11–9. [*Hint*: Note that $\vec{a} \cdot \vec{r} = \vec{a} \cdot \vec{s}$ implies that $\vec{a} \cdot (\vec{r} - \vec{s}) = 0$.]
3. Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are non-$\vec{0}$ translations and that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$. Show that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent. [*Hint*: Suppose that $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$. Dot multiply on both sides with '$\vec{a}$'.]
4. Let $\vec{r} = \vec{a}a + \vec{b}b + \vec{c}c$, where $\vec{a}$, $\vec{b}$, and $\vec{c}$ are non-$\vec{0}$ translations as described in Exercise 3.
   - (a) Show that $a = (\vec{r} \cdot \vec{a})/(\vec{a} \cdot \vec{a})$. Obtain similar results for 'b' and 'c'.
   - (b) Show that $\vec{r} = \text{proj}_{[\vec{a}]}(\vec{r}) + \text{proj}_{[\vec{b}]}(\vec{r}) + \text{proj}_{[\vec{c}]}(\vec{r})$

*

Any 3-termed sequence of linearly independent translations is a basis for $\mathscr{T}$. So, what has just been established in Exercise 3, above, is:

$$(**)\quad \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0 \longrightarrow (\vec{a}, \vec{b}, \vec{c}) \text{ is a basis} \quad [\vec{a}, \vec{b}, \vec{c} \neq \vec{0}]$$

In our earlier work, we said that $\vec{a}$ and $\vec{b}$ are orthogonal if their dot product is zero. So, (**) tells us that any 3-termed sequence of non-$\vec{0}$ translations which are *pairwise orthogonal* is a basis for $\mathscr{T}$. Such bases – those whose terms are pairwise orthogonal – are called *orthogonal bases.*

> Definition 11–3 $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis
>
> $\longleftrightarrow$
>
> $(\vec{a}, \vec{b}$, and $\vec{c}$ are non-$\vec{0}$ and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0)$

Using this definition and the results in Exercise 4, above, we have:

> Theorem 11–10 $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis
> $\longrightarrow \vec{r} = \text{proj}_{[\vec{a}]}(\vec{r}) + \text{proj}_{[\vec{b}]}(\vec{r}) + \text{proj}_{[\vec{c}]}(\vec{r})$

The exercises on page 59 together with those on page 60 provide important illustrations of the use of orthogonal bases. We recommend that these exercises be used as a basis for class discussion. Students should then be well prepared to treat Parts A and B (pp. 61-62) for homework.

Answers for Exercises

1. The answers to the questions in the proof of the lemma are, in turn: Replacement rule for equations; Postulate 4_{12}; Postulate 4_{13}; Theorem 11-1(c).
2. Suppose that $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent and that $\vec{a} \cdot \vec{r} = \vec{a} \cdot \vec{s}$, $\vec{b} \cdot \vec{r} = \vec{b} \cdot \vec{s}$, and $\vec{c} \cdot \vec{r} = \vec{c} \cdot \vec{s}$. Then, $\vec{a} \cdot (\vec{r} - \vec{s}) = \vec{b} \cdot (\vec{r} - \vec{s}) = \vec{c} \cdot (\vec{r} - \vec{s}) = 0$ so that, by the lemma, $\vec{r} - \vec{s} = \vec{0}$. Thus, $\vec{r} = \vec{s}$. Hence, Theorem 11-9.
3. Following the hint, we see that $(\vec{a} \cdot \vec{a})a = 0$. Since $\vec{a} \neq \vec{0}$, it follows that $a = 0$. Dot multiplying the given equation on both sides with '$\vec{b}$' and, in turn, with '$\vec{c}$', we see that $b = 0$ and $c = 0$. So, $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent.
4. (a) $\vec{r} \cdot \vec{a} = (\vec{a} \cdot \vec{a})a$, $\vec{r} \cdot \vec{b} = (\vec{b} \cdot \vec{b})b$, and $\vec{r} \cdot \vec{c} = (\vec{c} \cdot \vec{c})c$. So, $a = (\vec{r} \cdot \vec{a})/(\vec{a} \cdot \vec{a})$, $b = (\vec{r} \cdot \vec{b})/(\vec{b} \cdot \vec{b})$, and $c = (\vec{r} \cdot \vec{c})/(\vec{c} \cdot \vec{c})$.
   - (b) This follows from (a), and Theorem 11-6, and Definition 11-1(c).

Exercises

Assume that $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis.

1. Let $\vec{r} = \vec{a}5 + \vec{b}2 + \vec{c} \cdot -3$ and $\vec{s} = \vec{a} \cdot -2 + \vec{b}6 + \vec{c}4$.
   - (a) What are $\text{proj}_{[\vec{a}]}(\vec{r})$, $\text{proj}_{[\vec{b}]}(\vec{r})$, and $\text{proj}_{[\vec{c}]}(\vec{r})$? Answer similar questions regarding $\vec{r}2 + \vec{s}5$ and $\vec{r}4 + \vec{s}3$.
   - (b) Compute $\vec{r} \cdot \vec{s}$. [*Hint*: $\vec{r} \cdot \vec{s} = (\vec{a}5 + \vec{b}2 + \vec{c} \cdot -3) \cdot (\vec{a} \cdot -2 + \vec{b}6 + \vec{c}4)$. Expand and simplify, remembering that $(\vec{a}, \vec{b}, \vec{c})$ is orthogonal.]
   - (c) Find values for '$\vec{a} \cdot \vec{a}$', '$\vec{b} \cdot \vec{b}$', and '$\vec{c} \cdot \vec{c}$' for which $\vec{r}$ and $\vec{s}$ are orthogonal. Is there more than one such set of values?
   - (d) Compute $\vec{r} \cdot \vec{r}$ and $\vec{s} \cdot \vec{s}$.
   - (e) Given that $\|\vec{a}\| = 2$, $\|\vec{b}\| = \frac{1}{2}$, and $\|\vec{c}\| = 1$, what are $\vec{r} \cdot \vec{s}$, $\|\vec{r}\|$, and $\|\vec{s}\|$?
2. (a) Give the components of both unit vectors in the direction of $\vec{r}$, where $\vec{r} = \vec{a}5 + \vec{b}2 + \vec{c} \cdot -3$. Which of these unit vectors is in the sense of $\vec{r}$? [*Hint*: For $\vec{a} \neq \vec{0}$, $\vec{a}/\|\vec{a}\|$ is a unit vector.]
   - (b) Express the unit vector in the sense of $\vec{s}$ [described in Exercise 1] as a linear combination of the basis vectors.
3. Let $\vec{t}_1 = \vec{a}4 + \vec{c} \cdot -3$ and $\vec{t}_2 = \vec{b} \cdot -2 + \vec{c}3$.
   - (a) What is $\vec{t}_1 \cdot \vec{t}_2$? Are there values of '$\|\vec{a}\|$', '$\|\vec{b}\|$', and '$\|\vec{c}\|$' such that $\vec{t}_1$ and $\vec{t}_2$ are orthogonal?
   - (b) Let $\vec{t}_3 = \vec{a}a + \vec{b}b + \vec{c}c$, for some a, b, and c. Show that $\vec{t}_1$ and $\vec{t}_3$ are orthogonal if $a = (\vec{c} \cdot \vec{c})/4$ and $c = (\vec{a} \cdot \vec{a})/3$. Describe other values for 'a' and 'c' for which $\vec{t}_1$ and $\vec{t}_3$ are orthogonal. Describe the values for 'b' for which $\vec{t}_1$ and $\vec{t}_3$ are orthogonal.
   - (c) Describe values for 'a', 'b', and 'c' for which $\vec{t}_2$ and $\vec{t}_3$ are orthogonal.
   - (d) Given that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are unit vectors, compute $\|\vec{t}_1\|$ and $\|\vec{t}_2\|$.
   - (e) Given that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are unit vectors, and $\vec{t}_3 = \vec{a}a + \vec{b}b + \vec{c}c$, describe $\|\vec{t}_3\|$ in terms of 'a', 'b', and 'c'.
4. Given that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are unit vectors [and that $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis], assume that $\vec{r} = \vec{a}3$, $\vec{s} = \vec{b}2$, and $\vec{t} = \vec{c} \cdot -5$. Compute each of the following norms.

(a) $\|\vec{r}\|$	(b) $\|\vec{s}\|$	(c) $\|\vec{t}\|$
(d) $\|\vec{r} + \vec{s}\|$	(e) $\|\vec{s} + \vec{t}\|$	(f) $\|\vec{r} - \vec{t}\|$
(g) $\|\vec{r} - \vec{s}\|$	(h) $\|\vec{r} + \vec{s} + \vec{t}\|$	(i) $\|\vec{r} + \vec{s} - \vec{t}\|$

*

From the exercises just completed, we see that much information about particular vectors can be obtained in terms of the components of these vectors with respect to an orthogonal basis. As a matter of fact, when the basis vectors are also unit vectors—in addition to being pairwise orthogonal—we are able to express the norm of any vector in terms of just its components and, further, we are able to express the

Answers for Exercises

1. (a) $\vec{a}5$, $\vec{b}2$, $\vec{c} \cdot -3$; Since $\vec{r}2 + \vec{s}5 = \vec{a}0 + \vec{b}34 + \vec{c}14$ it follows, by Theorem 11-10, that the projections of $\vec{r}2 + \vec{s}5$ on $[\vec{a}]$, $[\vec{b}]$, and $[\vec{c}]$ are $\vec{0}$, $\vec{b}34$, and $\vec{c}14$, respectively. Also, since $\vec{r}4 + \vec{s}3 = \vec{a}14 + \vec{b}26 + \vec{c}0$, it follows that the projections of $\vec{r}4 + \vec{s}3$ on $[\vec{a}]$, $[\vec{b}]$, and $[\vec{c}]$ are $\vec{a}14$, $\vec{b}26$, and $\vec{0}$, respectively.
   - (b) Since $(\vec{a}, \vec{b}, \vec{c})$ is orthogonal, $\vec{r} \cdot \vec{s} = (\vec{a} \cdot \vec{a}) \cdot -10 + (\vec{b} \cdot \vec{b})12 + (\vec{c} \cdot \vec{c}) \cdot -12$.
   - (c) $\vec{r}$ and $\vec{s}$ are orthogonal if and only if $\vec{r} \cdot \vec{s} = 0$. By (b), this is the case if and only if $(\vec{a} \cdot \vec{a}) \cdot -10 + (\vec{b} \cdot \vec{b})12 + (\vec{c} \cdot \vec{c}) \cdot -12 = 0$. The latter is the case for many — in fact, infinitely many — triples $(\vec{a} \cdot \vec{a}, \vec{b} \cdot \vec{b}, \vec{c} \cdot \vec{c})$. To find a particular solution, choose values for, say, '$\vec{a} \cdot \vec{a}$' and '$\vec{b} \cdot \vec{b}$' and solve for '$\vec{c} \cdot \vec{c}$'.
   - (d) $\vec{r} \cdot \vec{r} = (\vec{a} \cdot \vec{a})25 + (\vec{b} \cdot \vec{b})4 + (\vec{c} \cdot \vec{c})9$; $\vec{s} \cdot \vec{s} = (\vec{a} \cdot \vec{a})4 + (\vec{b} \cdot \vec{b})36 + (\vec{c} \cdot \vec{c})16$.
   - (e) $\vec{r} \cdot \vec{s} = 4 \cdot -10 + \frac{1}{4} \cdot 12 + 1 \cdot -12 = -49$;
     $\|\vec{r}\|^2 = 4 \cdot 25 + \frac{1}{4} \cdot 4 + 1 \cdot 9 = 110$ so that $\|\vec{r}\| = \sqrt{110}$;
     $\|\vec{s}\|^2 = 4 \cdot 4 + \frac{1}{4} \cdot 36 + 1 \cdot 16 = 41$ so that $\|\vec{s}\| = \sqrt{41}$.

2. (a) $\vec{r}/\|\vec{r}\|$ and $-\vec{r}/\|\vec{r}\|$ are the unit vectors in $[\vec{r}]$ and $\vec{r}/\|\vec{r}\| \in [\vec{r}]^{\prime}$. Note that

$$\vec{r} \cdot \vec{r} = (\vec{a} \cdot \vec{a})25 + (\vec{b} \cdot \vec{b})4 + (\vec{c} \cdot \vec{c})9$$

so that $\|\vec{r}\| = \sqrt{(\vec{a} \cdot \vec{a})25 + (\vec{b} \cdot \vec{b})4 + (\vec{c} \cdot \vec{c})9}$. The components of $\vec{r}/\|\vec{r}\|$ are $(5/\|\vec{r}\|, 2/\|\vec{r}\|, -3/\|\vec{r}\|)$ and those of $-\vec{r}/\|\vec{r}\|$ are $(-5/\|\vec{r}\|, -2/\|\vec{r}\|, 3/\|\vec{r}\|)$.

   - (b) By 1(d), we know that $\|\vec{s}\| = \sqrt{(\vec{a} \cdot \vec{a})4 + (\vec{b} \cdot \vec{b})36 + (\vec{c} \cdot \vec{c})16}$. So,

$$\vec{s}/\|\vec{s}\| = \vec{a} \cdot -2/\|\vec{s}\| + \vec{b}6/\|\vec{s}\| + \vec{c}4/\|\vec{s}\|.$$

3. (a) $\vec{t}_1 \cdot \vec{t}_2 = (\vec{a}4 + \vec{c} \cdot -3) \cdot (\vec{b} \cdot -2 + \vec{c}3) = (\vec{c} \cdot \vec{c}) \cdot -9$. Since $\vec{c} \neq \vec{0}$, $\|\vec{c}\| \neq 0$ so that there are no values for '$\|\vec{a}\|$', $\|\vec{b}\|$', and '$\|\vec{c}\|$' for which $\vec{t}_1 \cdot \vec{t}_2 = 0$.
   - (b) $\vec{t}_1 \cdot \vec{t}_3 = (\vec{a}4 + \vec{c} \cdot -3) \cdot (\vec{a}a + \vec{b}b + \vec{c}c)$
     $= (\vec{a} \cdot \vec{a})4a + (\vec{c} \cdot \vec{c}) \cdot -3c$

     So, $\vec{t}_1 \cdot \vec{t}_3 = 0$ if and only if $(\vec{a} \cdot \vec{a})4a - (\vec{c} \cdot \vec{c})3c = 0$. Clearly, if $a = (\vec{c} \cdot \vec{c})/4$ and $c = (\vec{a} \cdot \vec{a})/3$ then $\vec{t}_1 \cdot \vec{t}_3 = 0$. The values for 'a' and 'c' for which $\vec{t}_1$ and $\vec{t}_3$ are orthogonal are such that

$$(a, c) = (t(\vec{c} \cdot \vec{c})/4,\ t(\vec{a} \cdot \vec{a})/3), \text{ for some } t.$$

The orthogonality of $\vec{t}_1$ and $\vec{t}_3$ does not depend on values for 'b'.

Answers for Exercises [cont.]

3. (c) $\vec{t}_2 \cdot \vec{t}_3 = (\vec{b} \cdot -2 + \vec{c}3) \cdot (\vec{a}a + \vec{b}b + \vec{c}c)$

$= (\vec{b} \cdot \vec{b}) \cdot -2b + (\vec{c} \cdot \vec{c})3c$

so that $\vec{t}_2 \cdot \vec{t}_3 = 0$ if and only if $(\vec{b} \cdot \vec{b})2b - (\vec{c} \cdot \vec{c})3c = 0$. The values for 'b' and 'c' for which the latter is true are such that

$$(b, c) = (t(\vec{c} \cdot \vec{c})/2,\ t(\vec{b} \cdot \vec{b})/3), \text{ for some } t.$$

The orthogonality of $\vec{t}_2$ and $\vec{t}_3$ does not depend on values for 'a'.

(d) $\vec{t}_1 \cdot \vec{t}_1 = (\vec{a} \cdot \vec{a})16 + (\vec{c} \cdot \vec{c})9$. If $\vec{a} \cdot \vec{a} = 1 = \vec{c} \cdot \vec{c}$, then $\vec{t}_1 \cdot \vec{t}_1 = 25$. So, $||\vec{t}_1|| = 5$.

$\vec{t}_2 \cdot \vec{t}_2 = (\vec{b} \cdot \vec{b})4 + (\vec{c} \cdot \vec{c})9$ so that if $\vec{b} \cdot \vec{b} = 1 = \vec{c} \cdot \vec{c}$, $\vec{t}_2 \cdot \vec{t}_2 = 13$. So, $||\vec{t}_2|| = \sqrt{13}$.

(e) $||\vec{t}_3|| = \sqrt{a^2 + b^2 + c^2}$

4. (a) 3 (b) 2 (c) 5

(d) $\sqrt{13}$ (e) $\sqrt{29}$ (f) $\sqrt{34}$

(g) $\sqrt{13}$ (h) $\sqrt{38}$ (i) $\sqrt{38}$

dot product of any two vectors in terms of their components alone. Thus, a most useful and convenient orthogonal basis is one whose terms are unit vectors. Such a basis is called *an orthonormal basis* and is defined formally in:

> Definition 11–4 $(\vec{a}, \vec{b}, \vec{c})$ is an orthonormal basis
>
> $\longleftrightarrow$
>
> $(\vec{a}, \vec{b}$, and $\vec{c}$ are unit vectors and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0)$

Recall that, for $\vec{a} \neq \vec{0}$, $\text{proj}_{[\vec{a}]}(\vec{r}) = \vec{a}\,\text{comp}_{\vec{a}}(\vec{r}) = \vec{a}[(\vec{r} \cdot \vec{a})/(\vec{a} \cdot \vec{a})]$. Now, if $\|\vec{a}\| = 1$, we have that $\text{proj}_{[\vec{a}]}(\vec{r}) = \vec{a}(\vec{r} \cdot \vec{a})$. [Explain.] Making use of this we obtain a most useful special case of Theorem 11–10, which we state in:

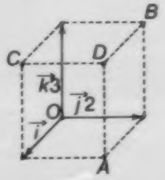
> Theorem 11–11 $(\vec{a}, \vec{b}, \vec{c})$ is an orthonormal basis
> $\longrightarrow \vec{r} = \vec{a}(\vec{r} \cdot \vec{a}) + \vec{b}(\vec{r} \cdot \vec{b}) + \vec{c}(\vec{r} \cdot \vec{c})$

Exercises

Part A

In these exercises, let $(\vec{i}, \vec{j}, \vec{k})$ be an orthonormal basis for $\mathscr{T}$.

1. Suppose that $\vec{a} = \vec{i}3 + \vec{j} \cdot -2 + \vec{k}$ and that $\vec{b} = \vec{i}3 + \vec{j}2 + \vec{k} \cdot -5$.
   (a) Compute $\vec{i} \cdot \vec{j}$, $\vec{i} \cdot \vec{k}$, $\vec{j} \cdot \vec{j}$, $\vec{j} \cdot \vec{k}$, and $\vec{k} \cdot \vec{k}$.
   (b) What are $\vec{a} \cdot \vec{i}$, $\vec{a} \cdot \vec{j}$, and $\vec{a} \cdot \vec{k}$? What are $\vec{b} \cdot \vec{i}$, $\vec{b} \cdot \vec{j}$, and $\vec{b} \cdot \vec{k}$?
   (c) Determine whether $\vec{a}$ and $\vec{b}$ are orthogonal.
   (d) Compute $\|\vec{a}\|$ and $\|\vec{b}\|$.
   (e) Express the unit vectors in the senses of $\vec{a}$ and $\vec{b}$, respectively, as linear combinations of the given basis vectors.
2. Assume that $A = O + \vec{i} + \vec{j}2$, $B = O + \vec{j}2 + \vec{k}3$, $C = O + \vec{i} + \vec{k}3$, and $D = A + \vec{k}3$, as pictured at the right.
   (a) Show that $D - A$ is orthogonal to both $B - C$ and $A - O$.
   (b) Compute $\|C - A\|$, $\|B - A\|$, and $\|D - O\|$.



3. Assume that $\vec{a} = \vec{i}a_1 + \vec{j}a_2 + \vec{k}a_3$ and that $\vec{b} = \vec{i}b_1 + \vec{j}b_2 + \vec{k}b_3$.
   (a) Express $\|\vec{a}\|$ and $\|\vec{b}\|$ in terms of the components of $\vec{a}$ and $\vec{b}$, respectively.
   (b) Express $\vec{a} \cdot \vec{b}$ in terms of the components of $\vec{a}$ and $\vec{b}$.
   (c) Give the components of a unit vector in $[\vec{a}]$, in terms of the components of $\vec{a}$, when $\vec{a} \neq \vec{0}$.

Explanation called for in text: If $||\vec{a}|| = 1$, we have $\vec{a}\cdot\vec{a} = 1$ so that $\text{proj}_{[\vec{a}]}(\vec{r}) = \vec{a}[(\vec{r}\cdot\vec{a})/(\vec{a}\cdot\vec{a})] = \vec{a}[(\vec{r}\cdot\vec{a})/1] = \vec{a}(\vec{r}\cdot\vec{a})$.

Answers for Part A

1. (a) 0, 0, 1, 0, 1
   (b) 3, −2, and 1; 3, 2, and −5
   (c) $\vec{a}\cdot\vec{b} = 3\cdot 3 + -2\cdot 2 + 1\cdot -5 = 0$. So, $\vec{a}$ and $\vec{b}$ are orthogonal.
   (d) $||\vec{a}||^2 = 9 + 4 + 1 = 14$, so that $||\vec{a}|| = \sqrt{14}$;
   $||\vec{b}||^2 = 9 + 4 + 25 = 38$, so that $||\vec{b}|| = \sqrt{38}$.
   (e) $\vec{a}/||\vec{a}|| = \vec{i}3/\sqrt{14} + \vec{j}\cdot -2/\sqrt{14} + \vec{k}/\sqrt{14}$;
   $\vec{b}/||\vec{b}|| = \vec{i}3/\sqrt{38} + \vec{j}2/\sqrt{38} + \vec{k}\cdot -5/\sqrt{38}$.
2. (a) $D - A = \vec{k}3$, $B - C = \vec{i}\cdot -1 + \vec{j}2$, and $A - O = \vec{i} + \vec{j}2$. So,

   $$(D - A)\cdot(B - C) = \vec{k}3\cdot(\vec{i}\cdot -1 + \vec{j}2) = 0$$

   and

   $$(D - A)\cdot(A - O) = \vec{k}3\cdot(\vec{i} + \vec{j}2) = 0.$$

   So, $D - A$ is orthogonal to both $B - C$ and $A - O$.
   (b) $C - A = \vec{k}3 - \vec{j}2$. So, $||C - A|| = ||\vec{k}3 - \vec{j}2|| = \sqrt{9 + 4} = \sqrt{13}$.
   $B - A = \vec{k}3 - \vec{i}$. So, $||B - A|| = ||\vec{k}3 - \vec{i}|| = \sqrt{9 + 1} = \sqrt{10}$.
   $D - O = \vec{i} + \vec{j}2 + \vec{k}3$. So, $||D - O|| = ||\vec{i} + \vec{j}2 + \vec{k}3|| = \sqrt{14}$.

3. (a) $||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, for $\vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2$.
   $||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2}$
   (b) $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
   (c) $\vec{a}/||\vec{a}||$ has components $(a_1/||\vec{a}||, a_2/||\vec{a}||, a_3/||\vec{a}||)$, where $||\vec{a}||$ is given in (a). $-\vec{a}/||\vec{a}||$ has components $(-a_1/||\vec{a}||, -a_2/||\vec{a}||, -a_3/||\vec{a}||)$.

Answers for Part A [cont.]

4.

(a)	(i) 19	(ii) $5\sqrt{2}$	(iii) $\sqrt{29}$	(iv) $\sqrt{117}$	(v) $\sqrt{41}$
(b)	(i) −59	(ii) $\sqrt{59}$	(iii) $\sqrt{59}$	(iv) 0	(v) $2\sqrt{59}$
(c)	(i) 76	(ii) $\sqrt{38}$	(iii) $2\sqrt{38}$	(iv) $3\sqrt{38}$	(v) $\sqrt{38}$
(d)	(i) 121	(ii) $\sqrt{194}$	(iii) $\sqrt{194}$	(iv) $3\sqrt{70}$	(v) $\sqrt{146}$

Answers for Part B

1. (a) (1, 3, 5)
   (b) $\sqrt{35}$ [for, $||P - O|| = \sqrt{1^2 + 3^2 + 5^2}$]
   (c) $(/\sqrt{35}, 3/\sqrt{35}, 5/\sqrt{35})$ and $(-/\sqrt{35}, -3/\sqrt{35}, -5/\sqrt{35})$
2. (a) (2, −4, −6)
   (b) $2\sqrt{14}$ [or, $\sqrt{56}$]
   (c) $(/\sqrt{14}, -2/\sqrt{14}, -3/\sqrt{14})$ and $(-/\sqrt{14}, 2/\sqrt{14}$, and $3/\sqrt{14})$
3. (a) Components of $P - Q$ are (−1, 7, 11); $(P - Q)\cdot(P - Q) = 171$
   (b) Since $||P - Q|| = 3\sqrt{19}$ [or, $\sqrt{171}$], the unit vectors in $[P - Q]$ have components $(-/(3\sqrt{19}), 7/(3\sqrt{19}), 11/(3\sqrt{19}))$ and $(/(3\sqrt{19}), -7/(3\sqrt{19}), -11/(3\sqrt{19}))$.
4. (a) $(s_1 - r_1, s_2 - r_2, s_3 - r_3)$
   (b) $\sqrt{(s_1 - r_1)^2 + (s_2 - r_2)^2 + (s_3 - r_3)^2}$
   (c) $\big((s_1 - r_1)/||S - R||, (s_2 - r_2)/||S - R||, (s_3 - r_3)/||S - R||\big)$ and $\big((r_1 - s_1)/||S - R||, (r_2 - s_2)/||S - R||, (r_3 - s_3)/||S - R||\big)$, where $||S - R||$ is given in part (b).
5. (a) (1, −9, 4)
   (b) (3/2, 1/2, 5), for since $M = A + (B - A)\frac{1}{2}$, the coordinates of M are $(1 + 1\cdot\frac{1}{2}, 5 + -9\cdot\frac{1}{2}, 3 + 4\cdot\frac{1}{2})$.
   (c) $M - C$ has components (−1/2, 1/2, 5). So, $||M - C|| = \sqrt{\frac{1}{4} + \frac{1}{4} + 25} = \sqrt{102}/2$.
   (d) N has coordinates (2, −2, 7/2). Thus, the components of $N - A$ are (1, −7, 1/2) so that $||N - A|| = \sqrt{201}/2$.
6. (a) Making use of the hint, the coordinates of D are (1, 1, 10).
   (b) E is such that $E = A + (C - B)$. So, E has coordinates (1, 9, −4).
   (c) $D - C$ has components (−1, 1, 10) so that $||D - C|| = \sqrt{102}$; $E - B$ has components (−1, 13, −11) so that $||E - B|| = \sqrt{291}$.

4. In each of the following parts, you are given $\vec{a}$ and $\vec{b}$ as linear combinations of the terms of $(\vec{i}, \vec{j}, \vec{k})$. You are to compute (i) $\vec{a} \cdot \vec{b}$, (ii) $\|\vec{a}\|$, (iii) $\|\vec{b}\|$, (iv) $\|\vec{a} + \vec{b}\|$, and (v) $\|\vec{a} - \vec{b}\|$.
   - **(a)** $\vec{a} = \vec{i}4 + \vec{j}3 + \vec{k}5, \vec{b} = \vec{i}2 + \vec{j} \cdot -3 + \vec{k}4$
   - **(b)** $\vec{a} = \vec{i}5 + \vec{j} \cdot -3 + \vec{k}5, \vec{b} = \vec{i} \cdot -5 + \vec{j}3 + \vec{k} \cdot -5$
   - **(c)** $\vec{a} = \vec{i}2 + \vec{j}3 + \vec{k}5, \vec{b} = \vec{i}4 + \vec{j}6 + \vec{k}10$
   - **(d)** $\vec{a} = \vec{i}3 + \vec{j}4 + \vec{k}13, \vec{b} = \vec{i}12 + \vec{j}5 + \vec{k}5$

Part B

In these exercises, all coordinates of points are given in terms of the coordinate system with origin O determined by the orthonormal basis $(\vec{i}, \vec{j}, \vec{k})$.

1. Suppose that P has coordinates $(1, 3, 5)$.
   - **(a)** What are the components of $P - O$?
   - **(b)** Compute $\|P - O\|$.
   - **(c)** Determine the components of each of the unit vectors in $[P - O]$.
2. Suppose that Q has coordinates $(2, -4, -6)$.
   - **(a)** What are the components of $Q - O$?
   - **(b)** Compute $\|Q - O\|$.
   - **(c)** Determine the components of each of the unit vectors in $[Q - O]$.
3. Consider the points P and Q described in Exercises 1 and 2.
   - **(a)** Determine the components of $P - Q$ and compute $(P - Q) \cdot (P - Q)$.
   - **(b)** Give the components of each of the unit vectors in $[P - Q]$.
4. Suppose that R and S have coordinates (r_1, r_2, r_3) and (s_1, s_2, s_3), respectively. Determine each of the following in terms of these coordinates.
   - **(a)** the components of $S - R$
   - **(b)** $\|S - R\|$
   - **(c)** the components of each unit vector in $[S - R]$, when $R \neq S$.
5. Suppose that A, B, and C have coordinates $(1, 5, 3)$, $(2, -4, 7)$, and $(2, 0, 0)$, respectively. Compute each of the following.
   - **(a)** the components of $B - A$
   - **(b)** the coordinates of M, the midpoint of $\overleftrightarrow{AB}$
   - **(c)** $\|M - C\|$
   - **(d)** $\|N - A\|$, where N is the midpoint of $\overleftrightarrow{BC}$
6. Given A, B, and C described in Exercise 5, determine the following.
   - **(a)** the coordinates of D, where $ACBD$ is a parallelogram [*Hint*: $D = A + (B - C)$.]
   - **(b)** the coordinates of E, where $ABCE$ is a parallelogram
   - **(c)** $\|D - C\|$ and $\|E - B\|$

*

Sample Quiz

Suppose that $(\vec{i}, \vec{j}, \vec{k})$ is an orthonormal basis for T, and that

$$\vec{a} = \vec{i}2 - \vec{j}2 + \vec{k}5, \quad \vec{b} = -\vec{i}2 - \vec{j}2 + \vec{k}3,$$
$$\vec{c} = \vec{i}3 - \vec{j}3, \quad \vec{d} = \vec{j}12 + \vec{k}5.$$

Complete the following. Show all of your work.

1. $\vec{a} \cdot \vec{b} =$ ___ 2. $\vec{c} \cdot \vec{d} =$ ___

3. $\|\vec{c}\| =$ ___ 4. $\|\vec{a} + \vec{b}\| =$ ___

5. $\vec{a}$ _________ orthogonal to $\vec{b}$ because...
(is/is not)

6. $\vec{c}$ _________ orthogonal to $\vec{b}$ because...
(is/is not)

7. The components of a unit vector in $[\vec{a}]$ are _____

Answers for Sample Quiz

1. 15 $[2 \cdot -2 - 2 \cdot -2 + 5 \cdot 3]$ 2. -36 $[3 \cdot 0 - 3 \cdot 12 + 0 \cdot 5]$

3. $3\sqrt{2}$ $[\sqrt{3^2 + 3^2}]$ 4. $4\sqrt{5}$ $[\sqrt{0 + 4^2 + 8^2}]$

5. is not, because $\vec{a} \cdot \vec{b} \neq 0$.

6. is, because $\vec{c} \cdot \vec{b} = 0$.

7. $(2/\sqrt{33}, -2/\sqrt{33}, 5/\sqrt{33})$, for $\|\vec{a}\| = \sqrt{4 + 4 + 25}$ and $\vec{a}/\|\vec{a}\|$ is a unit vector in $[\vec{a}]$. An alternate answer is $(-2/\sqrt{33}, 2/\sqrt{33}, -5/\sqrt{33})$, for $-\vec{a}/\|\vec{a}\|$ is another unit vector in $[\vec{a}]$.

We conclude this section by summarizing two of the results concerning components of vectors and coordinates of points proved in the exercises just completed.

> **Theorem 11-12** Given an orthonormal basis $(\vec i, \vec j, \vec k)$, if $\vec a = \vec i a_1 + \vec j a_2 + \vec k a_3$ and $\vec b = \vec i b_1 + \vec j b_2 + \vec k b_3$ then
> (a) $\vec a \cdot \vec b = a_1 b_1 + a_2 b_2 + a_3 b_3$, and
> (b) $\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

11.12 Chapter Summary

Vocabulary Summary

- perpendicular
- orthogonal
  - projection of a point
  - projection of a vector
  - complement of a direction
  - vectors
  - basis for $\mathscr{T}$
  - coordinate system for $\mathscr{E}$
- orthonormal basis for $\mathscr{T}$
- distance
- norm
- unit vector
- dot multiplication
- inner product space
- perpendicular bisector
- equidistant
- altitude

Postulates

1. (a) $B - A \in \mathscr{T}$ (b) $A + \vec a \in \mathscr{E}$
2. (a) $A + (B - A) = B$ (b) $\vec a = (A + \vec a) - A$
3. $(B - A) + (C - B) = C - A$

$4'$. $\mathscr{T}$ is a 3-dimensional vector space over $\mathscr{R}$.

4_0. (e) $\vec a \cdot \vec b \in \mathscr{R}$

4_{11}. $\vec a \cdot \vec a > 0$ $[\vec a \neq \vec 0]$

4_{12}. $(\vec a + \vec b) \cdot \vec c = \vec a \cdot \vec c + \vec b \cdot \vec c$

4_{13}. $(\vec a a) \cdot \vec b = (\vec a \cdot \vec b)a$

4_{14}. $\vec a \cdot \vec b = \vec b \cdot \vec a$

4. $\mathscr{T}$ is a 3-dimensional inner product space over $\mathscr{R}$.

Definitions

11-1. (a) $\|\vec a\| = \sqrt{\vec a \cdot \vec a}$
(b) $[\vec a]^{\perp} = \{\vec x : \forall_{\vec y \in [\vec a]}\ \vec x \cdot \vec y = 0\}$
(c) $\text{comp}_{\vec b}(\vec a) = (\vec a \cdot \vec b)/(\vec b \cdot \vec b)$

11-2. $P + \text{proj}_{[\vec a]}(\vec b) = Q \Longleftrightarrow \left(Q - P \in [\vec a] \text{ and } \vec b - (Q - P) \in [\vec a]^{\perp}\right)$

11-3. $(\vec a, \vec b, \vec c)$ is an orthogonal basis $\Longleftrightarrow$ $(\vec a, \vec b,$ and $\vec c$ are non-$\vec 0$ and $\vec a \cdot \vec b = \vec b \cdot \vec c = \vec c \cdot \vec a = 0)$

11-4. $(\vec a, \vec b, \vec c)$ is an orthonormal basis $\Longleftrightarrow$ $(\vec a, \vec b,$ and $\vec c$ are unit vectors and $\vec a \cdot \vec b = \vec b \cdot \vec c = \vec c \cdot \vec a = 0)$

Other Theorems

11-1. (a) $\vec a \cdot (\vec b b) = (\vec a \cdot \vec b)b$ (b) $\vec 0 \cdot \vec a = 0 = \vec a \cdot \vec 0$
(c) $\vec a = \vec 0 \Longleftrightarrow \vec a \cdot \vec a = 0$ (d) $\vec a \cdot (\vec b + \vec c) = \vec a \cdot \vec b + \vec a \cdot \vec c$
(e) $\vec a \cdot -\vec b = -(\vec a \cdot \vec b) = -\vec a \cdot \vec b$ (f) $-\vec a \cdot -\vec b = \vec a \cdot \vec b$
(g) $(\vec a - \vec b) \cdot \vec c = \vec a \cdot \vec c - \vec b \cdot \vec c$ (h) $\vec a \cdot (\vec b - \vec c) = \vec a \cdot \vec b - \vec a \cdot \vec c$
(i) $(\vec a + \vec b) \cdot (\vec a - \vec b) = \vec a \cdot \vec a - \vec b \cdot \vec b$
(j) $(\vec a + \vec b) \cdot (\vec a + \vec b) = \vec a \cdot \vec a + \vec b \cdot \vec b + 2(\vec a \cdot \vec b)$
(k) $(\vec a - \vec b) \cdot (\vec a - \vec b) = \vec a \cdot \vec a + \vec b \cdot \vec b - 2(\vec a \cdot \vec b)$

11-2. $\vec b \in [\vec a]^{\perp} \Longleftrightarrow \vec b \cdot \vec a = 0$

11-3. (a) $\|\vec a\| \in \mathscr{R}$ (b) $\vec a \neq \vec 0 \Longleftrightarrow \|\vec a\| > 0$
(c) $\|\vec a a\| = \|\vec a\|\,|a|$ (d) $\vec a + \vec b \in [\vec a - \vec b]^{\perp} \Longleftrightarrow \|\vec a\| = \|\vec b\|$
(e) $\vec a_1 = \vec a/\|\vec a\| \Longrightarrow \left(\|\vec a_1\| = 1 \text{ and } \vec a = \vec a_1\|\vec a\|\right)$ $[\vec a \neq \vec 0]$.

Lemma 1. (a) $[\vec 0]^{\perp} = \mathscr{T}$
(b) $\vec a - \vec b\left((\vec a \cdot \vec b)/(\vec b \cdot \vec b)\right) \in [\vec b]^{\perp}$
(c) $\{\vec b, \vec c\} \subseteq [\vec a]^{\perp} \Longrightarrow [\vec b, \vec c] \subseteq [\vec a]^{\perp}$

Lemma 2. (a) $\vec u \cdot \vec v = 1 \Longleftrightarrow \vec u = \vec v$ $[\|\vec u\| = 1 = \|\vec v\|]$
(b) $\vec u \cdot \vec v = -1 \Longleftrightarrow \vec u = -\vec v$ $[\|\vec u\| = 1 = \|\vec v\|]$

11-4. (a) $[\vec a]^{\perp}$ is a proper bidirection $[\vec a \neq \vec 0]$
(b) $[\vec a] \cap [\vec a]^{\perp} = \{\vec 0\}$
(c) $[\vec a]^{\perp} = [\vec b]^{\perp} \Longleftrightarrow [\vec a] = [\vec b]$
(d) $\vec b \in [\vec a]^{\perp} \Longleftrightarrow \vec a \in [\vec b]^{\perp}$

11-5. For $\vec a \neq \vec 0$,
(a) $\text{comp}_{\vec a}(\vec a) \in \mathscr{R}$ and $(\vec b \in [\vec a] \Longrightarrow \text{comp}_{\vec a}(\vec b) = \vec b : \vec a)$
(b) $\text{comp}_{\vec a}(\vec b) = 0 \Longleftrightarrow \vec b \in [\vec a]^{\perp}$
(c) $\text{comp}_{\vec a}(\vec b + \vec c) = \text{comp}_{\vec a}(\vec b) + \text{comp}_{\vec a}(\vec c)$
(d) $\text{comp}_{\vec a}(\vec b b) = \text{comp}_{\vec a}(\vec b)b$, and
(e) $\text{comp}_{(\vec a a)}(\vec b) = \text{comp}_{\vec a}(\vec b)/a$ $[a \neq 0]$

11-6. $\text{proj}_{[\vec a]}(\vec b) = \vec a\,\text{comp}_{\vec a}(\vec b)$

Corollary. (a) $\text{proj}_{[\vec a]}(\vec b) \in \mathscr{T}$
(b) $\text{proj}_{[\vec a]}(\vec b) = \vec c \Longleftrightarrow (\vec c \in [\vec a] \text{ and } \vec b - \vec c \in [\vec a]^{\perp})$

11-7. (a) $\text{proj}_{[\vec a]}(\vec b) \in [\vec a]$ and $\left(\text{proj}_{[\vec a]}(\vec b) = \vec b \Longleftrightarrow \vec b \in [\vec a]\right)$
(b) $\text{proj}_{[\vec a]}(\vec b) = \vec 0 \Longleftrightarrow \vec b \in [\vec a]^{\perp}$
(c) $\text{proj}_{[\vec a]}(\vec b + \vec c) = \text{proj}_{[\vec a]}(\vec b) + \text{proj}_{[\vec a]}(\vec c)$
(d) $\text{proj}_{[\vec a]}(\vec b b) = \text{proj}_{[\vec a]}(\vec b)b$

11-8. (a) $\|\vec{a}\|\,\|\vec{b}\| \ge |\vec{a} \cdot \vec{b}|$

(b) $(\vec{a}, \vec{b})$ is linearly dependent if and only if $\|\vec{a}\|\,\|\vec{b}\| = |\vec{a} \cdot \vec{b}|$

Lemma. If $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent then $[\vec{a} \cdot \vec{r} = \vec{b} \cdot \vec{r} = \vec{c} \cdot \vec{r} = 0 \longrightarrow \vec{r} = \vec{0}]$

11-9. If $(\vec{a}, \vec{b}, \vec{c})$ is linearly independent then $[(\vec{a} \cdot \vec{r} = \vec{a} \cdot \vec{s}, \vec{b} \cdot \vec{r} = \vec{b} \cdot \vec{s}$, and $\vec{c} \cdot \vec{r} = \vec{c} \cdot \vec{s}) \longrightarrow \vec{r} = \vec{s}]$.

11-10. $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis $\longrightarrow \vec{r} = \text{proj}_{[\vec{a}]}(\vec{r}) + \text{proj}_{[\vec{b}]}(\vec{r}) + \text{proj}_{[\vec{c}]}(\vec{r})$

11-11. $(\vec{a}, \vec{b}, \vec{c})$ is an orthonormal basis $\longrightarrow \vec{r} = \vec{a}(\vec{r} \cdot \vec{a}) + \vec{b}(\vec{r} \cdot \vec{b}) + \vec{c}(\vec{r} \cdot \vec{c})$

11-12. Given an orthonormal basis $(\vec{i}, \vec{j}, \vec{k})$, if $\vec{a} = \vec{i}a_1 + \vec{j}a_2 + \vec{k}a_3$ and $\vec{b} = \vec{i}b_1 + \vec{j}b_2 + \vec{k}b_3$ then

(a) $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$, and

(b) $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

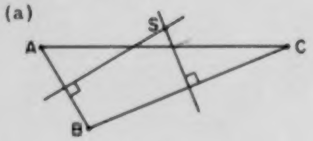
Chapter Test

1. Prove that if $\vec{a}$ and $\vec{b}$ are proper vectors and $\vec{b} \in [\vec{a}]^\perp$ then $(\vec{a}, \vec{b})$ is linearly independent.
2. Suppose that $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis for $\mathcal{T}$, $\vec{r} = \vec{a}r_1 + \vec{b}r_2 + \vec{c}r_3$, and $\vec{s} = \vec{a}s_1 + \vec{b}s_2 + \vec{c}s_3$. Complete these sentences.
   (a) $\text{proj}_{[\vec{a}]}(\vec{r}) : \vec{a} =$ ________
   (b) $\text{proj}_{[\vec{b}]}(\vec{s}) \in$ ________
   (c) $\text{proj}_{[\vec{c}]}(\vec{r} - \vec{s}) =$ ________
   (d) The unit vector in the sense of $-\vec{r}$ is ________
3. Suppose that $\overleftrightarrow{ABC}$ is a plane, that $S \in \overleftrightarrow{ABC}$, and that $d(A, S) = d(B, S) = d(C, S)$.
   (a) Draw a picture of these conditions.
   (b) $\text{proj}_{[C - B]}(S - B) =$ ________
   (c) Given that the point D, $D \ne S$, is equidistant from A, B, and C, then $[S - D]^\perp =$ ________.
   (d) Describe the set of points equidistant from A, B, and C.
4. For each of the following sentences, (1) give a counterexample to show that, as stated, the sentence is not true for all translations and, (2) state a restriction under which the sentence should be a nontrivial theorem.
   (a) $\|\vec{a} - \vec{b}\| > \|\vec{a}\| - \|\vec{b}\|$
   (b) $\|\vec{a}\| = \|\vec{b}\| \longrightarrow \vec{a} \in [\vec{b}]$
   (c) $|\vec{a} \cdot \vec{b}| < \|\vec{a}\|\,\|\vec{b}\|$
5. Assume that $(\vec{i}, \vec{j}, \vec{k})$ is an orthonormal basis for $\mathcal{T}$. Suppose that $\vec{a} = \vec{i} + \vec{j} \cdot -2 + \vec{k}$ and $\vec{b} = \vec{i} + \vec{j}2 + \vec{k}3$.
   (a) Compute $\vec{a} \cdot \vec{b}$.
   (b) Compute $\|\vec{b} - \vec{a}\|$.
   (c) Express $\vec{b}/\|\vec{b}\|$ as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Answers for Chapter Test

1. Suppose that $\vec{a}$ and $\vec{b}$ are proper vectors and that $\vec{b} \in [\vec{a}]^\perp$. Let a and b be numbers such that $\vec{a}a + \vec{b}b = \vec{0}$. Since $\vec{b} \in [\vec{a}]^\perp$, $\vec{b} \cdot \vec{a} = 0$. So, $0 = \vec{b} \cdot \vec{0} = \vec{b} \cdot (\vec{a}a + \vec{b}b) = (\vec{b} \cdot \vec{a})a + (\vec{b} \cdot \vec{b})b = (\vec{b} \cdot \vec{b})b$. Since $\vec{b} \ne \vec{0}$, $b = 0$. So, $\vec{a}a = \vec{0}$ and, since $\vec{a} \ne \vec{0}$, $a = 0$. Thus, both a and b are zero. Hence, $(\vec{a}, \vec{b})$ is linearly independent.

   [Here is an alternate, and shorter, proof: Suppose that $\vec{a}$ and $\vec{b}$ are proper vectors and that $(\vec{a}, \vec{b})$ is linearly dependent. Then, $\vec{b} = \vec{a}b$ for some $b \ne 0$. So, $\vec{b} \cdot \vec{a} = (\vec{a}b) \cdot \vec{a} = (\vec{a} \cdot \vec{a})b$. Since neither $\vec{a} \cdot \vec{a}$ nor b is 0, $(\vec{a} \cdot \vec{a})b \ne 0$ and, so, $\vec{b} \cdot \vec{a} \ne 0$. Thus, $\vec{b} \notin [\vec{a}]^\perp$. Hence, if $(\vec{a}, \vec{b})$ is linearly dependent then $\vec{b} \notin [\vec{a}]^\perp$. By contraposition, we obtain the required result.
2. (a) r_1
   (b) $[\vec{b}]$ [or: $\mathcal{T}$]
   (c) $\vec{c}(r_3 - s_3)$
   (d) $-\vec{r}/\|\vec{r}\|$
3. (a)



   S is the point of $\overleftrightarrow{ABC}$ which is common to the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$.
   (b) $(C - B)\frac{1}{2}$
   (c) $[B - A, C - A]$ [or: $[\overleftrightarrow{ABC}]$]
   (d) The set of points equidistant from A, B, and C consists of the line which contains S and is perpendicular to $\overleftrightarrow{ABC}$.
4. (a) (1) Not true when $\vec{a} = \vec{b}$.
   (2) $(\vec{a}, \vec{b})$ is linearly independent.
   (b) (1) Not true unless $(\vec{a}, \vec{b})$ is linearly dependent; so, any pair of linearly independent vectors with the same norm will yield a counterexample.
   (2) $(\vec{a}, \vec{b})$ is linearly dependent.
   (c) (1) Choose any linearly dependent vectors $\vec{a}$ and $\vec{b}$.
   (2) $(\vec{a}, \vec{b})$ is linearly independent.
5. (a) $\vec{a} \cdot \vec{b} = (\vec{i} + \vec{j} \cdot -2 + \vec{k}) \cdot (\vec{i} + \vec{j}2 + \vec{k}3) = 1 - 4 + 3 = 0$
   (b) $\|\vec{b} - \vec{a}\| = \|\vec{j}4 + \vec{k}2\| = \sqrt{16 + 4} = 2\sqrt{5}$
   (c) $\vec{b}/\|\vec{b}\| = \vec{i} \cdot /\sqrt{14} + \vec{j} \cdot 2/\sqrt{14} + \vec{k} \cdot 3/\sqrt{14}$

(d) Assume that $\vec{c} = \vec{i}c_1 + \vec{j}c_2 + \vec{k}c_3$. Describe the values of 'c_1', 'c_2', and 'c_3' for which $(\vec{a}, \vec{b}, \vec{c})$ is an orthogonal basis for $\mathscr{T}$.

6. Suppose that, with respect to a coordinate system with origin O determined by an orthonormal basis, the coordinates of the points A, B, and C are $(3, -1, 4)$, $(2, 2, 0)$, and $(-1, 3, -4)$, respectively.
   (a) The components of $B - C$ are ________.
   (b) The coordinates of D such that $ABCD$ is a parallelogram are ________.
   (c) Determine whether or not $B - A$ is orthogonal to $B - C$.
   (d) Show that for M, the midpoint of $\overleftrightarrow{AC}$, $B - M \in [C - A]^{\perp}$.

Background Topic

Recall that, given an origin $O \in \mathscr{E}$, each point X has a position vector, $X - O$, with respect to O. It will be convenient to use corresponding letters, 'A' and '$\vec{a}$', 'P' and '$\vec{p}$', etc., to refer to a point and its position vector with respect to O. So, for each point X, $\vec{x} = X - O$, and $X = O + \vec{x}$. Given, in addition, a basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ for $\mathscr{T}$, each vector $\vec{x}$ has components (x_1, x_2, x_3) with respect to this basis,

$$\vec{x} = \vec{u}_1x_1 + \vec{u}_2x_2 + \vec{u}_3x_3$$

and these components are, by definition, the coordinates of the point X whose position vector is $\vec{x}$. So, the point X whose coordinates are (x_1, x_2, x_3) is given by:

$$X = O + (\vec{u}_1x_1 + \vec{u}_2x_2 + \vec{u}_3x_3)$$

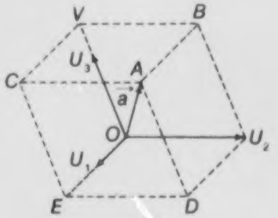


Fig. 11–16

The figure indicates a coordinate system with origin O and basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$, and shows, among others, the point A whose coordinates are $(2, 1, 3/2)$. The lines $\overleftrightarrow{O[\vec{u}_1]}$, $\overleftrightarrow{O[\vec{u}_2]}$, and $\overleftrightarrow{O[\vec{u}_3]}$ are the first, second, and third coordinate axes. The planes $\overleftrightarrow{O[\vec{u}_2, \vec{u}_3]}$, $\overleftrightarrow{O[\vec{u}_3, \vec{u}_1]}$, and $\overleftrightarrow{O[\vec{u}_1, \vec{u}_2]}$ are the first, second, and third coordinate planes.

Answers for Chapter Test [cont.]

5. (d) From part (a), we know that $\vec{a}$ and $\vec{b}$ are orthogonal. For $\vec{a}$ and $\vec{c}$, and $\vec{b}$ and $\vec{c}$, to be orthogonal, we must have $c_1 - 2c_2 + c_3 = 0$ and $c_1 + 2c_2 + 3c_3 = 0$. So, any triple $(c_1, \frac{1}{4}c_1, -\frac{1}{2}c_1)$, for some c_1, will be such that $\vec{a}$ is orthogonal to $\vec{c}$ and $\vec{b}$ is orthogonal to $\vec{c}$.

6. (a) $(3, -1, 4)$
   (b) $(0, 0, 0)$
   (c) $B - A$ has components $(-1, 3, -4)$. So, $(B - A) \cdot (B - C) = -1 \cdot 3 + 3 \cdot -1 + -4 \cdot 4 = -22 \neq 0$. So, $(B - A) \not\perp (B - C)$.
   (d) $B - M$ has components $(1, 1, 0)$ and $C - A$ has components $(-4, 4, -8)$. So, $(B - M) \cdot (C - A) = 1 \cdot -4 + 1 \cdot 4 + 0 \cdot -8 = 0$. Thus, by definition, $B - M \in [C - A]^{\perp}$.

Exercises

Part A

The following questions refer to Fig. 11–16. The coordinates of A are $(2, 1, 3/2)$.

1. (a) What are the coordinates of U_1? Of E?
   (b) What can you say about the coordinates of any point of the first coordinate axis?
   (c) What can you say about the location of any point whose first and third coordinates are 0?
2. (a) What are the coordinates of B? Of C? Of D?
   (b) What can you say about the coordinates of any point in the third coordinate plane?
   (c) What can you say about the location of any point whose third coordinate is 3/2? [*Hint*: The figure contains notation which you can use in giving a quick answer.]
   (d) Explain why it is reasonable to consider the equation '$x_3 = 3/2$' to be an equation of the plane $\overleftrightarrow{ABC}$.
3. The bidirection of the plane containing the points A, U_1, and U_2 is $[\overrightarrow{p}, \overrightarrow{q}]$, where $\overrightarrow{p} = U_1 - A$ and $\overrightarrow{q} = U_2 - A$. So, the plane $\overleftrightarrow{AU_1U_2}$ is $A[\overrightarrow{p}, \overrightarrow{q}]$. Hence, a point X belongs to this plane if and only if there are numbers r and s such that $X = A + \overrightarrow{p}r + \overrightarrow{q}s$. [Explain.]
   (a) What are the components of $\overrightarrow{p}$? Of $\overrightarrow{q}$?
   (b) Explain why the equations:

$$\begin{cases} x_1 = 2 - r - 2s \\ x_2 = 1 - r \\ x_3 = \frac{3}{2} - \frac{3}{2}r - \frac{3}{2}s \end{cases}$$

   are said to be parametric equations for $\overleftrightarrow{AU_1U_2}$.
4. We are dealing with a "big" coordinate system, for all of $\mathscr{E}$, determined by O and $(\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3})$. But, the equations in Exercise 3 suggest a "small" coordinate system for $\overleftrightarrow{AU_1U_2}$, determined by using A as the origin and $(\overrightarrow{p}, \overrightarrow{q})$ as a basis.
   (a) What values for 'r' and 's' give the coordinates of A, the origin of the "small" coordinate system? Of U_1? Of U_2?
   (b) There is a point of $\overleftrightarrow{AU_1U_2}$ whose coordinates in the "small" coordinate system are $(2, -3)$. What are the coordinates of this point in the "large" coordinate system?
   (c) The point with coordinates $(-1, 2, 0)$ happens to belong to $\overleftrightarrow{AU_1U_2}$. Use the equations in Exercise 3(b) to show that it does and to find its coordinates in the "small" system.
   (d) Determine which of the points whose coordinates are given here belong to $\overleftrightarrow{AU_1U_2}$. For those which belong to $\overleftrightarrow{AU_1U_2}$, find their coordinates in the "small" system.
   $(3, 2, 3)$, $(-4, -1, -9)$, $(-4, -1, -\frac{9}{2})$
   $(2, 1, \frac{3}{2})$, $(5, 2, \frac{9}{2})$, $(-5, 2, 3)$

Answers for Part A

1. (a) $U_1 = O + \vec{u}_1$; so, the coordinates of U_1 are $(1, 0, 0)$.
   $E = O + \text{proj}_{[\vec{u}_1]}(\vec{a})$; so, the coordinates of E are $(2, 0, 0)$.
   (b) The coordinates are $(a, 0, 0)$, for some $a \in \mathcal{R}$.
   (c) It is on the second coordinate axis.
2. (a) $B = O + \text{proj}_{[\vec{u}_2]}(\vec{a}) + \text{proj}_{[\vec{u}_3]}(\vec{a})$; so the coordinates of B are $(0, 1, 3/2)$.
   $C = O + \text{proj}_{[\vec{u}_1]}(\vec{a}) + \text{proj}_{[\vec{u}_3]}(\vec{a})$; so the coordinates of C are $(2, 0, 3/2)$.
   $D = O + \text{proj}_{[\vec{u}_1]}(\vec{a}) + \text{proj}_{[\vec{u}_2]}(\vec{a})$; so the coordinates of D are $(2, 1, 0)$.
   (b) The coordinates are $(p, q, 0)$ for some p and q.
   (c) If the coordinates of a point P are $(p, q, 3/2)$, for some p and q, then $P \in \overleftrightarrow{ABC}$.
   (d) $\overleftrightarrow{ABC}$ is the set of all points with third coordinate 3/2. That is, given that P has coordinates (p_1, p_2, p_3), $P \in \overleftrightarrow{ABC}$ if and only if $p_3 = 3/2$. So, in this sense, the equation '$x_3 = 3/2$' is an equation of the plane $\overleftrightarrow{ABC}$.
3. (a) Since A, U_1, and U_2 have coordinates $(2, 1, 3/2)$, $(1, 0, 0)$, and $(0, 1, 0)$, respectively, $\vec{p}$ has components $(-1, -1, -3/2)$; $\vec{q}$ has components $(-2, 0, -3/2)$.
   (b) The numbers x_1, x_2, and x_3 are coordinates of a point of $\overleftrightarrow{AU_1U_2}$ if and only if they are obtainable from these equations for properly chosen values of 'r' and 's'. [$X \in \overleftrightarrow{AU_1U_2}$ if and only if $X = A + \vec{p}r + \vec{q}s$ for some r and s, and this is the case if and only if $x_1 = a_1 + p_1r + q_1s$, etc. — that is, if and only if $x_1 = 2 + -1r + -2s$, etc.]
4. (a) $r = 0$ and $s = 0$ [for, then, $x_1 = 2$, $x_2 = 2$, and $x_3 = 3/2$];
   $r = 1$ and $s = 0$ [for, then, $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$];
   $r = 0$ and $s = 1$ [for, then, $x_1 = 0$, $x_2 = 1$, and $x_3 = 0$].
   (b) The given point is such that $r = 2$ and $s = -3$. So, $x_1 = 6$, $x_2 = -1$, and $x_3 = 3$. So, the given point has coordinates $(6, -1, 3)$ in the large coordinate system.
   (c) The point with coordinates $(-1, 2, 0)$ belongs to $\overleftrightarrow{AU_1U_2}$ if and only if $-1 = 2 - r - 2s$, $2 = 1 - r$, and $0 = \frac{3}{2} - \frac{3}{2}r - \frac{3}{2}s$, for some r and s. The latter equations are satisfied if and only if $r = -1$ and $s = 2$. So, there are numbers r and s which satisfy the required equations and this shows that $(-1, 2, 0)$ are the coordinates of a point in $\overleftrightarrow{AU_1U_2}$. Also, this point has coordinates $(-1, 2)$ in the "small" system.

Answers for Part A [cont.]

4. (d) (3, 2, 3) — Yes [choose r = -1 and s = 0]; (-1, 0)
(-4, -1, -9) — No [for the equations '$-4 = 2 - r - 2s$', '$-1 = 1 - r$', and '$-9 = \frac{3}{2} - \frac{3}{2}r - \frac{3}{2}s$' have no common solution].
$(-4, -1, -\frac{9}{2})$ — Yes [choose r = 2 and s = 2]; (2, 2)
(2, 1, 3/2) — Yes [choose r = 0 and s = 0]; (0, 0)
(5, 2, 9/2) — Yes [choose r = -1 and s = -1]; (-1, -1)
(-5, 2, 3) — No [for the equations '$-5 = 2 - r - 2s$', '$2 = 1 - r$', and '$3 = \frac{3}{2} - \frac{3}{2}r - \frac{3}{2}s$' have no common solution].

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5. The solutions to the equations in 3(b) for any chosen pair (r, s) are the coordinates of a point in $\overleftrightarrow{AU_1U_2}$. Noting that the second of those equations is equivalent to '$r = 1 - x_2$', and using this with the other equations, we obtain the equivalent system:

$$\begin{cases} x_1 - x_2 = 1 - 2s \\ 2x_3 - 3x_2 = -3s \end{cases}$$

Solving the second of these equations for 's' and substituting in the first of the equations yields the equivalent equation:

$$x_1 - x_2 = 1 - 2(x_2 - \tfrac{2}{3}x_3)$$

Simplifying the latter we obtain:

$$x_1 + x_2 - \tfrac{4}{3}x_3 = 1$$

5. The equations in Exercise 3(b) can be used to find coordinates of as many points as you wish in $\overleftrightarrow{AU_1U_2}$. [How?] They can also be used to find a single equation in 'x_1', 'x_2', and 'x_3' [but not 'r' and 's'] which is satisfied by the coordinates of just those points which belong to $\overleftrightarrow{AU_1U_2}$. Find such an equation. [*Hint*: Your work in Exercise 4(c) may suggest how.]

6. In Volume 1 you learned that any equation like:

$$(*) \qquad a_1x_1 + a_2x_2 + a_3x_3 = c \qquad [(a_1, a_2, a_3) \neq (0, 0, 0)]$$

is an equation of some plane.

(a) Use what you know about the representation of planes by parametric equations to show that for $a_1 \neq 0$, (*) represents a plane. [*Hint*: Solve (*) for 'x_1' and take 'x_2' and 'x_3' as parameters.]

(b) From your work in (a), obtain the coordinates of a point on the first coordinate axis which belongs to the plane π which is represented by (*). Also, obtain the components of vectors $\vec{p}$ and $\vec{q}$ such that $[\pi] = [\vec{p}, \vec{q}]$.

(c) Here is an equation like (*):

$$x_1 + 2x_2 = 2$$

which is somewhat special. Still, since it is like (*), it represents a plane. What is special about this plane? [*Hint*: The plane is related in a special way to one of the coordinate axes. It may help you to note that just four of the labeled points in Fig. 11–16 belong to this plane. Which four?]

(d) Make a conjecture concerning planes represented by equations like (*) in which $a_3 = 0$. Verify your conjecture. [*Hint*: A point belongs to the third coordinate axis just if its first and second coordinates are both 0. Which of these points satisfy an equation like '$a_1x_1 + a_2x_2 = c$' if the value of 'c' is not 0? If the value of 'c' is 0?]

(e) What can you say about a plane represented by an equation like (*) in which $a_2 = 0$? In which both $a_2 = 0$ and $a_3 = 0$?

(f) Write at least six equations like (*) in each of which either one or two of 'a_1', 'a_2', and 'a_3' has the value 0 [but not all three have the value 0]. For each equation, name the coordinate axes and the coordinate planes [if any] which are parallel to the plane represented by that equation.

*

In choosing a coordinate system it is often—but not always—preferable to choose the basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ to be orthonormal. We shall fre-

Answers for Part A [cont.]

6. (a) For $a_1 \neq 0$, (*) is equivalent to:

$$x_1 = c/a_1 - (a_2/a_1)x_2 - (a_3/a_1)x_3$$

and, so, to the system:

$$\begin{cases} x_1 = c/a_1 - (a_2/a_1)s - (a_3/a_1)t \\ x_2 = s \\ x_3 = t \end{cases}$$

Since the latter system is a [parametric] representation of a plane it follows that, for $a_1 \neq 0$, (*) represents a plane.

(b) $(c/a_1, 0, 0)$ [for, any point on the first coordinate axis has coordinates $(p, 0, 0)$, for some p]

Let $\vec{p}$ and $\vec{q}$ be vectors whose components are $(-a_2/a_1, 1, 0)$ and $(-a_3/a_1, 0, 1)$, respectively. Then $[\pi] = [\vec{p}, \vec{q}]$.

(c) It is parallel to the third coordinate axis. [Looking at Figure 11-16, the points in question are E, U_2, B, and C.]

(d) Any such plane has an equation like:

$$a_1x_1 + a_2x_2 = c \quad [(a_1, a_2) \neq (0, 0)]$$

This plane is parallel to the third coordinate axis. If $c = 0$ then the plane contains each point of the third coordinate axis; if $c \neq 0$ then the plane contains no point of the third coordinate axis. In either case the plane is parallel to the third coordinate axis.

(e) Parallel to the second coordinate axis; parallel to both the second and third coordinate axes — and, so, to the first coordinate plane.

(f) [There is obviously a great variety of answers. We give but a few.]

$x_2 + x_3 = 5$; parallel to the first coordinate axis, does not contain the origin, and is not parallel to a coordinate plane

$x_2 + x_3 = 0$; parallel to the first coordinate axis, contains the origin, and is not parallel to a coordinate plane

$x_2 = 5$; parallel to both the first and third coordinate axes and, so, is parallel to the second coordinate plane

$x_1 = 0$; parallel to both the second and third coordinate axes and, so, is parallel to the first coordinate plane [In fact, this equation describes the first coordinate plane.]

quently have occasion to do this, and to make appropriate drawings. In making such drawings you will find it helpful to use squared paper [¼" squares are about right]. Also, the following conventions are helpful:

(1) Represent the first coordinate axis by a line along the diagonals of the squares, from lower left to upper right.

(2) Represent the second and third coordinate axes by a horizontal line and a vertical line, respectively.

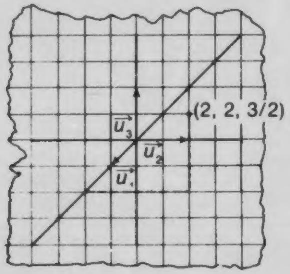

Fig. 11–17

In the resulting drawing you are seeing the first coordinate plane "head on", but the first coordinate axis is pointing below and to your left and, so, your view of it is foreshortened. So, since $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ have the same norm, $\vec{u}_2$ and $\vec{u}_3$ should be represented by arrows of the same length and $\vec{u}_1$ should be pictured by a shorter arrow:

(3) If you use n times the side of a square as the length of the arrows for $\vec{u}_2$ and $\vec{u}_3$, use $n/2$ times the diagonal of a square as the length of the arrow for $\vec{u}_1$. [In the figure, we have chosen $n = 2$.]

In the figure, we have indicated the point whose coordinates are $(2, 2, \frac{3}{2})$. Note the use of dashed lines to help show what point is meant. The point with coordinates $(1, \frac{3}{2}, 1)$ could be represented by the same dot. Give the coordinates of another point represented by this dot. What are the coordinates of a point in the first coordinate plane which is represented by this dot? Of such a point in the third coordinate plane? This same dot can be used to represent any point on a certain line. [You are seeing the line "end on".] Try to find parametric equations for this line. [*Hint*: Find coordinates of several points on the line, and look for a pattern.]

With reference to the figure, a point whose coordinates are all positive can be described as being in front of the first coordinate plane, to the right of the second, and above the third. Where is a point whose third coordinate is negative? Whose first and second coordinates are

Drawing other dashed lines to end at the same dot one finds that (2, 2, 3/2), (1, 3/2, 1) (0 1, 1/2), and (-1, 1/2, 0) are all represented by the same dot. The fairly obvious pattern suggests that all points of the line whose parametric equations are:

$$\begin{cases} x_1 = 2 - r \\ x_2 = 2 - r/2 \\ x_3 = \frac{3}{2} - r/2 \end{cases}$$

are represented by this same dot. Generalizing, each possible dot represents all points on the line whose equations, for some values of 'a_1', 'a_2', 'a_3', are:

$$\begin{cases} x_1 = a_1 - r \\ x_2 = a_2 - r/2 \\ x_3 = a_3 - r/2 \end{cases}$$

A point whose third coordinate is negative is below the third coordinate plane; one whose first and second coordinates are negative is behind the first coordinate plane and to the left of the second coordinate plane.

You might remark that the eight regions into which a set of coordinate planes divides $\mathcal{E}$ are called octants.

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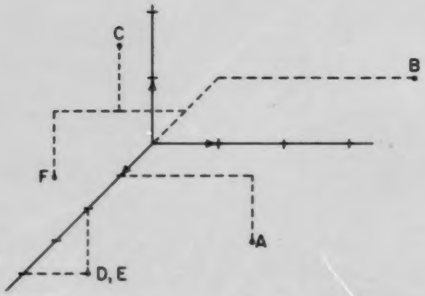
Answers for Part B

1. Here is an appropriate picture for Exercise 1:



both negative? Choose three numbers at random and describe the location of a point whose coordinates are these numbers, in the order chosen. While doing this, look at the figure and imagine where the corresponding dot should be placed. Repeat this "exercise" until you feel quite familiar with this way of picturing points.

Part B

1. Draw an orthonormal coordinate system and mark and label a dot for each of the points whose coordinates are given below. Draw dashed lines to help "bring out" the picture. [As a help in choosing the scale for your picture, note first all the numbers given as coordinates.]

A: (1, 2, −1) B: (−2, 3, 0) C: (−1, −1, 1)
D: (2, 0, −1) E: (4, 1, 0) F: (−1, −2, −1)

2. **(a)** Draw an orthonormal coordinate system and, on your drawing, plot the points A: (3, 0, 0), B: (0, 1, 0), and C: (0, 0, 2). Draw $\triangle ABC$.

(b) As you did in Exercise 3 of Part A, find parametric equations for $\overleftrightarrow{ABC}$.

(c) As in Exercise 5 of Part A, find a single equation for $\overleftrightarrow{ABC}$.

(d) The nonzero coordinates of the points in which a plane intersects the coordinate axes are called *the intercepts* of the plane, with respect to the coordinate system. For example, the first intercept of $\overleftrightarrow{ABC}$ is 3. What are its other intercepts?

(e) Transform the equation you obtained in part (c) into the form:

$$(*) \qquad \frac{x_1}{a} + \frac{x_2}{b} + \frac{x_3}{c} = 1$$

Make a conjecture as to how to obtain an equation for a plane whose intercepts you know. Verify your conjecture. [*Hint*: You know from Volume 1 that any equation like (*), with nonzero values of 'a', 'b', and 'c', represent a plane. You also know that a plane is determined by any three of its points which are noncollinear.]

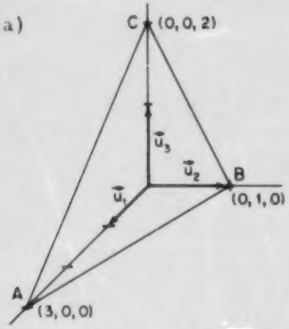
3. **(a)** On the picture you drew for Exercise 2, plot the points D: (5, 0, 0), E: (0, 3, 0), and F: (0, 0, 3), and draw $\triangle DEF$.

(b) Use what you learned in Exercise 2 to write an equation for $\overleftrightarrow{DEF}$. Then, simplify it by "clearing of fractions".

(c) Draw the line in which $\overleftrightarrow{ABC}$ and $\overleftrightarrow{DEF}$ intersect. [*Hint*: The line in which a given plane intersects a coordinate plane is called *the trace* of the given plane on that coordinate plane. A point which is common to the traces of two planes on some coordinate plane belongs, of course, to the intersection of the given planes.]

Answers for Part B [cont.]

2. (a)



(b) We know that $X \in \overline{ABC}$ if and only if $X = A + (B - A)r + (C - A)s$, for some r and s. So, parametric equations for $\overline{ABC}$ are:

$$\begin{cases} x_1 = 3 - 3r - 3s \\ x_2 = r \\ x_3 = 2s \end{cases}$$

(c) From (b) we see that $x_1 = 3 - 3x_2 - 3 \cdot x_3/2$. So, a single equation for $\overline{ABC}$ is:

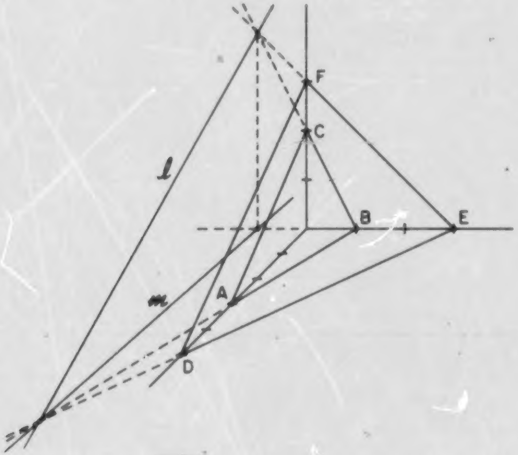
$$x_1 + 3x_2 + \frac{3}{2}x_3 = 3$$

(d) 1 and 2

(e) $\frac{x_1}{3} + \frac{x_2}{1} + \frac{x_3}{2} = 1$

Given that the plane is known to have intercepts a, b, and c, respectively, then an equation for the plane is (*). To verify this, simply note that (a, 0, 0), (0, b, 0), and (0, 0, c) satisfy (*) and are the coordinates of three noncollinear points.

3. (a), (c), (d)



(b) Since the intercepts of $\overline{DEF}$ are 5, 3, and 3 an equation for this plane is:

$$\frac{x_1}{5} + \frac{x_2}{3} + \frac{x_3}{3} = 1$$

which simplifies to: $3x_1 + 5x_2 + 5x_3 = 15$

Answers for Part B [cont.]

4. Since ℓ is in both $\overline{ABC}$ and $\overline{DEF}$, the coordinates of the points of ℓ must satisfy the equations for $\overline{ABC}$ and for $\overline{DEF}$. And, the points common to $\overline{ABC}$ and $\overline{DEF}$ must be precisely those whose coordinates satisfy both equations.

(a) Suppose that (x_1, x_2, x_3) satisfies both of the given equations. Then, $2x_1 + 6x_2 + 3x_3 = 6$ and $3x_1 + 5x_2 + 5x_3 = 15$, so that $(2x_1 + 6x_2 + 3x_3)a = 6a$ and $(3x_1 + 5x_2 + 5x_3)b = 15b$. Thus,

$$(2x_1 + 6x_2 + 3x_3)a + (3x_1 + 5x_2 + 5x_3)b = 6a + 15b.$$

(b) The equation of part (a) represents a plane unless $2a + 3b = 0$, $6a + 5b = 0$, and $3a + 5b = 0$. It is obvious that the last two equations are equivalent to '$a = 0$ and $b = 0$'. So, unless a and b are both 0 the equation of part (a) represents a plane. In any case ℓ is a subset of the set represented by the equation of part (a).

(c) An equation for such a plane is obtained by choosing a and b not both 0 and so that $3a + 5b = 0$. The most obvious choice is $a = 5$, $b = -3$. The resulting equation is '$x_1 + 15x_2 = -15$'. Other choices of a and b yield other equations for the same plane.

(d) Since the given coordinate system is orthonormal, any such plane is perpendicular to the third coordinate plane.

(e) The plane which contains ℓ and is perpendicular to the second coordinate plane is such that $6a + 5b = 0$, for some nonzero a and b. Using this and the equation from 4(b), we see that an appropriate equation is:

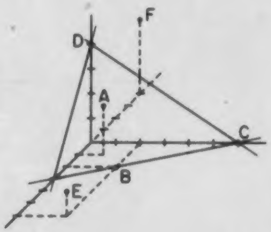
$$8x_1 + 15x_3 = 60$$

(f) Using the equation from 4(b) and the fact that 'a' and 'b' must be chosen so that $2a + 3b = 0$, we see that an appropriate equation is:

$$8x_2 - x_3 = -12$$

Answers for Part C

1. Here is an appropriate picture for Exercises 1 and 2(e).



(d) Draw the orthogonal projection in the third coordinate plane, of the line of intersection of $\overleftrightarrow{ABC}$ and $\overleftrightarrow{DEF}$. [*Hint*: Locate the orthogonal projections of two points of $\overleftrightarrow{ABC} \cap \overleftrightarrow{DEF}$.]

4. In Exercises 2 and 3 you should have found equations equivalent to:

$$2x_1 + 6x_2 + 3x_3 = 6 \qquad [\text{for } \overleftrightarrow{ABC}]$$
$$3x_1 + 5x_2 + 5x_3 = 15 \qquad [\text{for } \overleftrightarrow{DEF}]$$

The line l in which $\overleftrightarrow{ABC}$ and $\overleftrightarrow{DEF}$ intersect consists of just those points whose coordinates satisfy both these equations. [Explain.] There are, of course, many other planes which contain l.

(a) Show that, for any values of 'a' and 'b' the equation:

$$a(2x_1 + 6x_2 + 3x_3) + b(3x_1 + 5x_2 + 5x_3) = 6a + 15b$$

is satisfied by all coordinates (x_1, x_2, x_3) which satisfy both the given equation for $\overleftrightarrow{ABC}$ and the given equation for $\overleftrightarrow{DEF}$.

(b) Show that unless one chooses the value 0 for both 'a' and 'b', the equation of part (a) represents a plane which contains l. [*Hint*: The equation of part (a) is equivalent to:

$$(2a + 3b)x_1 + (6a + 5b)x_2 + (3a + 5b)x_3 = 6a + 15b$$

Under what conditions would such an equation *not* represent a plane? Can these conditions be satisfied for nonzero values of 'a' and 'b'? Assuming that they could be, what would this tell you about the vectors with components (2, 6, 3) and (3, 5, 5)?]

(c) Find an equation for the plane which contains l and is parallel to the third coordinate axis. [*Hint*: Recall Exercise 6 of Part A.]

(d) How would you describe the relation of the plane of part (c) and the third coordinate plane?

(e) Find an equation for the plane which contains l and is perpendicular to the second coordinate plane.

(f) Find an equation for the plane which projects l orthogonally into the first coordinate plane.

Part C

The coordinate system referred to in these exercises is an orthonormal one with origin O and orthonormal basis $(\vec{u_1}, \vec{u_2}, \vec{u_3})$. Use as many drawings of such a coordinate system as you need to in order to illustrate your answers.

1. Draw an orthonormal coordinate system and mark and label a dot for each of the points whose coordinates are given below.

A: (1, 1, 2) B: (2, 2, 0) C: (0, 6, 0)
D: (0, 0, 4) E: (6, 2, 1) F: (−4, 0, 3)

Answers for Part C [cont.]

2. (a) $\begin{cases} x_1 = 2 - 2r - 2s \\ x_2 = 2 + 4r - 2s \\ x_3 = 4s \end{cases}$

(b) $4x_1 + 2x_2 + 3x_3 = 12$

(c) Yes, for $4 \cdot 1 + 2 \cdot 1 + 3 \cdot 2 = 12$; No, for $4 \cdot 6 + 2 \cdot 2 + 3 \cdot 1 \neq 12$.

(d) Let P be the required point. Then P has coordinates (p, 0, 0), for some p. Using the equation in 2(b), we see that p = 3. So, P has coordinates (3, 0, 0).

(e) [See picture for Exercise 1.]

3. (a) Yes, the second coordinate plane;

Yes, the first and third coordinate axes.

(b) Yes; to the first and third coordinate planes.

(c) (i) $x_2 = 2$ (ii) $\begin{cases} x_1 = 6 + r \\ x_2 = 2 \\ x_3 = 1 + s \end{cases}$ For these equations, $\vec{p}$ and $\vec{q}$ were chosen with components (1, 0, 0) and (0, 0, 1).

4. (a) Equations for $\overleftrightarrow{BCD}$ and π are:

$$4x_1 + 2x_2 + 3x_3 = 12$$
$$x_2 = 2$$

So, for any a and b such that $(a, b) \neq (0, 0)$, the equation:

$$(4x_1 + 2x_2 + 3x_3)a + x_2 b = 12a + 2b$$

is an equation for a plane which contains ℓ.

(b) Substituting the coordinates (1, 2, 3) into the equation obtained in part (a) gives us '$17a + 2b = 12a + 2b$'. This is satisfied if and only if $a = 0$. Hence, taking $b = 1$ for simplicity, the plane in question has the equation '$x_2 = 2$'.

(c) Yes. For appropriate values for 'a' and 'b' can be found by substituting the coordinates of any point which is on the given plane and is not on ℓ. [Substituting the coordinates of a point on ℓ would yield the equation '$12a + 2b = 12a + 2b$'.]

2. (a) Find parametric equations for $\overleftrightarrow{BCD}$.
   (b) Find a single equation for $\overleftrightarrow{BCD}$. [When you have found the equation, check to make sure that it is satisfied by the coordinates of B, C, and D.]
   (c) Does the point A belong to $\overleftrightarrow{BCD}$? How about E?
   (d) Find the coordinates of the point in which $\overleftrightarrow{BCD}$ intersects the first coordinate axis.
   (e) On your figure for Exercise 1, draw the lines in which $\overleftrightarrow{BCD}$ intersects the three coordinate planes.
3. Let π be the plane which contains E and is perpendicular to the second coordinate axis.
   (a) Is π parallel to any of the coordinate planes? To any of the coordinate axes?
   (b) Do you think that π is perpendicular to any of the coordinate planes? If so, to which?
   (c) Find (i) a single equation for π, and (ii) parametric equations for π. [*Hint*: Whichever of (i) and (ii) you do first should help you do the other.]
4. Let l be the line of intersection of $\overleftrightarrow{BCD}$ and π.
   (a) Write an equation which you can use to find equations of many planes which contain l. [*Hint*: See Exercise 4 of Part B.]
   (b) Use your answer for part (a) to find an equation for the plane which contains l and the point whose coordinates are (1, 2, 3).
   (c) Your answer for part (a) contains two parameters. [In Exercise 4(a) of Part B, the parameters are 'a' and 'b'.] Do you think that each plane which contains l has an equation which can be obtained from yours by choosing appropriate values for the two parameters? Explain.
5. (a) What plane containing l is parallel to the first coordinate axis? [Remember that $l = \overleftrightarrow{BCD} \cap \pi$, where π is the plane of Exercise 3.]
   (b) What plane containing l is perpendicular to the first coordinate plane? To the third coordinate plane?
   (c) Find an equation for the plane which contains l and is perpendicular to the second coordinate plane. [*Hint*: Use your answer for Exercise 4(a) and recall Exercise 6 of Part A.]
   (d) Give the intercepts of $\overleftrightarrow{BCD}$. How many intercepts does the plane of part (c) have? How many does π have?
6. (a) Show that the points A, E, and F are collinear.
   (b) Find a single equation for the plane $\overleftrightarrow{AFO}$. [O is, of course, the origin.]
   (c) Find a single equation for $\overleftrightarrow{AFC}$.
   (d) Find an equation for the plane which contains $\overleftrightarrow{AF}$ and is perpendicular to the first coordinate plane.

Answers for Part C [cont.]

5. (a) The plane π.
   (b) π
   (c) The equation in the answer for Exercise 4(a) is equivalent to:

$$4ax_1 + (2a + b)x_2 + 3ax_3 = 12a + 2b$$

An equation for the required plane is obtained by choosing a and b, not both 0, such that $2a + b = 0$. Taking $a = 1$ and $b = -2$ we obtain '$4x_1 + 3x_3 = 8$'.

   (d) Since an equation for $\overleftrightarrow{BCD}$ is:

$$\frac{x_1}{3} + \frac{x_2}{6} + \frac{x_3}{4} = 1$$

the intercepts of $\overleftrightarrow{BCD}$ are 3, 6, and 4; the plane of part (c) has two intercepts, 2 and 8/3; π has one intercept, namely, 2.

6. (a) E - A and F - A have components (5, 1, −1) and (−5, −1, 1) respectively. Clearly, then $E - A = -(F - A)$ so that $(E - A, F - A)$ is linearly dependent. Hence, $\{A, E, F\}$ is collinear.
   (b) Parametric equations for $\overleftrightarrow{AFO} = \overleftrightarrow{OAF}$ are:

$$\begin{cases} x_1 = r - 4s \\ x_2 = r \\ x_3 = 2r + 3s \end{cases}$$

So, a single equation for $\overleftrightarrow{AFO}$ is:

$$3x_1 - 11x_2 + 4x_3 = 0$$

   (c) Parametric equations for $\overleftrightarrow{AFC}$ are:

$$\begin{cases} x_1 = r - 4s \\ x_2 = 6 - 5r - 6s \\ x_3 = 2r + 3s \end{cases}$$

So, a single equation for $\overleftrightarrow{AFC}$ is:

$$3x_1 + 11x_2 + 26x_3 = 66$$

   (d) $\overleftrightarrow{AF}$ is the intersection of $\overleftrightarrow{AFO}$ and $\overleftrightarrow{AFC}$. So, any plane which contains $\overleftrightarrow{AF}$ has an equation like:

$$(3a + 3b)x_1 + (-11a + 11b)x_2 + (4a + 26b)x_3 = 66b$$

So, the required plane must be such that $3a + 3b = 0$. So, an equation for the required plane is:

$$x_2 + x_3 = 3$$

Part D

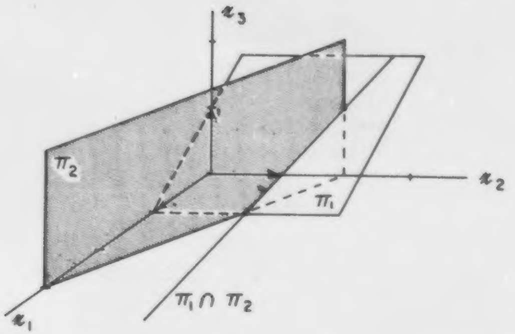
Here are the equations for planes π_1, π_2, and π_3:

$$\pi_1 : x_1 + x_3 = 1$$
$$\pi_2 : 2x_1 + 3x_2 = 6$$
$$\pi_3 : 3x_1 + 4x_2 + 4x_3 = 12$$

1. (a) Make sketches of π_1 and π_2 on the same picture of an orthogonal coordinate system.
   (b) Draw the line of intersection of π_1 and π_2.
   (c) Write an equation for the plane which contains $\pi_1 \cap \pi_2$ and is perpendicular to the first coordinate plane.
2. (a) Make sketches of π_1 and π_3 on the same picture of an orthogonal coordinate system. [Use a different picture of the coordinate system than the one drawn for Exercise 1.]
   (b) Draw the line of intersection of π_1 and π_3.
   (c) Write an equation for the plane which contains $\pi_1 \cap \pi_3$ and is perpendicular to the second coordinate plane.
   (d) How many planes contain $\pi_1 \cap \pi_3$ and are perpendicular to the third coordinate plane?
   (e) Find an equation for the plane which contains $\pi_1 \cap \pi_3$ and the point whose coordinates are (3, −2, 3).
3. Repeat Exercise 1 for π_2 and π_3.

Answers for Part D

1. (a), (b)



(c) A plane containing $\pi_1 \cap \pi_2$ has an equation of the form:

$$(x_1 + x_3)a + (2x_1 + 3x_2)b = a + 6b$$

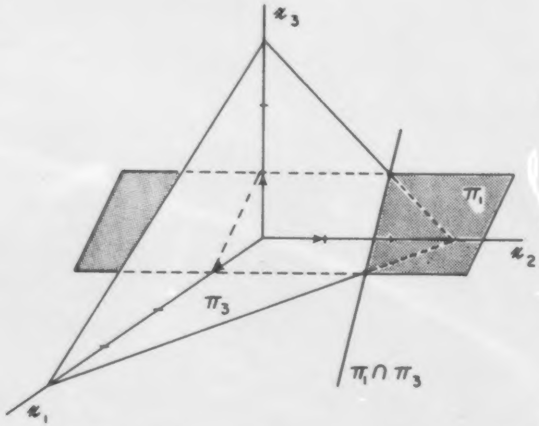
so that

$$(a + 2b)x_1 + 3bx_2 + ax_3 = a + 6b.$$

Such a plane is perpendicular to the first coordinate plane if $a + 2b = 0$. In this event, an equation of such a plane is:

$$3x_2 - 2x_3 = 4$$

2. (a), (b)



(c) The plane which contains $\pi_1 \cap \pi_3$ and is perpendicular to the second coordinate plane is such that

$$(a + 3b)x_1 + 4bx_2 + (a + 4b)x_3 = a + 12b$$

where $4b = 0$. So, an equation for this plane is:

$$x_1 + x_3 = 1$$

[That such a plane would have an equation the same as that given for π_1 may occur to some students prior to working the problem.]

TC 73 (2)

Answers for Part D [cont.]

2. (d) Any plane which contains $\pi_1 \cap \pi_3$ and is perpendicular to the third coordinate plane is such that

$$(a + 3b)x_1 + 4bx_2 + (a + 4b)x_3 = a + 12b$$

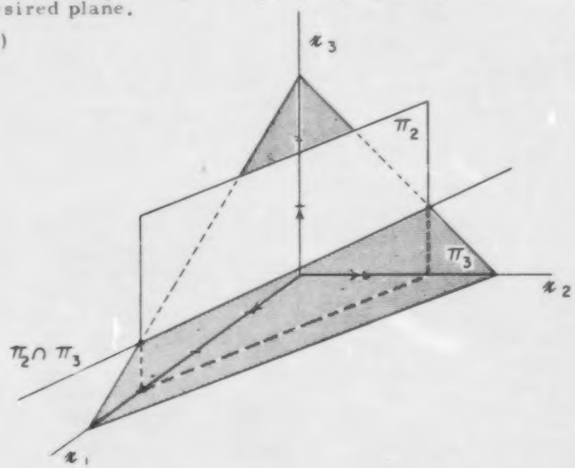
where $a + 4b = 0$; hence

$$x_1 - 4x_2 = -8.$$

Therefore there is one and only one plane.

(e) An equation of the desired plane is to be obtained from '$(2x_1 + 3x_2)a + (3x_1 + 4x_2 + 4x_3)b = 6a + 12b$'. Substituting the given coordinates we obtain '$13b = 6a + 12b$' and, so, '$6a - b = 0$'. Choosing $a = 1$ and $b = 6$ and simplifying we obtain '$20x_1 + 27x_2 + 24x_3 = 78$' as an equation of the desired plane.

3. (a), (b)



(c) The plane which contains $\pi_2 \cap \pi_3$ and is perpendicular to the first coordinate plane is such that $(2a + 3b)x_1 + (3a + 4b)x_2 + 4bx_3 = 6a + 12b$ where $2a + 3b = 0$. So, an equation of this plane is:

$$x_2 - 8x_3 = -6$$

TC 74

Answer to questions. Yes, the converse of (3) is a theorem. As a matter of fact, it is a principle of logic. Here is a paragraph proof of the converse of (3):

Suppose that $[m] = [\ell]$. Then, since $[m]^{\perp} = [m]^{\perp}$, it follows by the replacement rule for equations that $[m]^{\perp} = [\ell]^{\perp}$. Hence, if $[m] = [\ell]$ then $[m]^{\perp} = [\ell]^{\perp}$.

Yes, the converse of (4) is a theorem. In fact, the converse of (4) follows directly from (4) by substitution.

Chapter Twelve
Perpendicularity

12.01 Perpendicularity Between Planes and Lines

In the preceding chapter we investigated some intuitive notions concerning the perpendicularity of planes to lines and the distance between points. This investigation eventually led us to the operation of dot multiplication of translations and to new Postulates 4_0(e) and $4_{11}-4_{14}$ concerning this operation. Our investigation had also led us to the idea of the orthogonal complement of a direction. On the basis of Notions 1–4 concerning perpendicularity we were able to show that

(1) $[l]^{\perp}$ is a proper bidirection,

(2) $[l] \subseteq [l]^{\perp}$,

(3) $[m]^{\perp} = [l]^{\perp} \longrightarrow [m] = [l]$,

(4) $[l] \subseteq [m]^{\perp} \longrightarrow [m] \subseteq [l]^{\perp}$, and

(*) $\pi \perp l \longleftrightarrow [\pi] = [l]^{\perp}$.

[Is the converse of (3) a theorem? If so, prove it; if not, explain. Answer a similar question regarding (4).] Conversely, we were able to show [in Section 11.01] that if we adopted (*) as a definition then we could derive Notions 1–4 from sentences (1)–(4) and earlier theorems. Finally, after adopting our new postulates we were able to define orthogonal complements [in terms of dot multiplication] in such a way that (1)–(4) are now theorems. [See Theorem 11–4.] So, Notions 1–4 are theorems, also, once we adopt (*) as a definition. In this section we shall adopt this definition and shall both review and extend our knowledge concerning perpendicularity. First, the definition:

Definition 12–1 (a) $\pi \perp l \longleftrightarrow [\pi] = [l]^{\perp}$
(b) $l \perp \pi \longleftrightarrow \pi \perp l$

[Part (b) is introduced merely to give us freedom to speak of a line as being perpendicular to a plane, as well as of a plane as being perpendicular to a line.]

To begin with we note that, since $l \parallel \pi$ if and only if $[l] \subseteq [\pi]$, and in view of Definition 12–1(a), the theorem (2) amounts to the statement that if $\pi \perp l$ then $l \nparallel \pi$. This is Notion 1 of Section 11.01. Since $\mathscr{E}$ is 3-dimensional we can reformulate this as:

Theorem 12–1 $\pi \perp l \Longrightarrow l$ is a transversal of π

Since parallel planes are planes which have the same bidirection, and parallel lines are lines which have the same direction, Definition 12–1 yields:

Theorem 12–2 $(\sigma \parallel \pi \text{ and } m \parallel l) \Longrightarrow [\sigma \perp m \Longleftrightarrow \pi \perp l]$

As a special case of Theorem 12–2 we have:

Corollary Parallel planes are perpendicular to the same lines and parallel lines are perpendicular to the same planes.

The proof of Theorem 12–2 depends on the converse of the theorem (3)–but, this is a principle of logic. Using both it and (3), Definition 12–1 yields:

Theorem 12–3 $(\sigma \perp m \text{ and } \pi \perp l) \Longrightarrow [\sigma \parallel \pi \Longleftrightarrow m \parallel l]$

In words:

Corollary Planes perpendicular to parallel lines are parallel and lines perpendicular to parallel planes are parallel.

Note that by combining the second parts of these two corollaries we obtain Notion 3 of Section 11.01:

$$(**) \qquad \pi \perp l \Longrightarrow [\pi \perp m \Longleftrightarrow m \parallel l]$$

Notion 4 of Section 11.01 amounts–as is shown there–to the theorem (4) on page 74. The latter can be restated in a theorem like Theorem 12–3:

Theorem 12–4 $(m \perp \sigma \text{ and } l \perp \pi) \Longrightarrow [l \parallel \sigma \Longleftrightarrow m \parallel \pi]$

The four preceding theorems tell us something of what results *if* given lines and planes are perpendicular. They leave open two important questions:

> Given a line, are there any planes perpendicular to it?
> Given a plane, are there any lines perpendicular to it?

As we know from Section 11.01 we have good intuitive evidence that the answer to each of these questions is 'Yes.'. The problem, now, is whether our postulates are adequate to show this. The key to the first is the theorem (1) on page 74. Using this, Theorem 9–11(a), and Definition 12–1, it is easy to prove:

Theorem 12–5 $\overline{P[l]^{\perp}}$ is the plane which contains P and is perpendicular to l.

Corollary There is one and only one plane which contains a given point and is perpendicular to a given line.

Note that the corollary is Notion 2 of Section 11.01. So, we have accounted for all four of our initial intuitions about perpendicularity.

The second question is answered affirmatively by:

Theorem 12–6 Planes which are perpendicular to two intersecting lines intersect in lines which are perpendicular to the plane of the two lines.

Since the argument given for this statement [see (5) on page 15] in Section 11.01 is based on Notions 2, 3, and 4 [and theorems which were proved in Volume 1], this argument is, now, a proof of Theorem 12–6.

Exercises

Part A

1. (a) By referring to Theorem 11–4, show that (1)–(4) on page 74 are theorems. [For one of these, you will need to make use of part of Lemma 1 on page 49 as well.]
   (b) Show that:

$$[l] \subseteq [m]^{\perp} \Longleftrightarrow [m] \subseteq [l]^{\perp}$$

   is a theorem.

2. What theorem from Chapter 10 is needed for the proof of Theorem 12–1?

Answers for Part A

1. (a) To prove (1): Let ℓ be any line. Then, there is a non-$\vec{0}$ translation — say, $\vec{a}$ — such that $[\ell] = [\vec{a}]$. By Theorem 11-4(a), $[\vec{a}]^{\perp}$ is a proper bidirection. So, by the replacement rule for equations, $[\ell]^{\perp}$ is a proper bidirection.

   To prove (2): Given a line ℓ, there is a non-$\vec{0}$ translation — say, $\vec{a}$ — in $[\ell]$. By Theorem 11-4(b), $[\ell] \cap [\ell]^{\perp} = \{\vec{0}\}$. So, $\vec{a} \in [\ell]$ and $\vec{a} \notin [\ell]^{\perp}$. Since there is a member of $[\ell]$, namely $\vec{a}$, which is not in $[\ell]^{\perp}$, it follows that $[\ell] \not\subseteq [\ell]^{\perp}$.

   To prove (3): This follows directly from Theorem 11-4(c).

   To prove (4): Suppose that $[\ell] \subseteq [m]^{\perp}$. Assume that $\vec{b} \in [m]$. We must show that $\vec{b} \in [\ell]^{\perp}$ [for, then it will be the case that $[m] \subseteq [\ell]^{\perp}$, which is what we wish to show]. Clearly, for $\vec{b} = \vec{0}$, $\vec{b} \in [\ell]^{\perp}$. So, assume that $\vec{b} \neq \vec{0}$. Let $\vec{a}$ be any non-$\vec{0}$ member of $[\ell]$. Then, $[\ell] = [\vec{a}]$ and $[m] = [\vec{b}]$. Also, $[\vec{a}] \subseteq [m]^{\perp}$ so that $\vec{a} \in [m]^{\perp} = [\vec{b}]^{\perp}$. Now, by Theorem 11-4(d), $\vec{a} \in [\vec{b}]^{\perp}$ if and only if $\vec{b} \in [\vec{a}]^{\perp}$. So, $\vec{b} \in [\vec{a}]^{\perp} = [\ell]^{\perp}$. Thus, in any case, if $\vec{b} \in [m]$ then $\vec{b} \in [\ell]^{\perp}$. Hence, if $[\ell] \subseteq [m]^{\perp}$ then $[m] \subseteq [\ell]^{\perp}$.

[Note to the teacher. Writing out sufficiently detailed paragraph proofs for theorems which so obviously follow from corresponding parts of a previously proved theorem is a time-consuming chore. By discussing the details of such theorems in class, one can allow the student to move quickly through parts of the exercises while practicing and strengthening his verbal skills and, at the same time, uncover specific weaknesses in these skills. Also, such a practice will enable the student to spend his "homework" time on some of the harder exercises. All of the exercises in Part A can be thoroughly covered during one class period which is devoted to the reading and discussion of text pp. 74-76. A reasonable homework assignment following such a session is Part B on pp. 77-78.]

1. (b) The given biconditional is equivalent to (4) and its converse. We proved (4) in part (a). Its converse is an instance of (4) and, so, is a theorem. Hence, the given biconditional is a theorem.

2. Theorem 10-4, which says that a line and a plane which are not parallel intersect at a single point.

TC 77 (1)

3. (a) The instance referred to in the hint is:

$$(\sigma \parallel \pi \text{ and } \ell \parallel \ell) \Longrightarrow [\sigma \perp \ell \Longleftrightarrow \pi \perp \ell]$$

   This is clearly equivalent to:

$$\sigma \parallel \pi \Longrightarrow [\sigma \perp \ell \Longleftrightarrow \pi \perp \ell]$$

   In words, the latter is the first part of the corollary.

3. **(a)** Show that the first part of the corollary to Theorem 12–2 is a special case of this theorem. [*Hint*: In the theorem, substitute 'l' for 'm'.]

(b) Do the same for the second part of the corollary.

4. Use rules of logic to show that the sentence:

$$[m] = [l] \longrightarrow [m]^{\perp} = [l]^{\perp}$$

is valid.

5. **(a)** The first part of the corollary to Theorem 12–3 can be reformulated as follows:

$$(\sigma \perp m \text{ and } \pi \perp l \text{ and } m \parallel l) \longrightarrow \sigma \parallel \pi$$

How does this follow from Theorem 12–3?

(b) Repeat part (a) for the second part of the same corollary.

6. Explain why (**) on page 75 is equivalent to the conjunction of:

$(*_1)$ $\qquad (\pi \perp l \text{ and } \pi \perp m) \longrightarrow m \parallel l$

and:

$(*_2)$ $\qquad (\pi \perp l \text{ and } m \parallel l) \longrightarrow \pi \perp m$

7. Show that $(*_1)$ and $(*_2)$ follow from the corollaries to Theorems 12–3 and 12–2, respectively.

8. **(a)** Use (4) on page 74 to prove Theorem 12–4.

(b) Use (1) on page 74 and Theorem 12–4 to prove (4).

9. **(a)** Prove Theorem 12–5.

(b) Prove the corollary to Theorem 12–5.

10. Check that the argument given for (5) on page 15 is, now, a proof of Theorem 12–6.

11. By (**) on page 75 and Definition 12–1(b) it follows that

(a) $\qquad l \perp \pi \longrightarrow [m \perp \pi \longleftrightarrow m \parallel l].$

Show that

(b) $\qquad \pi \perp l \longrightarrow [\sigma \perp l \longleftrightarrow \sigma \parallel \pi].$

Part B

Justify each of the following statements by referring either to Theorem 12–6, or to one of the two parts of one of the corollaries, or to statements in this part which precede it.

1. There are lines which are perpendicular to any given plane π.

2. All such lines are parallel. [That is, all lines which are perpendicular to a given plane π are parallel.]

Answers for Part A [cont.]

(b) Consider this instance of Theorem 12-2:

$$(\pi \parallel \pi \text{ and } m \parallel \ell) \Longrightarrow [\pi \perp m \Longleftrightarrow \pi \perp \ell]$$

This is equivalent to:

$$m \parallel \ell \Longrightarrow [\pi \perp m \Longleftrightarrow \pi \perp \ell]$$

In words, the latter is the second part of the corollary.

4. On TC 74, we wrote a paragraph proof for the given sentence. Here is a tree-diagram of this proof:

$$\dfrac{\dfrac{[m] \overset{*}{=} [\ell] \qquad [m]^{\perp} = [m]^{\perp}}{[m]^{\perp} = [\ell]^{\perp}}\;\text{(RRE)}}{[m] = [\ell] \Longrightarrow [m]^{\perp} = [\ell]^{\perp}}\;*$$

5. (a) Here is a tree-diagram which shows how the given sentence follows from Theorem 12-3:

$$\dfrac{\dfrac{\dfrac{\sigma \perp m \text{ and } \pi \overset{*}{\perp} \ell \text{ and } m \parallel \ell}{m \parallel \ell} \qquad \dfrac{\dfrac{\dfrac{\sigma \perp m \text{ and } \pi \overset{*}{\perp} \ell \text{ and } m \parallel \ell}{\sigma \perp m \text{ and } \pi \perp \ell} \quad \text{Theorem 12-3}}{\sigma \parallel \pi \Longleftrightarrow m \parallel \ell}}{m \parallel \ell \Longrightarrow \sigma \parallel \pi}}{\sigma \parallel \pi}}{(\sigma \perp m \text{ and } \pi \perp \ell \text{ and } m \parallel \ell) \Longrightarrow \sigma \parallel \pi}\;*$$

(b) The formalization of the second part of the corollary is:

$$(\sigma \perp m \text{ and } \pi \perp \ell \text{ and } \sigma \parallel \pi) \Longrightarrow m \parallel \ell$$

A tree-diagram which shows how this sentence follows from Theorem 12-3 is:

$$\dfrac{\dfrac{\dfrac{\sigma \perp m \text{ and } \pi \overset{*}{\perp} \ell \text{ and } \sigma \parallel \pi}{\sigma \parallel \pi} \qquad \dfrac{\dfrac{\dfrac{\sigma \perp m \text{ and } \pi \overset{*}{\perp} \ell \text{ and } \sigma \parallel \pi}{\sigma \perp m \text{ and } \pi \perp \ell} \quad \text{Theorem 12-3}}{\sigma \parallel \pi \Longleftrightarrow m \parallel \ell}}{\sigma \parallel \pi \Longrightarrow m \parallel \ell}}{m \parallel \ell}}{(\sigma \perp m \text{ and } \pi \perp \ell \text{ and } \sigma \parallel \pi) \Longrightarrow m \parallel \ell}\;*$$

Answers for Part A [cont.]

6. One way to see that (**) is equivalent to the conjunction of $(*_1)$ and $(*_2)$ is to note that $(*_1)$ is equivalent to:

$$\pi \perp \ell \Longrightarrow [\pi \perp m \Longrightarrow m \parallel \ell]$$

and $(*_2)$ is equivalent to:

$$\pi \perp \ell \Longrightarrow [m \parallel \ell \Longrightarrow \pi \perp m]$$

And, the conjunction of these sentences is equivalent to:

$$\pi \perp \ell \Longrightarrow [\pi \perp m \Longleftrightarrow m \parallel \ell]$$

which is (**).

To formally establish the above-mentioned equivalence, we should derive each from the other. We do this as follows:

To derive (**) from the conjunction of $(*_1)$ and $(*_2)$:

$$\dfrac{\dfrac{\dfrac{\overset{\dagger}{\pi \perp \ell} \quad \dfrac{\dfrac{(*_1) \text{ and } (*_2)}{(*_2)}}{\pi \perp \ell \Longrightarrow [m \parallel \ell \Longrightarrow \pi \perp m]}}{m \parallel \ell \Longrightarrow \pi \perp m} \qquad \dfrac{\overset{\dagger}{\pi \perp \ell} \quad \dfrac{\dfrac{(*_1) \text{ and } (*_2)}{(*_1)}}{\pi \perp \ell \Longrightarrow [\pi \perp m \Longrightarrow m \parallel \ell]}}{\pi \perp m \Longrightarrow m \parallel \ell}}{\pi \perp m \Longleftrightarrow m \parallel \ell}}{(**)}\;\dagger$$

To derive the conjunction of $(*_1)$ and $(*_2)$ from (**):

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\dagger}{\pi \perp \ell \text{ and } \pi \perp m}}{\pi \perp m} \qquad \dfrac{\dfrac{\dfrac{\overset{\dagger}{\pi \perp \ell \text{ and } \pi \perp m}}{\pi \perp \ell} \quad (**)}{\pi \perp m \Longleftrightarrow m \parallel \ell}}{\pi \perp m \Longrightarrow m \parallel \ell}}{m \parallel \ell}}{(*_1)}\;\dagger \qquad \dfrac{\dfrac{\dfrac{\overset{\dagger\dagger}{\pi \perp \ell \text{ and } m \parallel \ell}}{m \parallel \ell} \qquad \dfrac{\dfrac{\dfrac{\overset{\dagger\dagger}{\pi \perp \ell \text{ and } m \parallel \ell}}{\pi \perp \ell} \quad (**)}{\pi \perp m \Longleftrightarrow m \parallel \ell}}{m \parallel \ell \Longrightarrow \pi \perp m}}{\pi \perp m}}{(*_2)}\;\dagger\dagger}{(*_1) \text{ and } (*_2)}$$

7. $(*_1)$ follows from the second part of the corollary to Theorem 12-3 and the fact that $\pi \parallel \pi$. [It can be derived directly from Theorem 12-3 by considering one of the instances of the theorem in which $\sigma = \pi$.]

$(*_2)$ follows directly from the second part of the corollary to Theorem 12-2, as it is a formalization of this part of the corollary.

Answers for Part A [cont.]

8. (a) Suppose that $m \perp \sigma$ and $\ell \perp \pi$. Then, $[m]^{\perp} = [\sigma]$ and $[\ell]^{\perp} = [\pi]$. Thus, $\ell \parallel \sigma$ if and only if $[\ell] \subseteq [\sigma] = [m]^{\perp}$ and $m \parallel \pi$ if and only if $[m] \subseteq [\pi] = [\ell]^{\perp}$. By (4), we know that $[\ell] \subseteq [m]^{\perp}$ if and only if $[m] \subseteq [\ell]^{\perp}$. Thus, by the replacement rule for biconditionals, $\ell \parallel \sigma$ if and only if $m \parallel \pi$. Hence, Theorem 12-4.

   (b) Suppose that $[\ell] \subseteq [m]^{\perp}$. Since each of ℓ and m are lines, it follows by (1) that $[\ell]^{\perp}$ and $[m]^{\perp}$ are proper bidirections. Let π and σ be planes such that $[\pi] = [\ell]^{\perp}$ and $[\sigma] = [m]^{\perp}$. Then, $m \perp \sigma$ and $\ell \perp \pi$ so that, by Theorem 12-4, we know that $\ell \parallel \sigma$ if and only if $m \parallel \pi$. The latter is equivalent, by definition, to saying that $[\ell] \subseteq [\sigma]$ if and only if $[m] \subseteq [\pi]$. So, $[\ell] \subseteq [m]^{\perp}$ if and only if $[m] \subseteq [\ell]^{\perp}$. Thus, $[m] \subseteq [\ell]^{\perp}$. Hence, if $[\ell] \subseteq [m]^{\perp}$ then $[m] \subseteq [\ell]^{\perp}$, which is (4).

9. (a) Theorem 9-11(b) tells us that $\overleftrightarrow{P[\pi]}$ is the plane through P with the direction of π. Consider any line ℓ. By (1), $[\ell]^{\perp}$ is a proper bidirection. So, $\overleftrightarrow{P[\ell]^{\perp}}$ is the plane through P with the bidirection $[\ell]^{\perp}$. By Definition 12-1(a) it is, then, the plane which contains P and is perpendicular to ℓ.

   (b) Since no two planes with the same direction can have a point in common, and since, by Theorem 12-5, there is a plane which contains a given point and is perpendicular to a given line, there can be only one such plane. Hence, the corollary.

10. [It is best to have the students turn back to page 15 and read through the argument. Clear up any questions as they arise.]

11. Suppose that $\pi \perp \ell$. Then, by Definition 12-1(a), $[\pi] = [\ell]^{\perp}$. Now, $\sigma \perp \ell$ if and only if $[\sigma] = [\ell]^{\perp}$. So, $\sigma \perp \ell$ if and only if $[\sigma] = [\pi]$. Since $[\sigma] = [\pi]$ if and only if $\sigma \parallel \pi$, we have that $\sigma \perp \ell$ if and only if $\sigma \parallel \pi$. Hence, (b).

    Here is a tree-diagram of this proof:

$$\cfrac{\cfrac{\sigma \parallel \pi \Longleftrightarrow [\sigma] = [\pi] \qquad \cfrac{\cfrac{\overset{*}{\pi \perp \ell} \qquad \pi \perp \ell \Longleftrightarrow [\pi] = [\ell]^{\perp}}{[\pi] = [\ell]^{\perp}} \qquad \sigma \perp \ell \Longleftrightarrow [\sigma] = [\ell]^{\perp}}{\sigma \perp \ell \Longleftrightarrow [\sigma] = [\pi]}}{\sigma \perp \ell \Longleftrightarrow \sigma \parallel \pi}}{\pi \perp \ell \Longrightarrow [\sigma \perp \ell \Longleftrightarrow \sigma \parallel \pi]}\,^{*}$$

Answers for Part B

1. Let π be any plane. π contains three noncollinear points — say, A, B, and C. Let $\ell = \overleftrightarrow{AB}$ and $m = \overleftrightarrow{BC}$. Then, for any point P, $\overleftrightarrow{P[\ell]^{\perp}}$ and $\overleftrightarrow{P[m]^{\perp}}$ are planes which are perpendicular to ℓ and m, respectively. By Theorem 12-6, these planes intersect in a line — say, n — which is perpendicular to π.

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2. By the second part of the corollary to Theorem 12-3, lines perpendicular to a given plane are parallel.
3. This follows from the second part of the corollary to Theorem 12-2.
4. By Exercises 1, 2, and 3.
5. By the first part of the corollary to Theorem 12-2.
6. By the second part of the corollary to Theorem 12-3.

3. All lines which are parallel to such lines are perpendicular to π.
4. There is a proper direction such that a line is perpendicular to π if and only if it has this direction.
5. The direction referred to in Exercise 4 depends only on the bidirection of π.
6. If π and σ have different bidirections then the direction of lines perpendicular to π is different from the direction of lines perpendicular to σ.

*

It will be convenient to have a notation for the proper direction referred to in Exercise 4 of Part B. Previous experience, and Exercise 5, suggests that we might call this direction *the orthogonal complement of* $[\pi]$ and denote it by '$[\pi]^\perp$'. This will turn out to be a good idea but, to make use of it we need a suitable definition. 'Suitable', here, means that we should be able to prove:

Theorem 12–7 $l \perp \pi \Longleftrightarrow [l] = [\pi]^\perp$

What the definition should be is suggested by Theorem 12–6. According to this theorem, if m and n are two intersecting lines in π then the direction which we wish to refer to as '$[\pi]^\perp$' is $[m]^\perp \cap [n]^\perp$. If $[m] = [\vec{a}]$ and $[n] = [\vec{b}]$ then, assuming that m and n are two intersecting lines, $(\vec{a}, \vec{b})$ is linearly independent. So, assuming that m and n are subsets of π, it follows that $[\pi] = [\vec{a}, \vec{b}]$. Our problem, then, is to define '$[\vec{a}, \vec{b}]^\perp$', so that

$$(5) \qquad [\vec{a}, \vec{b}]^\perp = [\vec{a}]^\perp \cap [\vec{b}]^\perp.$$

If this is a theorem then, by the preceding argument, so is Theorem 12–7. Since '$[\vec{a}, \vec{b}]^\perp$' should be defined in terms of '$[\vec{a}, \vec{b}]$', (5) is not suitable as a definition. But, it tells us that the definition should be such that $\vec{c} \in [\vec{a}, \vec{b}]^\perp$ if and only if $\vec{c} \in [\vec{a}]^\perp$ and $\vec{c} \in [\vec{b}]^\perp$ – that is, if and only if $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$. Since it is easily shown that

$$(*) \quad (\vec{c} \cdot \vec{a} = 0 \text{ and } \vec{c} \cdot \vec{b} = 0) \Longleftrightarrow \forall_{\vec{y} \in [\vec{a}, \vec{b}]}\ \vec{c} \cdot \vec{y} = 0,$$

this suggests that the following is suitable for the definition we have been seeking:

Definition 12–2 $[\vec{a}, \vec{b}]^\perp = \{\vec{x} \colon \forall_{\vec{y} \in [\vec{a}, \vec{b}]}\ \vec{x} \cdot \vec{y} = 0\}$

[Note that, since $[\vec{a}, \vec{a}] = [\vec{a}]$, Definition 12–2 includes Definition 11–1(b).] By (*) and Definition 12–2, (5) is a theorem and hence, as previously noted, so is Theorem 12–7. By Theorem 12–6 and the first

After the discussion on pages 78-79, we recommend that Parts B and C (pages 79-80) be used for class discussion and demonstration. Part A can be used as one homework assignment and Part D will provide more stimulating discussion for the following class period.

Answers for Part A

1. We prove (☆) in two parts:

   only if: Suppose that $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$. Assume that $\vec{p} \in [\vec{a}, \vec{b}]$. Then, $\vec{p} = \vec{a}a + \vec{b}b$, for some a and b, and $\vec{c} \cdot \vec{p} = \vec{c} \cdot (\vec{a}a + \vec{b}b) = \vec{c} \cdot (\vec{a}a) + \vec{c} \cdot (\vec{b}b) = (\vec{c} \cdot \vec{a})a + (\vec{c} \cdot \vec{b})b = 0$. So, for each $\vec{y}$ in $[\vec{a}, \vec{b}]$, $\vec{c} \cdot \vec{y} = 0$. Hence, the only if-part of (☆).

   if: Suppose that $\forall_{\vec{y} \in [\vec{a}, \vec{b}]}\ \vec{c} \cdot \vec{y} = 0$. Since both $\vec{a}$ and $\vec{b}$ are in $[\vec{a}, \vec{b}]$, $\vec{c} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{b} = 0$. Hence, the if-part of (☆).

2. We have, by Theorem 12-7, that $[\ell] = [\pi]^\perp$ if and only if $\ell \perp \pi$. And, by Definition 12-1, $\pi \perp \ell$ if and only if $[\pi] = [\ell]^\perp$. Hence, by the replacement rule for biconditionals, $[\ell] = [\pi]^\perp$ if and only if $[\pi] = [\ell]^\perp$.

3. Since $[\ell]^\perp$ is a proper bidirection and $[\pi]^\perp$ is a proper direction, it follows, by Exercise 2, that $[\ell] = [\ell]^{\perp\perp} \Longleftrightarrow [\ell]^\perp = [\ell]^\perp$ and $[\pi]^\perp = [\pi]^\perp \Longleftrightarrow [\pi] = [\pi]^{\perp\perp}$. And, since $[\ell]^\perp = [\ell]^\perp$ and $[\pi]^\perp = [\pi]^\perp$, we have that $[\ell]^{\perp\perp} = [\ell]$ and $[\pi]^{\perp\perp} = [\pi]$.

4. (a) In case $(\vec{a}, \vec{b})$ is linearly independent, $[\vec{a}, \vec{b}]$ is a proper bidirection by definition. In case $(\vec{a}, \vec{b})$ is linearly dependent, $[\vec{a}, \vec{b}]$ is either a proper direction [in case at least one of $\vec{a}$ and $\vec{b}$ is non-$\vec{0}$] or is $\{\vec{0}\}$ [in case both $\vec{a}$ and $\vec{b}$ are $\vec{0}$].

   (b) A reasonable definition is: $\mathcal{T}^\perp = \{\vec{0}\}$.

5. By (6), each of $[\sigma]^\perp$ and $[\pi]^\perp$ is a proper direction. Let $[m] = [\sigma]^\perp$ and $[\ell] = [\pi]^\perp$. By Exercise 2, we have that $[m]^\perp = [\sigma]$ and $[\ell]^\perp = [\pi]$. Also, from Exercise 1(b) on page 76, we know that $[m] \subseteq [\ell]^\perp$ if and only if $[\ell] \subseteq [m]^\perp$. So, it follows that $[\sigma]^\perp \subseteq [\pi]$ if and only if $[\pi]^\perp \subseteq [\sigma]$.

6. (a) Let $\vec{p} \in [\vec{a}, \vec{b}] \cap [\vec{a}, \vec{b}]^\perp$. Since $\vec{p} \in [\vec{a}, \vec{b}]^\perp$, $\vec{p} \cdot \vec{a} = \vec{p} \cdot \vec{b} = 0$. Since $\vec{p} \in [\vec{a}, \vec{b}]$, it follows, by (☆), that $\vec{p} \cdot \vec{p} = 0$. Thus, $\vec{p} = \vec{0}$. So $[\vec{a}, \vec{b}] \cap [\vec{a}, \vec{b}]^\perp \subseteq \{\vec{0}\}$. Clearly, $\{\vec{0}\} \subseteq [\vec{a}, \vec{b}] \cap [\vec{a}, \vec{b}]^\perp$. Hence, $[\vec{a}, \vec{b}] \cap [\vec{a}, \vec{b}]^\perp = \{\vec{0}\}$.

   (b) By (a), $[\pi]^\perp \cap [\pi] = \{\vec{0}\}$. Since $[\pi]^\perp$ contains at least one non-$\vec{0}$ translation, it follows that there is at least one translation in $[\pi]^\perp$ which is not in $[\pi]$. Hence, $[\pi]^\perp \not\subseteq [\pi]$.

part of the corollary to Theorem 12–3, respectively, we have:

(6) $[\pi]^{\perp}$ is a proper direction,

and

(7) $[\pi]^{\perp} = [\sigma]^{\perp} \longrightarrow [\pi] = [\sigma]$

By (6) and Theorem 12–6 we have:

> **Theorem 12–8** $\overleftrightarrow{P[\pi]^{\perp}}$ is the line which contains P and is perpendicular to π.
>
> **Corollary** There is one and only one line which contains a given point and is perpendicular to a given plane.

Exercises

Part A

1. Prove (*).
2. Show that $[l] = [\pi]^{\perp} \longleftrightarrow [\pi] = [l]^{\perp}$.
3. Show that $[l]^{\perp\perp} = [l]$ and that $[\pi]^{\perp\perp} = [\pi]$. [*Hint*: By (1) on page 74, it is permissible to replace '$[\pi]$' in Exercise 2 by '$[l]^{\perp}$'.]
4. (a) In case $(\vec{a}, \vec{b})$ is linearly independent $[\vec{a}, \vec{b}]$ is a proper bidirection. [Why?] What is $[\vec{a}, \vec{b}]$ in case $(\vec{a}, \vec{b})$ is linearly dependent? [Consider two cases.]
   (b) What would be a reasonable definition for '$\mathscr{T}^{\perp}$'? [In answering, consider Exercise 3 and Lemma 1(a) on page 49; also, consider Definitions 12–2 and 11–1(b) and the corollary to Theorem 12–8.]
5. Prove: $[\sigma]^{\perp} \subsetneq [\pi] \longleftrightarrow [\pi]^{\perp} \subsetneq [\sigma]$ [*Hint*: Exercise 1(b) of Part A on page 76 and Exercise 2, above, will be helpful. And, recall the hint for Exercise 3.]
6. Show that
   (a) $[\vec{a}, \vec{b}] \cap [\vec{a}, \vec{b}]^{\perp} = \{\vec{0}\}$ (b) $[\pi]^{\perp} \not\subseteq [\pi]$

Part B

Make use of your classroom to locate models of the following:

1. two planes which intersect. [What must the intersection be?]
2. two parallel planes.
3. two lines which intersect
4. three concurrent lines.
5. two skew lines.
6. two parallel lines.
7. three parallel lines which are not contained in a single plane.

With Exercise 4 of Part A we complete the definition of orthogonal complementing for subspaces of $\mathcal{T}$ — that is, for $\{\vec{0}\}$, for proper directions or bidirections, and for $\mathcal{T}$. Note that the orthogonal complement of any subspace is a subspace whose dimension is that of $\mathcal{T}$ minus that of the given subspace. If we use '$\mathcal{X}$', with or without subscripts, as a variable ranging over subspaces of $\mathcal{T}$ we can define orthogonal complementing by:

$$\mathcal{X}^{\perp} = \{\vec{x} : \forall_{\vec{y} \in \mathcal{X}}\ \vec{x} \cdot \vec{y} = 0\}$$

Aside from the results previously mentioned:

$$\mathcal{X}^{\perp} \text{ is a subset of } \mathcal{T} \text{ and: } \dim(\mathcal{X}^{\perp}) = 3 - \dim(\mathcal{X})$$

the important properties of this operation are given by:

$$\mathcal{X}^{\perp\perp} = \mathcal{X} \text{ and: } \mathcal{X}_1 \subseteq \mathcal{X}_2 \Longleftrightarrow \mathcal{X}_2^{\perp} \subseteq \mathcal{X}_1^{\perp}$$

From the first of these we can derive:

$$\mathcal{X}_1 = \mathcal{X}_2^{\perp} \Longleftrightarrow \mathcal{X}_2 = \mathcal{X}_1^{\perp}$$

and from both we can derive:

$$\underset{*}{\mathcal{X}_1^{\perp} \subseteq \mathcal{X}_2 \Longleftrightarrow \mathcal{X}_2^{\perp} \subseteq \mathcal{X}_1}$$

Answers for Part B

[This set of exercises is probably best done as a class exercise. There are obviously many answers for each exercise.]

Answer to questions: 1. A line.

8. They are parallel.

8. a plane and four lines perpendicular to it. [What do you observe about these lines?]
9. one line which is perpendicular to each of two intersecting lines. [Find one such line which intersects the given lines and one such line which doesn't intersect the given lines.]
10. two planes which are perpendicular to the same line. [What do you observe about these planes?]
11. two planes which are perpendicular to the same plane.
12. three planes, each two of which are perpendicular.
13. one plane which is perpendicular to each of two given planes. [What do you observe about the intersection of the given planes?]
14. two planes whose line of intersection is perpendicular to a given plane. [What do you observe about the given plane and either of the two planes?]

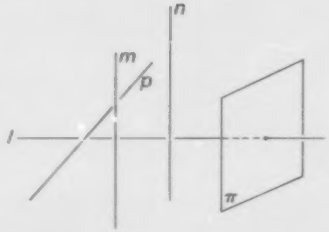
Part C

Answer the following questions and be prepared to illustrate your answers with objects in the classroom, if possible.

1. Are each two lines coplanar?
2. Do each two nonparallel lines intersect?
3. Are there four noncoplanar points?
4. Will a table with three legs always stand firmly on a floor? How about a table with four legs?
5. How many lines are there which are perpendicular to a given plane and through a given point? How many such planes are there?
6. Are all lines which are perpendicular to a horizontal line vertical lines?
7. Are all planes which are perpendicular to a horizontal line vertical planes?
8. Can a line be perpendicular to each of two intersecting lines and not intersect these lines?
9. If two lines are parallel to the same line, must they be parallel?
10. If two planes are parallel to the same line, must they be parallel?
11. If a line is parallel to a plane, is that line parallel to each line in the plane?
12. If a line is perpendicular to a plane, is that line perpendicular to each line in the plane?

Part D

1. Here is a picture of a horizontal line *l* and a vertical line *m*. It seems reasonable to say that *m* is perpendicular to *l*—for short: $m \perp l$. Suppose that π is perpendicular to *l*. What is the relation between *m* and π?



Answers for Part B [cont.]

10. They are parallel.
13. It is empty or a line perpendicular to the third plane.
14. They are perpendicular.

Answers for Part C

1. No. [See Exercise 5, Part B.]
2. No.
3. Yes.
4. Yes. [Provided that the feet of the legs are noncollinear.]; Just if the feet of the legs are coplanar.
5. One. Infinitely many.
6. No.
7. Yes.
8. Yes. [This is so provided that 'perpendicular' is defined in such a way as not to restrict skew lines from being perpendicular.]
9. Yes.
10. No.
11. No.
12. Yes.

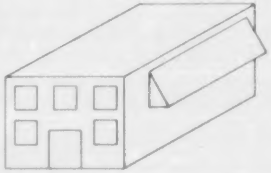
*

Exercises 8 and 12 of Part C bring up the question of whether perpendicular lines should be required to intersect. Exercise 2 of Part D brings up the same question. Exercise 3 of Part D should incline students to our point of view — that it is convenient to make perpendicularity of lines depend only on their directions. Our definition of perpendicularity of lines [and of planes] is given in words with the first paragraph of section 12.02, and in symbols in Definition 12-3.

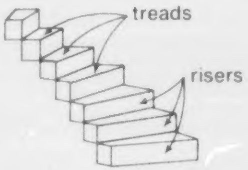
Answers for Part D

1. $m \parallel \pi$

2. Suppose that the line n is parallel to m. Do you think that $n \perp l$?
3. If you answered 'No.' to Exercise 2 it may be because, as the picture shows, n does not appear to intersect l. As with any definition, we have a choice. One thing to consider in making this choice is that, so far, it has been a convenience that whether things are parallel or perpendicular depends only on their directions. Now, how do you feel about Exercise 2?
4. Suppose that the line p is parallel to π but is not, like m, a vertical line. Thus, $[p] \subseteq [l]^{\perp}$ [just as $[m] \subseteq [l]^{\perp}$]. Do you think that $p \perp l$?
5. Here is a picture of a building. It is not unreasonable to say that the planes containing the walls of the building are perpendicular to the plane containing the first floor of the building.



   (a) Describe some other planes to which the planes of the walls are perpendicular. What is the relation between each of these planes and the plane of the first floor?
   (b) Describe some obvious lines in the planes of the walls which are perpendicular to the plane of the first floor. Are there other such lines? Try to make use of some of these lines to describe other planes which are perpendicular to the plane of the first floor.
   (c) Consider the plane of the awning on the side of the building. Is this plane perpendicular to the plane of the first floor? Is there a line in the plane of the awning which is perpendicular to the plane of the first floor?
6. Suppose that there is a flag pole in front of the building described in Exercise 5 which is perpendicular to the plane of the first floor.
   (a) What can you say about the flag pole and any of the planes which are perpendicular to the first floor?
   (b) What can you say about the flag pole and any plane which is parallel to the first floor?
7. In the building described in Exercise 5, there is a circular stairway, part of which is pictured at the right.



   (a) Let π be the plane of the first floor. Describe some planes which are perpendicular to π. What is the bidirection of any one of these planes?
   (b) Let σ be any one of the planes you described in part (a). Is there a line in σ which is perpendicular to π? If you think so, what is its direction? If not, explain.

Answers for Part D [cont.]

2. Yes. [Some students may feel that the answer is 'No.'. We have no real basis for argument until we adopt a definition. When we do so, we will try to give a reasonable justification for having skew perpendicular lines.]
3. [Some may still not wish to change their answer. But, at least we have reminded them that definitions are somewhat arbitrary.]
4. Yes.
5. (a) The plane of any floor is [presumably] such that the walls of the building are perpendicular to it. Any such plane is parallel to the plane of the first floor.
   (b) The lines of intersection of any two walls are perpendicular to the plane of the first floor.; Yes.; Any line which is parallel to one of these lines is also perpendicular to the plane of the first floor. Any two of these lines determine a plane, and any plane so determined is perpendicular to the plane of the first floor.
   (c) No.; No.
6. (a) They are parallel.
   (b) It is perpendicular to that plane.
7. (a) The plane of any riser in the circular staircase is perpendicular to π. Given a line ℓ in π, $[\ell]^{\perp}$ is the bidirection of a plane perpendicular to π.
   (b) Yes. Its direction is $[\pi]^{\perp}$.
8. (a) No.
   (b) No.

Sample Quiz

1. Which of the following sentences is a reformulation of:

   (*) Parallel planes are perpendicular to the same lines.

   (a) ($\pi \parallel \sigma$ and $\pi \perp \ell$ and $\sigma \perp m$) $\Longrightarrow \ell = m$

   (b) $\pi \parallel \sigma \Longrightarrow (\pi \perp \ell \Longleftrightarrow \sigma \perp \ell)$

2. Prove the reformulation of (*) that you selected in Exercise 1.

3. Reformulate the sentence:

   Lines perpendicular to parallel planes are parallel.

   in a [conditional] sentence in 'ℓ', 'm', 'π', and 'σ'.

4. Write a paragraph proof for the sentence you wrote in Exercise 3.

Key to Sample Quiz

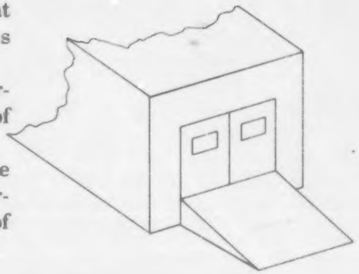
1. (b)

2. Suppose that $\pi \parallel \sigma$. Then, $[\pi] = [\sigma]$ so that $[\pi] = [\ell]^{\perp}$ if and only if $[\sigma] = [\ell]^{\perp}$. Thus, $\pi \perp \ell$ if and only if $\sigma \perp \ell$. Hence, $\pi \parallel \sigma \Longrightarrow [\pi \perp \ell \Longleftrightarrow \sigma \perp \ell]$

3. ($\ell \perp \pi$ and $m \perp \sigma$ and $\pi \parallel \sigma$) $\Longrightarrow \ell \parallel m$ [There are, of course, other correct answers.]

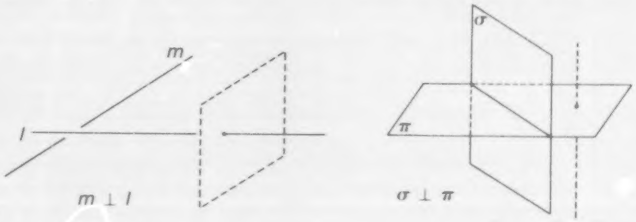
4. Suppose that $\ell \perp \pi$, $m \perp \sigma$, and $\pi \parallel \sigma$. Then, $[\ell]^{\perp} = [\pi]$, $[m]^{\perp} = [\sigma]$, and $[\pi] = [\sigma]$ so that $[\ell]^{\perp} = [m]^{\perp}$. The latter is the case if and only if $[\ell] = [m]$, that is, $\ell \parallel m$. Hence, if $\ell \perp \pi$ and $m \perp \sigma$ and $\pi \parallel \sigma$ then $\ell \parallel m$.

8. Suppose that there is a ramp at the rear of the building, as is pictured at the right.
   (a) Is the plane of the ramp perpendicular to the plane of the first floor?
   (b) Are there lines in the plane of the ramp which are perpendicular to the plane of the first floor?



12.02 Perpendicularity of Lines and of Planes

As suggested by the preceding exercises, we shall say that one line is perpendicular to another if the first is parallel to a plane which is perpendicular to the second line. Similarly, we shall say that one plane is perpendicular to another if the first is parallel to a line which is perpendicular to the second plane. We do this formally in:



$m \perp l$ $\sigma \perp \pi$

Fig. 12–1

Definition 12–3 (a) $m \perp l \Longleftrightarrow [m] \subseteq [l]^{\perp}$
(b) $\sigma \perp \pi \Longleftrightarrow [\pi]^{\perp} \subseteq [\sigma]$

In Exercise 1(b) of Part A on page 76, it was established that

$$[l] \subseteq [m]^{\perp} \Longleftrightarrow [m] \subseteq [l]^{\perp}.$$

Making use of this result together with Definition 12–3(a), it is not difficult to establish that the relation of perpendicularity among lines is symmetric – that is, that

$$(*) \qquad l \perp m \Longleftrightarrow m \perp l.$$

[Do so.] And, making use of Exercise 5 of Part A on page 79 – that

$$[\sigma]^{\perp} \subseteq [\pi] \longleftrightarrow [\pi]^{\perp} \subseteq [\sigma]$$

– it is not difficult to establish that

$$(**) \qquad \sigma \perp \pi \longleftrightarrow \pi \perp \sigma.$$

[Do so.] We summarize the results in (*) and (**) in:

Theorem 12-9 (a) $l \perp m \longleftrightarrow m \perp l$
(b) $\pi \perp \sigma \longleftrightarrow \sigma \perp \pi$

Since perpendicular planes are not parallel [Why?] and perpendicular lines are not parallel [Why?] we have, in analogy with Theorem 12-1:

Theorem 12-10 (a) $\sigma \perp \pi \longrightarrow \sigma \cap \pi$ is a line
(b) Coplanar perpendicular lines intersect.

[Can you imagine two perpendicular lines which have no point in common?]

In analogy with the corollary to Theorem 12-2 we have:

Theorem 12-11 Parallel lines are perpendicular to the same lines and parallel planes are perpendicular to the same planes.

[Explain.] Must lines which are perpendicular to the same line be parallel? How about planes which are perpendicular to the same plane? How about lines which are perpendicular to the same plane? How about planes which are perpendicular to the same line?

What can you say about a plane and a line which are perpendicular to the same line? About a plane and a line which are perpendicular to the same plane? Your answers to these questions may suggest:

Theorem 12-12 (a) $\pi \perp l \longrightarrow [m \perp l \longleftrightarrow m \parallel \pi]$
(b) $l \perp \pi \longrightarrow [\sigma \perp \pi \longleftrightarrow \sigma \parallel l]$

Answers to Questions

To establish (*) from the result of Exercise 1(b) in Part A on page 76, and Definition 12-3(a) requires only two replacements in the former of instances of the latter. Thus, according to Definition 12-3(a), $m \perp \ell \Longleftrightarrow [m] \subseteq [\ell]^{\perp}$ and $\ell \perp m \Longleftrightarrow [\ell] \subseteq [m]^{\perp}$. So, since $[\ell] \subseteq [m]^{\perp} \Longleftrightarrow [m] \subseteq [\ell]^{\perp}$, we have that $\ell \perp m \Longleftrightarrow m \perp \ell$. (**) follows from Exercise 5, Part A, page 79, and Definition 12-3(b) in the same manner.

Suppose that π and σ are perpendicular planes. Since $[\pi]$ and $[\sigma]$ are proper bidirections, $[\pi]^{\perp}$ is a proper direction, and since, by Definition 12-3(b), $[\pi]^{\perp} \subseteq [\sigma]$, it follows that $[\pi]^{\perp} \cap [\sigma]$ contains a non-$\vec{0}$ translation. Now, suppose that $\pi \parallel \sigma$. Then $[\pi] = [\sigma]$, and, by replacement, $[\pi]^{\perp} \cap [\pi]$ contains a non-$\vec{0}$ translation. But, by Exercise 6(a), Part A, page 79, $[\pi]^{\perp} \cap [\pi] = \{\vec{0}\}$. Hence, if $\pi \perp \sigma$ then $\pi \nparallel \sigma$. In a similar manner, using Definition 12-3(a) and the fact that $[\ell] \cap [\ell]^{\perp} = \{\vec{0}\}$, it may be shown that perpendicular lines are not parallel.

Yes, two skew lines may be perpendicular and do not intersect.

Theorem 12-11 states that if line ℓ is perpendicular to lines m_1, m_2, m_3, ..., then any line parallel to ℓ will be perpendicular to m_1, m_2, m_3, ... Similarly, if plane π is perpendicular to planes σ_1, σ_2, σ_3, ..., then any plane parallel to π will be perpendicular to σ_1, σ_2, σ_3, ... It is not the case that lines which are perpendicular to the same line must be parallel, nor is it the case that planes which are perpendicular to the same plane must be parallel. [Students should be able to suggest models for counterexamples which are in the classroom.] Planes which are perpendicular to the same line are parallel.

A plane and a line which are perpendicular to the same line are parallel. A plane and a line perpendicular to the same plane are parallel.

TC 84 (1)

Parts A-D contain too many exercises for one homework assignment. Part A can be used as supervised class exercises or for class discussion. If you permit students to team up, Part B can be used for homework. Part C is good for class discussion and Part D gives a second homework assignment.

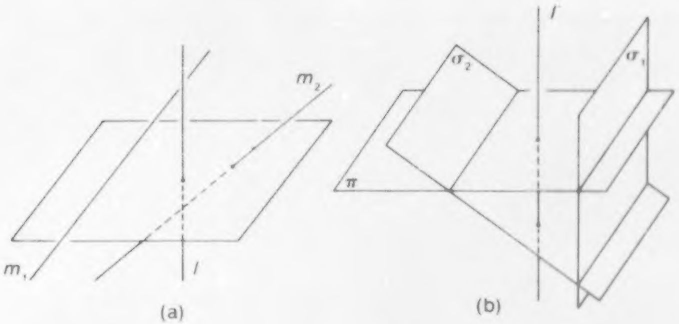


Fig. 12–2

Parts (a) and (b) differ very little from the corresponding parts of Definition 12–3. [Explain.] Compare these with (a) and (b) in Exercise 11 of Part A on page 77.

Exercises

Part A

Prove:

1. Theorem 12–10(a)
2. Theorem 12–10(b)
3. Theorem 12–11
4. Theorem 12–12
5. $(m \perp \sigma$ and $\pi \perp l \longrightarrow [\sigma \perp \pi \longleftrightarrow m \perp l]$

Part B

1. Show that
   (a) a line is perpendicular to l if and only if it is contained in a plane which is perpendicular to l, and
   (b) a plane is perpendicular to π if and only if it contains a line which is perpendicular to π.
2. Prove:

> Theorem 12–13 If $l \perp \pi$ then
> (a) l is perpendicular to each line contained in π, and
> (b) each plane which contains l is perpendicular to π.

3. Prove the following corollary of Theorem 12–13:

> Corollary $(l \subseteq \pi$ and $\sigma \perp l) \longrightarrow (\sigma \cap \pi) \perp l$

Answers for Part A

1. Suppose that $\sigma \perp \pi$. Since $\sigma \nparallel \pi$, by Theorem 10-2, $\sigma \cap \pi$ is a line. Hence, Theorem 12-10(a).

2. Suppose that ℓ and m are coplanar perpendicular lines. Then ℓ and m are coplanar nonparallel lines and, by Theorem 9-7, ℓ and m intersect. Hence, Theorem 12-10(b).

3. We will prove that parallel lines are perpendicular to the same lines. Suppose that the line ℓ is perpendicular to each of the lines $m_1, m_2, \ldots, m_r, \ldots$. Then, for ℓ and any of these lines, say m_k, $[\ell] \subset [m_k]^{\perp}$, by Definition 12-3(a). Now suppose that each of the lines $\ell_1, \ell_2, \ldots, \ell_s, \ldots$ is parallel to ℓ. Then, for any of these lines, say ℓ_j, $[\ell_j] = [\ell]$, and so $[\ell_j] \subseteq [m_k]^{\perp}$. Thus $\ell_j \perp m_k$. Hence, the first part of Theorem 12-11. The second part of the theorem is proved in a similar way, using Definition 12-3(b) and the fact that parallel planes have the same bidirections.

4. Here is a tree-diagram of a proof of Theorem 12-12(a):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{**}{\pi \perp \ell}}{[\pi] = [\ell]^{\perp}} \qquad \dfrac{\overset{*}{m \perp \ell}}{[m] \subseteq [\ell]^{\perp}}}{[m] \subseteq [\pi]}}{m \parallel \pi}}{m \perp \ell \Longrightarrow m \parallel \pi}\,{}^{*} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{**}{\pi \perp \ell}}{[\pi] = [\ell]^{\perp}} \qquad \dfrac{\overset{*}{m \parallel \pi}}{[m] \subseteq [\pi]}}{[m] \subseteq [\ell]^{\perp}}}{m \perp \ell}}{m \parallel \pi \Longrightarrow m \perp \ell}\,{}^{*}}{m \perp \ell \Longleftrightarrow m \parallel \pi}}{\pi \perp \ell \Longrightarrow [m \perp \ell \Longleftrightarrow m \parallel \pi]}\,{}^{**}$$

Here is a tree-diagram of a proof of Theorem 12-12(b):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{**}{\ell \perp \pi}}{[\ell] = [\pi]^{\perp}} \qquad \dfrac{\overset{*}{\sigma \perp \pi}}{[\pi]^{\perp} \subseteq [\sigma]}}{[\ell] \subseteq [\sigma]}}{\sigma \parallel \ell}}{\sigma \perp \pi \Longrightarrow \sigma \parallel \ell}\,{}^{*} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{**}{\ell \perp \pi}}{[\ell] = [\pi]^{\perp}} \qquad \dfrac{\overset{*}{\sigma \parallel \ell}}{[\ell] \subseteq [\sigma]}}{[\pi]^{\perp} \subseteq [\sigma]}}{\sigma \perp \pi}}{\sigma \parallel \ell \Longrightarrow \sigma \perp \pi}\,{}^{*}}{\sigma \perp \pi \Longleftrightarrow \sigma \parallel \ell}}{\ell \perp \pi \Longrightarrow [\sigma \perp \pi \Longleftrightarrow \sigma \parallel \ell]}\,{}^{**}$$

5. Suppose that $m \perp \sigma$ and $\pi \perp \ell$. It follows that $[m] = [\sigma]^{\perp}$ and $[\pi] = [\ell]^{\perp}$. It follows that if $\sigma \perp \pi$ — that is, if $[\sigma]^{\perp} \subseteq [\pi]$ — then $[\sigma]^{\perp} \subseteq [\ell]^{\perp}$ and, so, $[m] \subseteq [\ell]^{\perp}$ — that is, $m \perp \ell$. Hence, if $\sigma \perp \pi$ then $m \perp \ell$. On the other hand, it follows that if $m \perp \ell$ — that is, if $[m] \subseteq [\ell]^{\perp}$ — then $[m] \subseteq [\pi]$ and, so, $[\sigma]^{\perp} \subseteq [\pi]$ — that is, $\sigma \perp \pi$. Hence, if $m \perp \ell$ then $\sigma \perp \pi$.

   So, assuming that $m \perp \sigma$ and $\pi \perp \ell$, it follows that $\sigma \perp m \Longleftrightarrow m \perp \ell$. Hence, Exercise 5. [It is easy, if you have enough horizontal space, to put this argument in tree-form.]

Answers for Part B

1. (a) To establish the only if-part of this theorem, suppose that m is perpendicular to ℓ. Then $[m] \subset [\ell]^\perp$ and, for $P \in m$, $\overleftrightarrow{P[\ell]^\perp}$ is a plane containing m and perpendicular to ℓ. Hence, if a line is perpendicular to a line ℓ then there is a plane perpendicular to ℓ which contains the line.

Here is a tree-diagram of the proof of the if-part:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{m \subset \pi \text{ and } \pi \perp \ell}}{m \subset \pi}}{[m] \subset [\pi]} \qquad \dfrac{\dfrac{\overset{*}{m \subset \pi \text{ and } \pi \perp \ell}}{\pi \perp \ell}}{[\pi] = [\ell]^\perp}}{[m] \subset [\ell]^\perp}}{m \perp \ell}}{(m \subset \pi \text{ and } \pi \perp \ell) \Longrightarrow m \perp \ell}\,{}^{*}}{\exists_\pi\ (m \subset \pi \text{ and } \pi \perp \ell) \Longrightarrow m \perp \ell}$$

(b) Suppose that σ is perpendicular to π. Then $[\pi]^\perp \subset [\sigma]$. Let $P \in \sigma$. Since $[\pi]^\perp$ is a proper direction, $\overleftrightarrow{P[\pi]^\perp}$ is a line which is contained in σ, and $\overleftrightarrow{P[\pi]^\perp}$ is perpendicular to π. Thus, if a plane is perpendicular to π then it contains a line which is perpendicular to π.

Suppose, now, that σ contains a line — say, ℓ — which is perpendicular to π. Then, $[\ell] \subset [\sigma]$ and $[\ell] = [\pi]^\perp$. So, by replacement, $[\pi]^\perp \subset [\sigma]$, which means that $\sigma \perp \pi$. Thus, if a plane contains a line which is perpendicular to π then that plane is perpendicular to π.

The results in the preceding two paragraphs imply the biconditional in (b).

2. (a) Suppose that $\ell \perp \pi$. Then $[\ell] = [\pi]^\perp$. Given that $m \subset \pi$, $[m] \subset [\pi]$. Since $[\ell] = [\pi]^\perp$, it follows that $[\ell]^\perp = [\pi]$. So, $[m] \subset [\ell]^\perp$, which means that $m \perp \ell$. Thus, $\ell \perp m$. Hence, Theorem 12-13(a).

(b) Suppose that $\ell \perp \pi$. Then $[\ell] = [\pi]^\perp$. Given that $\ell \subset \sigma$, $[\ell] \subset [\sigma]$. So, $[\pi]^\perp \subset [\sigma]$, which means that $\sigma \perp \pi$. Hence, Theorem 12-13(b).

3. Suppose that $\ell \subset \pi$ and $\sigma \perp \ell$. Then $[\ell] \subset [\pi]$ and $[\sigma] = [\ell]^\perp$. Since $[\sigma] = [\ell]^\perp$, it follows that $[\sigma]^\perp = [\ell] \subset [\pi]$. So, since $[\sigma]^\perp \not\subset [\sigma]$, $[\sigma] \neq [\pi]$ which means that $\sigma \cap \pi$ is a line. Since $\sigma \cap \pi$ is a line in σ and $\ell \perp \sigma$, by Theorem 12-13(a), ℓ is perpendicular to $\sigma \cap \pi$. So, $(\sigma \cap \pi) \perp \ell$. Hence, the corollary.

Answers for Part B [cont.]

4. (a) Suppose that m and n are intersecting lines and that ℓ is perpendicular to both m and n. Then, by Exercise 1(a), ℓ is contained in a plane — say, π — which is perpendicular to m, and ℓ is contained in a plane — say, σ — which is perpendicular to n. Since $m \nparallel n$, $\pi \nparallel \sigma$ so that $\pi \cap \sigma = \ell$. By Theorem 12-6, ℓ is perpendicular to the plane containing m and n.

(b) Suppose that σ_1 and σ_2 are intersecting planes and that π is perpendicular to each of them. Then, there is a line in π — say, ℓ_1 — which is perpendicular to σ_1, and there is a line in π — say, ℓ_2 — which is perpendicular to σ_2. Since $\sigma_1 \nparallel \sigma_2$, $\ell_1 \nparallel \ell_2$ so that ℓ_1 and ℓ_2 are intersecting lines in π. By Theorem 12-13(a), ℓ_1 and ℓ_2 are perpendicular to the line of intersection of σ_1 and σ_2. So, by Theorem 12-14(a), π is perpendicular to the line of intersection of σ_1 and σ_2.

5. (a) A line is parallel to π if and only if it is contained in a plane which is parallel to π.

(b) A plane is parallel to ℓ if and only if it contains a line which is parallel to ℓ.

6. (a) No. For if $\sigma \perp \ell$ and $\sigma \parallel \pi$ then $[\ell] = [\sigma]^\perp$ and $[\sigma] = [\pi]$ so that $[\ell] = [\pi]^\perp$.

No. For if $m \perp \pi$ and $m \parallel \ell$ then $[m] = [\pi]^\perp$ and $[m] = [\ell]$ so that $[\ell] = [\pi]^\perp$.

7. (a) Suppose that $\ell \not\perp \pi$. Then, $[\ell]^\perp \neq [\pi]$. Now, since both $[\ell]^\perp$ and $[\pi]$ are proper bidirections, it follows that $[\ell]^\perp \cap [\pi]$ is a proper direction. Let m be any line which contains P, is perpendicular to ℓ, and is parallel to π. Then, $[m] \subset [\ell]^\perp$ and $[m] \subset [\pi]$ so that $[m] \subset [\ell]^\perp \cap [\pi]$. Since $[m]$ is a proper direction, $[m] = [\ell]^\perp \cap [\pi]$. So, any line which contains P, is perpendicular to ℓ, and is parallel to π has direction $[\ell]^\perp \cap [\pi]$. Thus, there is at most one such line. Since there is at least one such line, namely the line through P in the direction $[\ell]^\perp \cap [\pi]$, the theorem is proved.

(b) Suppose that $\ell \not\perp \pi$. Then, $[\ell]^\perp \neq [\pi]$ so that $[\ell] \neq [\pi]^\perp$. Let σ be any plane which is perpendicular to π and parallel to ℓ. Then, $[\pi]^\perp \subset [\sigma]$ and $[\ell] \subset [\sigma]$. Now, both $[\ell]$ and $[\pi]^\perp$ are proper directions. Let $\vec{a} \in [\ell]$ and $\vec{b} \in [\pi]^\perp$, where $\vec{a} \neq \vec{0} \neq \vec{b}$. Then, since $[\ell] \neq [\pi]^\perp$, $(\vec{a}, \vec{b})$ is linearly independent. So, $[\sigma] = [\vec{a}, \vec{b}]$. Thus, any plane which is perpendicular to π and parallel to ℓ has bidirection $[\vec{a}, \vec{b}]$, so that all such planes are parallel. Since no two parallel planes contain a given point P, there is at most one plane through P which is perpendicular to π and parallel to ℓ. And, since there is a plane through P whose bidirection is $[\vec{a}, \vec{b}]$, where $\vec{a}$ and $\vec{b}$ are as described above, the theorem is proved.

8. (a) Suppose that $\ell \subset \pi$ and $P \in \pi$. Then, $\ell \parallel \pi$ so that $\ell \not\perp \pi$. So, by Theorem 12-15(a), there is one and only one line through P which is perpendicular to ℓ and parallel to π. Since any line through P which is parallel to π is contained in π, the corollary is proved.

4. Prove:

> Theorem 12-14
> (a) A line which is perpendicular to each of two intersecting lines is perpendicular to the plane containing them.
> (b) A plane which is perpendicular to each of two intersecting planes is perpendicular to their line of intersection.

[*Hint*: For part (a), use Exercise 1(a) and Theorem 12-6.]

5. Use your knowledge of parallelism of lines and planes to complete the following analogues of (a) and (b) in Exercise 1.
   (a) A line is parallel to π if and only if it . . . .
   (b) A plane is parallel to l if and only if it . . . .
6. Suppose that $l \perp \pi$.
   (a) Is there a plane σ such that $\sigma \perp l$ and $\sigma \parallel \pi$?
   (b) Is there a line m such that $m \perp \pi$ and $m \parallel l$?
7. Prove:

> Theorem 12-15 If $l \not\perp \pi$ then
> (a) there is one and only one line through P which is perpendicular to l and parallel to π, and
> (b) there is one and only one plane through P which is perpendicular to π and parallel to l.

8. Prove the following corollaries of Theorem 12-15:
   (a) If $l \subseteq \pi$ and $P \in \pi$ then there is one and only one line in π which contains P and is perpendicular to l.
   (b) If $P \notin l$ then there is one and only one line which contains P, intersects l, and is perpendicular to l.
   (c) Coplanar lines which are perpendicular to a given line of the same plane are parallel.
   (d) If $l \not\perp \pi$ then there is one and only one plane which contains l and is perpendicular to π.
9. Suppose that σ_1 and σ_2 are two intersecting planes and that $l \perp \sigma_1, \subseteq \pi$, and $\pi \perp \sigma_2$. Show that $(\pi \cap \sigma_2) \perp (\sigma_1 \cap \sigma_2)$.
10. Suppose that l is *oblique* to π—that is, l is neither parallel nor perpendicular to π. Suppose that $l \cap \pi = \{P\}$ and that n is the line through P which is perpendicular to l and contained in π. [How do you know that there is such a line?] Finally, suppose that σ is a plane which contains l. Show that $\sigma \perp \pi$ if and only if $(\sigma \cap \pi) \perp n$. [*Hint*: If $\sigma \perp \pi$ then the line through P and perpendicular to σ is a subset of π. Show that this line is n.]
11. Suppose that σ_1 and σ_2 are perpendicular to π. Show that $\sigma_1 \perp \sigma_2$ if and only if $(\sigma_1 \cap \pi) \perp (\sigma_2 \cap \pi)$.

Answers for Part B [cont.]

8. (b) Suppose that $P \notin l$. Let π be the plane which contains P and l. Since $l \subset \pi$, $l \not\perp \pi$. By Theorem 12-15(a), there is one and only one line — say, m — through P which is perpendicular to l and parallel to π. Thus, $m \subset \pi$ and $m \nparallel l$, so that m intersects l. Hence, the corollary.

   (c) Suppose that l_1, l_2, and m are lines of a plane — say, π — and that $l_1 \perp m$ and $l_2 \perp m$. Then $l_1 \not\perp \pi$ and $l_2 \not\perp \pi$ so that, by Theorem 12-15(b), there is one and only one plane — say, σ_1 — through $P_1 \in l_1$ which is perpendicular to π and parallel to l_1, and one and only one plane — say, σ_2 — through $P_2 \in l_2$ which is perpendicular to π and parallel to l_2. Since σ_1 and σ_2 are perpendicular to m, $\sigma_1 \parallel \sigma_2$. Also, $l_1 = \sigma_1 \cap \pi$ and $l_2 = \sigma_2 \cap \pi$. So, $l_1 \parallel l_2$.

   (d) Suppose that $l \not\perp \pi$ and that $P \in l$. By Theorem 12-15(b), there is one and only one plane — say, σ — through P which is perpendicular to π and parallel to l. Since $P \in l \cap \sigma$ and $l \parallel \sigma$, σ contains l. Hence, the corollary.

9. Since σ_1 and σ_2 are two intersecting planes, $\sigma_1 \cap \sigma_2$ is a line. Also, $\pi \perp \sigma_2$ so that $\pi \cap \sigma_2$ is a line. Since $l \perp \sigma_1$ and $l \subseteq \pi$, we have that $\pi \perp \sigma_1$. So, by Theorem 12-14(b), π is perpendicular to $\sigma_1 \cap \sigma_2$. This means that $\sigma_1 \cap \sigma_2$ is perpendicular to π so that, by Theorem 12-13(a), $\sigma_1 \cap \sigma_2$ is perpendicular to each line contained in π. Since $\pi \cap \sigma_2$ is such a line, we have that $(\pi \cap \sigma_2) \perp (\sigma_1 \cap \sigma_2)$.

10. Suppose that $\sigma \perp \pi$. By Theorem 12-15(a), there is one and only one line through P which is perpendicular to l and parallel to π. Since $P \in \pi$, this line is contained in π. Now, n is one such line and the theorem tells us that there is no other. So, $n \perp \sigma$. By Theorem 12-13(a), $(\sigma \cap \pi) \perp n$. Suppose, next, that $(\sigma \cap \pi) \perp n$. Since $n \subset \pi$ and n is perpendicular to each of two intersecting lines, $\pi \cap \sigma$ and l, of σ it follows that n is perpendicular to σ. So, by Exercise 1(b), $\sigma \perp \pi$.

11. Suppose that $\sigma_1 \perp \pi$ and $\sigma_2 \perp \pi$. Assume that $\sigma_1 \perp \sigma_2$. By Theorem 12-14(b), we have that $\sigma_2 \perp (\sigma_1 \cap \pi)$. So, $\sigma_1 \cap \pi$ is perpendicular to each line in σ_2. Since $\sigma_2 \cap \pi$ is a line in σ_2, $(\sigma_1 \cap \pi) \perp (\sigma_2 \cap \pi)$. Thus, if $\sigma_1 \perp \sigma_2$ then $(\sigma_1 \cap \pi) \perp (\sigma_2 \cap \pi)$. Next, assume that $(\sigma_1 \cap \pi) \perp (\sigma_2 \cap \pi)$. Then, $(\sigma_1 \cap \pi) \nparallel (\sigma_2 \cap \pi)$ and, since $\sigma_1 \cap \pi$ and $\sigma_2 \cap \pi$ are coplanar lines, they intersect in a point — say, P. By Theorem 12-15(b), σ_1 is the only plane through P which is perpendicular to π and parallel to $\sigma_1 \cap \pi$; and, σ_2 is the only plane through P which is perpendicular to π and parallel to $\sigma_2 \cap \pi$. Since $(\sigma_1 \cap \pi) \nparallel (\sigma_2 \cap \pi)$, $\sigma_1 \nparallel \sigma_2$ so that $\sigma_1 \cap \sigma_2$ is a line. Also, $\pi \perp (\sigma_1 \cap \sigma_2)$. So, $(\sigma_1 \cap \pi) \perp (\sigma_1 \cap \sigma_2)$ and $(\sigma_1 \cap \pi) \perp (\sigma_2 \cap \pi)$ which means that a line of σ_1, namely $\sigma_1 \cap \pi$, is perpendicular to each of two intersecting lines of σ_2, namely $\sigma_1 \cap \sigma_2$ and $\sigma_2 \cap \pi$. So, $\sigma_1 \perp \sigma_2$. Thus, if $(\sigma_1 \cap \pi) \perp (\sigma_2 \cap \pi)$ then $\sigma_1 \perp \sigma_2$. Hence, the biconditional is established.

Sample Quiz

1. (a) Given a line ℓ, describe $[\ell]$.
   (b) Give the details in explaining that $[\ell]$, under function composition, is a vector space over the real numbers.
   (c) What is the dimension of the vector space described in (b)?
2. (a) Suppose that $\ell \parallel m$. Can ℓ be perpendicular to m? Explain.
   (b) Can the set of points consisting of the union of two parallel lines be associated with the vector space in 1(b) in the same way that we associate $\mathcal{E}$ and $\mathcal{T}$? Explain.

Key to Sample Quiz

1. (a) $[\ell] = \{\vec{x}: \exists_{X \epsilon \ell} \exists_{Y \epsilon \ell}\ \vec{x} = Y - X\}$. [There are other equally adequate answers.]
   (b) [It is easy to check that $[\ell]$ under function composition satisfies all of the postulates for a vector space over the real numbers.]
   (c) Its dimension is 1.
2. (a) No. Here is a "brute force" demonstration: Let $\vec{l} \,\epsilon\, [\ell]$ and $\vec{m} \,\epsilon\, [m]$ such that $\vec{l} \neq \vec{0} \neq \vec{m}$. Then, since $\ell \parallel m$, $\vec{l} = \vec{m}a$, for some nonzero a. So, given that $\vec{l} \cdot \vec{m} = 0$, $(\vec{m} \cdot \vec{m})a = 0$ and, since $a \neq 0$, $\vec{m} = \vec{0}$. But, $\vec{m} \neq \vec{0}$ so that $\vec{l} \cdot \vec{m} \neq 0$. Hence, $\ell \not\perp m$ when $\ell \parallel m$.
   (b) No. Choose one point from each of the two lines. The translation from one to the other of those points is not in the direction of the lines. So, the counterpart of Postulate 1 is not satisfied for the given set of points.

Part C

1. **(a)** We know that

$$l \perp \sigma \longrightarrow [\sigma \perp \pi \longleftrightarrow l \parallel \pi]. \qquad \text{[Why?]}$$

If $\sigma \perp \pi$ and $l \parallel \pi$ does it follow that $l \perp \sigma$?

(b) Suppose that π and σ are two intersecting planes. We know that

$$l \perp \sigma \longrightarrow l \perp (\pi \cap \sigma). \qquad \text{[Explain.]}$$

If $l \perp (\pi \cap \sigma)$ does it follow that $l \perp \sigma$?

(c) Does it follow that $l \perp \sigma$
   (i) if $\sigma \perp \pi$ and $l \perp (\pi \cap \sigma)$?
   (ii) if $l \parallel \pi$ and $l \perp (\pi \cap \sigma)$?

2. Suppose that π and σ are two intersecting planes, that $l \parallel \pi$, and that $l \perp (\pi \cap \sigma)$.
   (a) Explain why it follows that $[l] \subseteq [\pi] \cap [\pi \cap \sigma]^{\perp}$.
   (b) What assumption tells you that $[\pi \cap \sigma]^{\perp}$ is a proper bidirection?
   (c) Can $[\pi]$ and $[\pi \cap \sigma]^{\perp}$ be the same set? Explain your answer.
   (d) What kind of set is $[\pi] \cap [\pi \cap \sigma]^{\perp}$? Explain your answer.
   (e) Does it follows that $[l] = [\pi] \cap [\pi \cap \sigma]^{\perp}$? Why?
3. Suppose that $\sigma \perp \pi$, $l \parallel \pi$, and $l \perp (\pi \cap \sigma)$.
   (a) Should it follow that $l \perp \sigma$?
   (b) In view of Exercise 2 and Theorem 11–4, what might we do to show that $l \perp \sigma$?
   (c) Why do we know that $[\sigma]^{\perp} \subseteq [\pi]$? That $[\sigma]^{\perp} \subseteq [\pi \cap \sigma]^{\perp}$?
   (d) Why do we know that $[\sigma]^{\perp} = [\pi] \cap [\pi \cap \sigma]^{\perp}$?
   (e) Show that $l \perp \sigma$.
4. Prove:

Theorem 12–16 $(\sigma \perp \pi \text{ and } l \perp (\pi \cap \sigma)) \longrightarrow [l \parallel \pi \longleftrightarrow l \perp \sigma]$

[You may, of course, use results proved in the preceding exercises.]

Part D

If l and m are two intersecting lines then, in any given direction, there is a line which intersects both l and m. If the given direction is not contained in the bidirection of the plane containing l and m then there is at most one such line. [Explain.] Suppose, now, that l and m are skew lines.

1. Imagine various lines which intersect both l and m, and make a conjecture as to the possible directions of such lines.
2. Are there any directions in which there certainly are no lines which intersect both l and m? Explain.

Answers for Part C

1. (a) No. For an easy counterexample, let $\ell = \pi \cap \sigma$.

   (b) No. For a counterexample, choose $P \in \pi \cap \sigma$ and let ℓ be the line through P which is perpendicular to $\pi \cap \sigma$ and parallel to σ.

   (c) (i) No.
   (ii) Yes.

2. (a) Since $\ell \parallel \pi$ and $\ell \perp (\pi \cap \sigma)$, $[\ell] \subseteq [\pi]$ and $[\ell] \subseteq [\pi \cap \sigma]^{\perp}$. So, by definition, $[\ell] \subseteq [\pi] \cap [\pi \cap \sigma]^{\perp}$.

   (b) Given that π and σ are two intersecting planes, we know that $\pi \cap \sigma$ is a line so that $[\pi \cap \sigma]$ is a proper direction.

   (c) No, for $[\pi]$ is the bidirection of planes parallel to π and $[\pi \cap \sigma]^{\perp}$ is the bidirection of planes perpendicular to $\pi \cap \sigma$ which is a line parallel to π, so that $[\pi \cap \sigma]^{\perp}$ is the bidirection of a class of planes perpendicular to π.

   (d) A proper direction, for T is 3-dimensional.

   (e) Yes. We know that $[\ell] \subseteq [\pi] \cap [\pi \cap \sigma]^{\perp}$ and that $[\pi] \cap [\pi \cap \sigma]^{\perp}$ is a proper direction. If $\vec{a}$ is a non-$\vec{0}$ member of $[\pi] \cap [\pi \cap \sigma]^{\perp}$ which is not in $[\ell]$, then there are [at least] two independent members of $[\pi] \cap [\pi \cap \sigma]^{\perp}$. But, a proper direction cannot contain two independent members. So, each non-$\vec{0}$ member of $[\pi] \cap [\pi \cap \sigma]^{\perp}$ belongs to $[\ell]$. Since $\vec{0}$ belongs to both sets in question, they are equal.

3. (a) Yes.

   (b) We might try to show that $[\sigma]^{\perp} = [\pi] \cap [\pi \cap \sigma]^{\perp}$.

   (c) $[\sigma]^{\perp} \subseteq [\pi]$, for $\sigma \perp \pi$; Since $[\pi \cap \sigma] \subseteq [\sigma]$, $[\sigma]^{\perp} \subseteq [\pi \cap \sigma]^{\perp}$.

   (d) As we argued in Exercise 2, since $[\sigma]^{\perp}$ and $[\pi] \cap [\pi \cap \sigma]^{\perp}$ are proper directions and $[\sigma]^{\perp} \subseteq [\pi] \cap [\pi \cap \sigma]^{\perp}$, it follows that $[\sigma]^{\perp} = [\pi] \cap [\pi \cap \sigma]^{\perp}$.

   (e) By the results in Exercises 2(e) and 3(d), we see that $[\ell] = [\sigma]^{\perp}$. So, $\ell \perp \sigma$.

4. Suppose that $\sigma \perp \pi$ and $\ell \perp (\pi \cap \sigma)$. Assume, first, that $\ell \parallel \pi$. Then, by Exercise 3, we know that $\ell \perp \sigma$. Assume, next, that $\ell \perp \sigma$. Then, $[\ell] = [\sigma]^{\perp}$ and, since $\sigma \perp \pi$, $[\sigma]^{\perp} \subseteq [\pi]$. So, by replacement, $[\ell] \subseteq [\pi]$. By definition, then, $\ell \parallel \pi$. Thus, $\ell \parallel \pi$ if and only if $\ell \perp \sigma$. Hence, Theorem 12-16.

Answers for Part D

1. No two of these lines have the same direction. [If you could manage to find two such lines with the same direction, then lines ℓ and m are coplanar. Since ℓ and m are given to be skew lines, the stated conjecture is a correct one.]

2. Yes. Suppose that n is a line whose direction is contained in the bidirection which contains $[\ell]$ and $[m]$. Suppose, also, that $n \cap \ell = \{P\}$ and $n \cap m = \{Q\}$. The plane through P in the specified bidirection will contain ℓ and n and, so, will contain Q and, hence, m. But ℓ and m are noncoplanar. Hence n does not intersect both ℓ and m.

3. Is there any direction in which there are two lines which intersect both l and m? [*Hint:* If both l and m were intersected by each of two parallel lines, what conclusion could you draw concerning l and m?]
4. Imagine all lines through a given point Q of m which intersect l. What can you say about the directions of these lines? [*Hint:* How many planes contain both Q and l?]
5. Repeat Exercise 4 with another point – say, R – of m.
6. Is there any plane containing l which contains no line intersecting both l and m? How many such planes are there?
7. Prove:

> **Theorem 12–17** If l, m, and n are not all parallel to the same plane then there is one and only one line which is parallel to n and intersects both l and m.

8. Prove:

> **Corollary** If $l \nparallel m$ then there is one and only one line which is perpendicular to both l and m and intersects both l and m.

12.03 Orthogonal Projections

We have proved that, given a point P and a line l, there is one and only one plane – the plane $P[l]^{\perp}$ – which contains P and is perpendicular to l. We have also proved that this plane intersects l in a single point. [What theorems justify the preceding statements?] Previously we have called the point of intersection of $P[l]^{\perp}$ and l *the orthogonal projection of P on l.*

We have also proved that, given a point P and a plane π, there is one and only one line – the line $\overleftrightarrow{P[\pi]^{\perp}}$ – which contains P and is perpendicular to π. We have also proved that this line intersects π in a single point. [Again, what theorems?] We shall call the point of intersection of $\overleftrightarrow{P[\pi]^{\perp}}$ and π *the orthogonal projection of P on π.*

> **Definition 12–4**
> (a) $\operatorname{proj}_l(P)$ = the point of intersection of l and $\overline{P[l]^{\perp}}$
> (b) $\operatorname{proj}_{\pi}(P)$ = the point of intersection of π and $\overleftrightarrow{P[\pi]^{\perp}}$

In Section 11.02 we studied, for a given proper direction $[l]$ and translation $\vec{b}$, a mapping $\operatorname{proj}_{[l]}(\vec{b})$ of $\mathscr{E}$ onto itself. Under this mapping,

Answers for Part D [cont.]

3. No, for then l and m would be coplanar lines.
4. The directions of the given lines are in the bidirection of the plane of Q and l.
5. The directions of the given lines are in the bidirection of the plane of R and l.
6. Yes, the plane whose bidirection contains the directions of l and m. There is exactly one such plane.
7. Suppose that l, m, and n are lines which are not all parallel to the same plane. Then, no two of the lines are parallel, for then the plane through one of them which is parallel to the third line would be parallel to all three lines. Let σ_1 be the plane which contains l and is parallel to n and let σ_2 be the plane which contains m and is parallel to n. σ_1 intersects m in a point — say, R — and σ_2 intersects l in a point — say, S. $\sigma_1 \nparallel \sigma_2$ so that $\sigma_1 \cap \sigma_2$ is a line which is parallel to n. Since both R and S are points of $\sigma_1 \cap \sigma_2$, $\overleftrightarrow{RS} = \sigma_1 \cap \sigma_2$. So, $\overleftrightarrow{RS}$ is a line which is parallel to n and intersects both l and m. If there were a second such line then l and m would be coplanar lines with n parallel to the plane which contains them so that all three would be parallel to the same plane. Since this is not the case, there is only one such line, and the theorem is proved.
8. Suppose that $l \nparallel m$ and that n is perpendicular to the plane containing l and parallel to m. By Theorem 12-17 there is one and only one line parallel to n and intersecting both l and m.

Answers to questions:

Theorem 12-5 and Theorem 12-1;

Theorem 12-8, Definition 12-1(b), and Theorem 12-1.

the image of any point P is the projection of $P + \overrightarrow{b}$ on $\overleftrightarrow{P[l]}$ – the line m in Fig. 12–3. Similarly, for a given proper bidirection $[\pi]$ and trans-

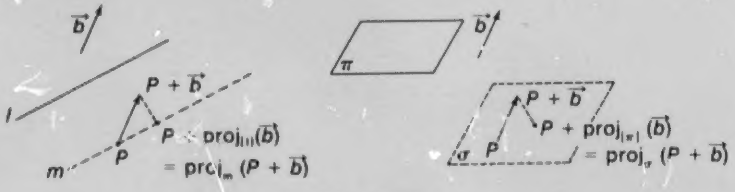

$\overrightarrow{b}$
l
$P + \overrightarrow{b}$
$P + \text{proj}_{[l]}(\overrightarrow{b})$
$= \text{proj}_m(P + \overrightarrow{b})$
m
P
π
$\overrightarrow{b}$
$P + \overrightarrow{b}$
$P + \text{proj}_{[\pi]}(\overrightarrow{b})$
$= \text{proj}_\sigma(P + \overrightarrow{b})$
σ
P

Fig. 12–3

lation $\overrightarrow{b}$, $\text{proj}_{[\pi]}(\overrightarrow{b})$ is the mapping of $\mathscr{E}$ into itself for which the image of any point P on the projection of $P + \overrightarrow{b}$ on $\overleftrightarrow{P[\pi]}$ – the plane σ in Fig. 12–3.

Using Definition 12–4(a) it follows that

$$P + \text{proj}_{[l]}(\overrightarrow{b}) = Q$$
$$\longleftrightarrow$$
$$\left(Q - P \in [l] \text{ and } (P + \overrightarrow{b}) - Q \in [l]^\perp\right).$$

This [with '$(P + \overrightarrow{b}) - Q$' replaced by '$b - (Q - P)$'] was adopted in Section 11–10 as the definition of '$\text{proj}_{[l]}(\overrightarrow{b})$'. A similar use of Definition 12–4(b) suggests:

> **Definition 12–5** $P + \text{proj}_{[\pi]}(\overrightarrow{b}) = Q$
> $$\longleftrightarrow$$
> $$\left(Q - P \in [\pi] \text{ and } \overrightarrow{b} - (Q - P) \in [\pi]^\perp\right)$$

To justify accepting this definition we need to show that, given $[\pi]$, $\overrightarrow{b}$, and P, there is one and only one point X such that

$$X - P \in [\pi] \text{ and } \overrightarrow{b} - (X - P) \in [\pi]^\perp.$$

Now, $[\pi]^\perp$ is a proper direction – say, $[l]$ – and if $[\pi]^\perp = [l]$ then $[\pi] = [l]^\perp$. So, what we have to show is that there is one and only one point X such that

$$X - P \in [l]^\perp \text{ and } \overrightarrow{b} - (X - P) \in [l].$$

Since we have previously shown [see (*) on page 53] that $\text{proj}_{[l]}(\overrightarrow{b})$ is the only translation $\overrightarrow{x}$ such that

$$\overrightarrow{b} - \overrightarrow{x} \in [l]^\perp \text{ and } \overrightarrow{x} \in [l]$$

it follows that $\overrightarrow{b} - \text{proj}_{[l]}(\overrightarrow{b})$ is the only translation $\overrightarrow{y}$ such that

$$\overrightarrow{y} \in [l]^\perp \text{ and } \overrightarrow{b} - \overrightarrow{y} \in [l].$$

In other words, $\overrightarrow{b} - \text{proj}_{[\pi]^\perp}(\overrightarrow{b})$ is the only translation $\overrightarrow{y}$ such that

$$(*) \qquad \overrightarrow{y} \in [\pi] \text{ and } \overrightarrow{b} - \overrightarrow{y} \in [\pi]^\perp.$$

It follows that $Q - P \in [\pi]$ and $\overrightarrow{b} - (Q - P) \in [\pi]^\perp$ if and only if $Q = P + (\overrightarrow{b} - \text{proj}_{[\pi]^\perp}(\overrightarrow{b}))$. This result justifies adopting Definition 12–5 and, at the same time, proves:

> **Theorem 12–18** $\text{proj}_{[\pi]}(\overrightarrow{b}) = \overrightarrow{b} - \text{proj}_{[\pi]^\perp}(\overrightarrow{b})$

From this, together with the fact that $\overrightarrow{b} - \text{proj}_{[\pi]^\perp}(\overrightarrow{b})$ is the only solution of (*), we have:

> **Corollary 1**
> (a) $\text{proj}_{[\pi]}(\overrightarrow{b}) \in \mathscr{T}$
> (b) $\text{proj}_{[\pi]}(\overrightarrow{b}) = \overrightarrow{c} \longleftrightarrow (\overrightarrow{c} \in [\pi] \text{ and } \overrightarrow{b} - \overrightarrow{c} \in [\pi]^\perp)$

Using Theorem 11–10 we obtain another:

> **Corollary 2** $(\overrightarrow{a}, \overrightarrow{b})$ is orthogonal
> $\longrightarrow \text{proj}_{[\overrightarrow{a}, \overrightarrow{b}]}(\overrightarrow{r}) = \text{proj}_{[\overrightarrow{a}]}(\overrightarrow{r}) + \text{proj}_{[\overrightarrow{b}]}(\overrightarrow{r})$

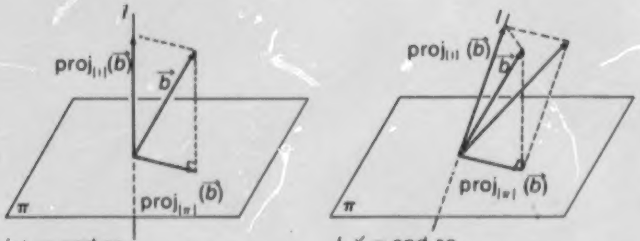
Definition 12–5 and the first corollary to Theorem 12–18 are analogous to Definition 11–2 and the corollary to Theorem 11–6. Naturally enough, there is an analogue to Theorem 11–7:

> **Theorem 12–19**
> (a) $\text{proj}_{[\pi]}(\overrightarrow{b}) \in [\pi]$ and $\text{proj}_{[\pi]}(\overrightarrow{b}) = \overrightarrow{b} \longleftrightarrow \overrightarrow{b} \in [\pi]$
> (b) $\text{proj}_{[\pi]}(\overrightarrow{b}) = \overrightarrow{0} \longleftrightarrow \overrightarrow{b} \in [\pi]^\perp$
> (c) $\text{proj}_{[\pi]}(\overrightarrow{b} + \overrightarrow{c}) = \text{proj}_{[\pi]}(\overrightarrow{b}) + \text{proj}_{[\pi]}(\overrightarrow{c})$
> (d) $\text{proj}_{[\pi]}(\overrightarrow{b}b) = \text{proj}_{[\pi]}(\overrightarrow{b})b$

This follows easily from Theorem 12–18 and Theorem 11–7. [For part (a), use Corollary 1(b) of Theorem 12–18.] In addition, we have:

> **Theorem 12–20** $l \perp \pi \longrightarrow \overrightarrow{b} = \text{proj}_{[l]}(\overrightarrow{b}) + \text{proj}_{[\pi]}(\overrightarrow{b})$

This theorem tells us that if a line is perpendicular to a plane then each vector is the sum of its orthogonal projections on the directions of the line and the plane. Fig. 12–4 pictures this.



l
$\text{proj}_{[l]}(\vec{b})$
$\vec{b}$
$\text{proj}_{[\pi]}(\vec{b})$
π
l
$\text{proj}_{[l]}(\vec{b})$
$\vec{b}$
$\text{proj}_{[\pi]}(\vec{b})$
π

$l \perp \pi$ and so
$\vec{b} = \text{proj}_{[l]}(\vec{b}) + \text{proj}_{[\pi]}(\vec{b})$

$l \not\perp \pi$ and so
$\vec{b} \neq \text{proj}_{[l]}(\vec{b}) + \text{proj}_{[\pi]}(\vec{b})$

Fig. 12–4

In Section 11.02 we established a connection between proj_l and $\text{proj}_{[l]}$ which can be stated as part (a) of:

> Theorem 12–21
> (a) $\text{proj}_l\,(P + \vec{b}) = \text{proj}_l\,(P) + \text{proj}_{[l]}\,(\vec{b})$
> (b) $\text{proj}_\pi\,(P + \vec{b}) = \text{proj}_\pi\,(P) + \text{proj}_{[\pi]}\,(\vec{b})$

The argument given in Section 11.02 to show that

$$\text{proj}_{[l]}\,(\vec{b}) = \text{proj}_l\,(P + \vec{b}) - \text{proj}_l\,(P)$$

can, now, serve as a proof of part (a). Another proof of part (a) can be carried out along the lines of that suggested for part (b) in the exercises which follow.

Recall that for any set $\mathscr{X}$ of points and mapping f of $\mathscr{E}$ into itself, $f(\mathscr{X})$ is the set whose members are the images under f of the members of $\mathscr{X}$. Using Theorem 12–21(b) and a couple of parts of Theorem 12–19 it is easy to prove:

> Theorem 12–22
> (a) $\text{proj}_\pi\,(l)$ is a line if and only if $l \not\perp \pi$.
> (b) If l and m are parallel lines which are not perpendicular to π then $\text{proj}_\pi\,(l) \parallel \text{proj}_\pi\,(m)$.

In the same order of ideas we have:

> Theorem 12–23 If $l \not\perp \pi$ then $\text{proj}_\pi\,(l)$ is the intersection with π of the plane which contains l and is perpendicular to π.

Sample Quiz

True or False?

1. Given that both ℓ and m are perpendicular to σ, it follows that $\ell \perp m$.
2. Given that both ℓ and m are parallel to σ, it follows that $\ell \parallel m$.
3. If both π and σ are parallel to ℓ, $\pi \parallel \sigma$.
4. If both π and σ are perpendicular to ℓ, $\pi \perp \sigma$.
5. Given that $\pi \nparallel \sigma$ and $\ell \perp \pi$, it follows that $\ell \perp (\pi \cap \sigma)$.
6. Given that $\pi \nparallel \sigma$ and $\ell \not\perp \pi$, it follows that $\ell \not\perp (\pi \cap \sigma)$.
7. Given that both ℓ and m are perpendicular to σ, it follows that $\ell \parallel m$.
8. Given that both ℓ and m are parallel to σ, it follows that $\ell \perp m$.
9. If ℓ is perpendicular to $\pi \cap \sigma$, where $\pi \nparallel \sigma$, ℓ is perpendicular to both π and σ.
10. If ℓ is perpendicular to $\pi \cap \sigma$, where $\pi \nparallel \sigma$, then either $\ell \perp \pi$ or $\ell \perp \sigma$.

Key to Sample Quiz

1. False. 2. False. 3. False. 4. False. 5. True.
6. False. 7. True. 8. False. 9. False. 10. False.

Exercises

Part A

1. Prove Theorem 12–19.
2. To prove Theorem 12–21(b) it is sufficient to show that

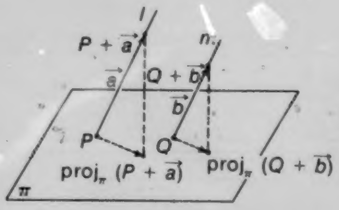
$$\text{proj}_{[\pi]}(\vec{b}) = \text{proj}_\pi(P + \vec{b}) - \text{proj}_\pi(P).$$

According to Corollary 1(b) of Theorem 12–18, this can be established by proving:
   - (i) $\text{proj}_\pi(P + \vec{b}) - \text{proj}_\pi(P) \in [\pi]$
   - (ii) $\vec{b} - \left(\text{proj}_\pi(P + \vec{b}) - \text{proj}_\pi(P)\right) \in [\pi]^\perp$

   Use Definition 12–4(b) to prove (i) and (ii).
3. What is $\text{proj}_\pi(l)$ in case $l \perp \pi$?
4. (a) Use Theorem 12–21(b) to prove Theorem 12–22(a).
   (b) Prove Theorem 12–22(b).
5. If $\text{proj}_\pi(l) \parallel \text{proj}_\pi(m)$, does it follow that $l \parallel m$?
6. (a) If $l \perp m$, does it follow that $\text{proj}_\pi(l) \perp \text{proj}_\pi(m)$?
   (b) If $\text{proj}_\pi(l) \perp \text{proj}_\pi(m)$, does it follow that $l \perp m$?
7. (a) Show that if $l \perp m$ and $l \parallel \pi$ and $m \not\perp \pi$ then $\text{proj}_\pi(l) \perp \text{proj}_\pi(m)$.
   (b) Make a conjecture:

   If $\text{proj}_\pi(l) \perp \text{proj}_\pi(m)$ then $l \perp m$ if and only if . . . .
8. Prove Theorem 12–23.

Part B


l
m
$P + \vec{a}$
$\vec{a}$
$Q + \vec{b}$
$\vec{b}$
P
Q
$\text{proj}_\pi(P + \vec{a})$
$\text{proj}_\pi(Q + \vec{b})$
π

1. Suppose that l and m are parallel lines which are not perpendicular to π, that $l \cap \pi = \{P\}$ and $m \cap \pi = \{Q\}$, and that $\vec{a} \in [l]$ and $\vec{b} \in [m]$, as shown in the picture at the right.
   (a) What is $\text{proj}_\pi(P + \vec{a}) - P$? $\text{proj}_\pi(Q + \vec{b}) - Q$?
   (b) Is either of the translations given in part (a) $\vec{0}$? Explain your answer.
   (c) Are the translations given in part (a) linearly dependent or independent? Explain.
2. Show that if $[\vec{a}] = [\vec{b}]$ and $[\vec{b}] \not\subseteq [\pi]^\perp$ then

$$\text{proj}_{[\pi]}(\vec{a}) : \text{proj}_{[\pi]}(\vec{b}) = \vec{a} : \vec{b}.$$

We recommend that Part A be used as supervised class exercises. This will ensure that students learn to apply the properties introduced in this section. Parts B and C are appropriate for homework or independent study but this should be followed by class discussion of the exercises.

Answers for Part A

1. Proof of Theorem 12-19:

(a) By Corollary 1 to Theorem 12-18, we know that if $\text{proj}_{[\pi]}(\vec{b}) = \text{proj}_{[\pi]}(\vec{b})$ then $(\text{proj}_{[\pi]}(\vec{b}) \in [\pi]$ and $\vec{b} - \text{proj}_{[\pi]}(\vec{b}) \in [\pi]^\perp)$. Since $\text{proj}_{[\pi]}(\vec{b}) = \text{proj}_{[\pi]}(\vec{b})$, it follows by modus ponens and a rule for conjunction sentences that $\text{proj}_{[\pi]}(\vec{b}) \in [\pi]$. It also follows from Corollary 1 that $\text{proj}_{[\pi]}(\vec{b}) = \vec{b}$ if and only if $(\vec{b} \in [\pi]$ and $\vec{b} - \vec{b} \in [\pi]^\perp)$. Since $\vec{b} - \vec{b} = \vec{0} \in [\pi]^\perp$, it follows that $\text{proj}_{[\pi]}(\vec{b}) = \vec{b}$ if and only if $\vec{b} \in [\pi]$.

(b) By Corollary 1(b), $\text{proj}_{[\pi]}(\vec{b}) = \vec{0}$ if and only if $(\vec{0} \in [\pi]$ and $\vec{b} - \vec{0} \in [\pi]^\perp)$. Since $\vec{0} \in [\pi]$ and $\vec{b} - \vec{0} = \vec{b}$, we have that $\text{proj}_{[\pi]}(\vec{b}) = \vec{0}$ if and only if $\vec{b} \in [\pi]^\perp$.

(c) By Theorems 12-18 and 11-7, we have that

$$\begin{aligned}\text{proj}_{[\pi]}(\vec{b} + \vec{c}) &= (\vec{b} + \vec{c}) - (\text{proj}_{[\pi]^\perp}(\vec{b} + \vec{c}))\\ &= (\vec{b} + \vec{c}) - (\text{proj}_{[\pi]^\perp}(\vec{b}) + \text{proj}_{[\pi]^\perp}(\vec{c}))\\ &= (\vec{b} - \text{proj}_{[\pi]^\perp}(\vec{b})) + (\vec{c} - \text{proj}_{[\pi]^\perp}(\vec{c}))\\ &= \text{proj}_{[\pi]}(\vec{b}) + \text{proj}_{[\pi]}(\vec{c}).\end{aligned}$$

(d) By Theorems 12-18 and 11-7, we have that

$$\begin{aligned}\text{proj}_{[\pi]}(\vec{b}b) &= \vec{b}b - \text{proj}_{[\pi]^\perp}(\vec{b}b)\\ &= \vec{b}b - \text{proj}_{[\pi]^\perp}(\vec{b})b\\ &= (\vec{b} - \text{proj}_{[\pi]^\perp}(\vec{b}))b\\ &= \text{proj}_{[\pi]}(\vec{b})b.\end{aligned}$$

2. (i) By Definition 12-4(b), both $\text{proj}_\pi(P + \vec{b})$ and $\text{proj}_\pi(P)$ are points of π. So, by Definition 9-4, $\text{proj}_\pi(P + \vec{b}) - \text{proj}_\pi(P) \in [\pi]$.

(ii) Note that $\vec{b} - (\text{proj}_\pi(P + \vec{b}) - \text{proj}_\pi(P)) = (\text{proj}_\pi(P) + \vec{b}) - \text{proj}_\pi(P + \vec{b})$. Now, by Definition 12-4(b), $\text{proj}_\pi(P + \vec{b})$ is on the line $\overleftrightarrow{(P + \vec{b})[\pi]^\perp}$. Furthermore, $\text{proj}_\pi(P) + \vec{b} = (P + \vec{b}) + (\text{proj}_\pi(P) - P)$ and $\text{proj}_\pi(P) - P \in [\pi]^\perp$. So, $\text{proj}_\pi(P) + \vec{b}$ is on the line $\overleftrightarrow{(P + \vec{b})[\pi]^\perp}$. Thus, $(\text{proj}_\pi(P) + \vec{b}) - \text{proj}_\pi(P + \vec{b}) \in [\pi]^\perp$. Hence, by what we noted first, $\vec{b} - (\text{proj}_\pi(P + \vec{b}) - \text{proj}_\pi(P)) \in [\pi]^\perp$.

3. In case $\ell \perp \pi$, $\text{proj}_\pi(\ell)$ = the point of intersection of ℓ and π. [This follows from Definition 12-4(b) and Theorem 12-19(b).]

Answers for Part A [cont.]

4. (a) Suppose that $A \in \ell$ and $\vec{0} \neq \vec{a} \in [\ell]$. It follows that $\ell = \overleftrightarrow{A[\vec{a}]}$. So, $P \in \text{proj}_\pi(\ell)$ if and only if there is some number — say, b — such that $P = \text{proj}_\pi(A + \vec{a}b)$. Now, $\text{proj}_\pi(A + \vec{a}b) = \text{proj}_\pi(A) + \text{proj}_{[\pi]}(\vec{a})b$, by Theorems 12-21(b) and 12-19(d). Hence, $\text{proj}_\pi(\ell) = \overleftrightarrow{\text{proj}_\pi(A)[\text{proj}_{[\pi]}(\vec{a})]}$ and is a line if and only if $\text{proj}_{[\pi]}(\vec{a}) \neq \vec{0}$. So, by Theorem 21-19(b), $\text{proj}_\pi(\ell)$ is a line if and only if $\vec{a} \notin [\pi]^\perp$ — that is, if and only if $\ell \not\perp \pi$.

(b) By the argument just given, if ℓ and m are parallel lines not perpendicular to π then $\text{proj}_\pi(\ell)$ and $\text{proj}_\pi(m)$ are lines whose direction is that of $\text{proj}_{[\pi]}(\vec{a})$ where $\vec{a}$ is any non-$\vec{0}$ vector in the common direction of ℓ and m.

5. No. For a counterexample take ℓ and m not perpendicular to π but in planes which are perpendicular to π.

6. (a) No. For a counterexample take ℓ and m perpendicular but neither parallel to π.

(b) No. Same counterexamples as in part (a).

7. (a) [We prove, more generally, that if $\ell \perp m$ and neither is perpendicular to π then $\text{proj}_{[\pi]}(\ell) \perp \text{proj}_{[\pi]}(m)$ if and only if $(\ell \parallel \pi$ or $m \parallel \pi)$.] Suppose $[\ell] = [\vec{a}]$ and $[m] = [\vec{b}]$. Suppose that $\ell \perp m$ — that is, that $\vec{a} \cdot \vec{b} = 0$. By Theorem 12-18, $\text{proj}_{[\pi]}(\vec{a}) = \vec{a} - \text{proj}_{[\pi]^\perp}(\vec{a})$ and $\text{proj}_{[\pi]}(\vec{b}) = \vec{b} - \text{proj}_{[\pi]^\perp}(\vec{b})$. So, $\text{proj}_{[\pi]}(\vec{a}) \cdot \text{proj}_{[\pi]}(\vec{b}) = \vec{a} \cdot \vec{b} + \text{proj}_{[\pi]^\perp}(\vec{a}) \cdot \text{proj}_{[\pi]^\perp}(\vec{b}) - (\vec{a} \cdot \text{proj}_{[\pi]^\perp}(\vec{b}) + \vec{b} \cdot \text{proj}_{[\pi]^\perp}(\vec{a}))$. Since $\vec{a} \cdot \vec{b} = 0$ it follows that $\text{proj}_{[\pi]}(\vec{a}) \cdot \text{proj}_{[\pi]}(\vec{b}) = 0$ — that is, that $\text{proj}_\pi(\ell) \perp \text{proj}_\pi(m)$ — if and only if $\text{proj}_{[\pi]^\perp}(\vec{a}) \cdot \text{proj}_{[\pi]^\perp}(\vec{b}) = \vec{a} \cdot \text{proj}_{[\pi]^\perp}(\vec{b}) + \vec{b} \cdot \text{proj}_{[\pi]^\perp}(\vec{a})$. This last is the case if and only if $\text{proj}_{[\pi]^\perp}(\vec{a}) \cdot (\text{proj}_{[\pi]^\perp}(\vec{b}) - \vec{b}) = \vec{a} \cdot \text{proj}_{[\pi]^\perp}(\vec{b})$ — that is, if and only if $\text{proj}_{[\pi]^\perp}(\vec{a}) \cdot \text{proj}_{[\pi]}(\vec{b}) + \vec{a} \cdot \text{proj}_{[\pi]^\perp}(\vec{b}) = 0$. This is the case if and only if $\vec{a} \cdot \text{proj}_{[\pi]^\perp}(\vec{b}) = 0$ — that is, if and only if $(\vec{a} \in [\pi]$ or $\vec{b} \in [\pi])$. Hence, $\text{proj}_{[\pi]}(\vec{a}) \cdot \text{proj}_{[\pi]}(\vec{b}) = 0$ — that is, $\text{proj}_\pi(\ell) \perp \text{proj}_\pi(m)$ — if and only if $(\ell \parallel \pi$ or $m \parallel \pi)$.

(b) If $\text{proj}_\pi(\ell) \perp \text{proj}_\pi(m)$ then $\ell \perp m$ if and only if at least one of ℓ and m is parallel to π. [The proof is like that in (a).]

Answers for Part A [cont.]

8. Suppose that $\ell \not\perp \pi$. Then, $\text{proj}_\pi(\ell)$ is a line. Let P and Q be two points of ℓ. Then $\text{proj}_\pi(P)$ and $\text{proj}_\pi(Q)$ are two points of $\text{proj}_\pi(\ell)$ which are by Definition 12-4(b), the points of intersection of π and $\overleftrightarrow{P[\pi]^\perp}$ and $\overleftrightarrow{Q[\pi]^\perp}$, respectively. The lines $\overleftrightarrow{P[\pi]^\perp}$ and $\overleftrightarrow{Q[\pi]^\perp}$ are parallel and, so, are contained in a plane — say, σ. Since P, Q, $\text{proj}_\pi(P)$, and $\text{proj}_\pi(Q)$ are points of σ, both ℓ and $\text{proj}_\pi(\ell)$ are contained in σ. Furthermore, the line $\overleftrightarrow{P[\pi]^\perp}$ is contained in σ and is perpendicular to π. So, $\sigma \perp \pi$. Thus, $\text{proj}_\pi(\ell)$ is the intersection of π and σ, and σ is the plane which contains ℓ and is perpendicular to π.

Answers for Part B

1. (a) $\text{proj}_{[\pi]}(\vec{a})$; $\text{proj}_{[\pi]}(\vec{b})$.

(b) No, for neither ℓ nor m is perpendicular to π.

(c) Linearly dependent, for both belong to the common direction of $\text{proj}_\pi(\ell)$ and $\text{proj}_\pi(m)$. [See answer for Exercise 4(b), Part A.]

2. Since $[\vec{a}] = [\vec{b}] \not\subset [\pi]^\perp$, we see that neither $\vec{a}$ nor $\vec{b}$ is $\vec{0}$ and that $\vec{a} = \vec{b}a$, for some $a \neq 0$. So, $a = \vec{a} : \vec{b}$. Also, neither $\text{proj}_{[\pi]}(\vec{a})$ nor $\text{proj}_{[\pi]}(\vec{b})$ is $\vec{0}$ so that

$$\begin{aligned} \text{proj}_{[\pi]}(\vec{a}) : \text{proj}_{[\pi]}(\vec{b}) &= \text{proj}_{[\pi]}(\vec{b}a) : \text{proj}_{[\pi]}(\vec{b}) \\ &= \text{proj}_{[\pi]}(\vec{b})a : \text{proj}_{[\pi]}(\vec{b}) \\ &= \big(\text{proj}_{[\pi]}(\vec{b}) : \text{proj}_{[\pi]}(\vec{b})\big)a \\ &= a = \vec{a} : \vec{b}. \end{aligned}$$

Hence, if $[\vec{a}] = [\vec{b}] \not\subset [\pi]^\perp$ then $\text{proj}_{[\pi]}(\vec{a}) : \text{proj}_{[\pi]}(\vec{b}) = \vec{a} : \vec{b}$.

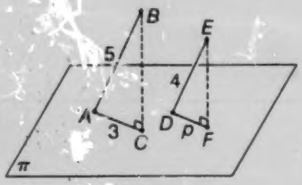
3. Prove:

Theorem 12–24

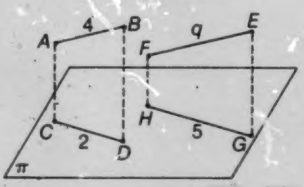
(a) If $\overline{AB}$ and $\overline{CD}$ are nondegenerate parallel intervals which are not perpendicular to π then $\overline{\text{proj}_\pi(\overline{CD})} : \overline{\text{proj}_\pi(\overline{AB})} = \overline{CD} : \overline{AB}$.

(b) If $\overline{AB}$ is a nondegenerate interval which is not perpendicular to π and $\sigma \parallel \pi$ then $\overline{\text{proj}_\sigma(\overline{AB})} : \overline{\text{proj}_\pi(\overline{AB})} = 1$.

4. (a) Given that $\overline{AB} \parallel \overline{DE}$ and that $\overline{AC}$ is the projection of $\overline{AB}$ on π and $\overline{DF}$ is the projection of $\overline{DE}$ on π, make use of the information in the picture at the right to compute the value of 'p'.



(b) Given that $\overline{AB} \parallel \overline{EF}$ and that $\overline{CD}$ is the projection of $\overline{AB}$ on π and $\overline{GH}$ is the projection of $\overline{EF}$ on π, use the information given in the picture at the right to compute the value of 'q'.



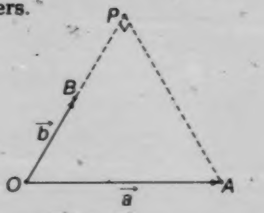
5. (a) In Exercise 4(a), compute a value of 'p' given that $\overline{AB} : \overline{DE} = \frac{7}{5}$ and $AB : AC = \frac{14}{9}$. Also, what is $\overline{EF} : \overline{BC}$ under these conditions?

(b) In Exercise 4(b), compute the value of 'q' given that $\overline{HG} : \overline{CD} = \frac{9}{5}$ and $\overline{CD} : \overline{AB} = \frac{3}{4}$.

(c) Is it possible to compute $\overline{BD} : \overline{EG}$ with the information given in Exercise 4(b)? How about with the information given in Exercise 5(b)? Explain your answers.

(d) Is it possible to compute $\overline{BC} : \overline{EF}$ with the information given in Exercise 4(a)? How about with the information given in Exercise 5(a)? Explain your answers.

Part C



Given points O, A, and B, with $B \neq O$, let $A - O = \vec{a}$ and $B - O = \vec{b}$. Suppose that $P \in \overleftrightarrow{OB}$. So, $P = O + \vec{b}b$ for some real number b.

1. Express $A - P$ as a linear combination of $\vec{a}$ and $\vec{b}$.
2. For what value of 'b' does $A - P$ belong to $[\vec{b}]^\perp$?
3. Compute $\|A - P\|^2$ when 'b' has the value obtained in Exercise 2.

Answers for Part B [cont.]

3. (a) Given that $\overline{AB}$ and $\overline{CD}$ are nondegenerate parallel intervals, which are not perpendicular to π, so are $\text{proj}_\pi(\overline{AB})$ and $\text{proj}_\pi(\overline{CD})$. So, $\overline{\text{proj}_\pi(\overline{CD})} : \overline{\text{proj}_\pi(\overline{AB})} = |\text{proj}_{[\pi]}(D - C) : \text{proj}_{[\pi]}(B - A)| = |(D - C) : (B - A)| = \overline{CD} : \overline{AB}$.

(b) Given that $\overline{AB}$ is a nondegenerate interval which is not perpendicular to π and $\sigma \parallel \pi$ then $\overline{AB} \not\perp \sigma$. Furthermore,

$$\begin{aligned} \overline{\text{proj}_\sigma(\overline{AB})} : \overline{\text{proj}_\pi(\overline{AB})} &= |\text{proj}_{[\sigma]}(B - A) : \text{proj}_{[\pi]}(B - A)| \\ &= |\text{proj}_{[\pi]}(B - A) : \text{proj}_{[\pi]}(B - A)| \\ &= 1. \end{aligned}$$

4. (a) By Theorem 12-24(a), $p/3 = 4/5$. So, $p = 12/5$.

(b) By Theorem 12-24(a), $q/4 = 5/2$. So, $q = 10$.

5. (a) $p/9 = 10/14$ so that $p = 45/7$; $\overline{EF} : \overline{BC} = 5/7$.

(b) $q/4 = 9/5$ so that $q = 36/5$.

(c) No. No. In each case, it is possible to satisfy the conditions in the problem for any choices whatever for $\|B - D\|$ and $\|E - G\|$.

(d) Yes. Yes. In each case, we are dealing with triangles whose corresponding sides are parallel.

Answers for Part C

1. $A - P = \vec{a} + \vec{b} \cdot -b$ [or: $\vec{a} - \vec{b}b$]
2. $A - P \in [\vec{b}]^\perp$ if and only if $(A - P) \cdot \vec{b} = 0$, that is, if and only if $(\vec{a} - \vec{b}b) \cdot \vec{b} = 0$. The latter is the case if and only if $b = (\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})$.
3. $$\begin{aligned} \|A - P\|^2 &= \left(\vec{a} - \vec{b}(\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})\right) \cdot \left(\vec{a} - \vec{b}(\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})\right) \\ &= \vec{a} \cdot \vec{a} + (\vec{a} \cdot \vec{b})^2/(\vec{b} \cdot \vec{b}) - 2(\vec{a} \cdot \vec{b})^2/(\vec{b} \cdot \vec{b}) \\ &= \vec{a} \cdot \vec{a} - (\vec{a} \cdot \vec{b})^2/(\vec{b} \cdot \vec{b}) \end{aligned}$$

4. Compute $\|P - O\|^2$ for the same value of 'b'.
5. Compare $\|A - P\|^2 + \|P - O\|^2$ and $\|(A - P) + (P - O)\|^2$.
6. What theorems in this section have as a consequence:

$$\vec{a} \in [\vec{b}]^{\perp} \longleftrightarrow \|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2?$$

7. Returning to the expression you found in Exercise 1, find the value of 'b' for which $\|A - P\|$ is as small a number as possible when P is restricted to $\overleftrightarrow{OB}$. [*Hint*: Note that

$$\vec{a} - \vec{b}b = \left(\vec{a} - \vec{b}\,\text{comp}_{\vec{b}}(\vec{a})\right) + \left(\vec{b}\,\text{comp}_{\vec{b}}(\vec{a}) - \vec{b}b\right)$$

and use the results obtained in Exercises 2 and 6.]

8. Recalling that we shall eventually define the distance between two points to be the norm of their difference—$d(A, B) = \|B - A\|$—Exercise 5 [or 6] may suggest a theorem about triangles of a special kind; and Exercises 3 and 7 may suggest results concerning the shortest distance between a point and a line. Can you figure out some of these theorems?

12.04 Orthogonality

As we saw earlier, $\vec{a} \in [\vec{b}]^{\perp}$ if and only if $\vec{a} \cdot \vec{b} = 0$. Now, when $\vec{a} \in [\vec{b}]^{\perp}$ it is customary to say that $\vec{a}$ is orthogonal to $\vec{b}$. In these terms, then, $\vec{a}$ is orthogonal to $\vec{b}$ if and only if $\vec{a} \cdot \vec{b} = 0$. As a matter of fact, since $\vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}$, it follows that if $\vec{a}$ is orthogonal to $\vec{b}$ then $\vec{b}$ is orthogonal to $\vec{a}$. So, it is also customary, when $\vec{a} \cdot \vec{b} = 0$, to say that $\vec{a}$ and $\vec{b}$ are orthogonal.

Suppose, now, that $\vec{a}$ and $\vec{b}$ are orthogonal and that $\vec{p} \in [\vec{a}]$ and $\vec{q} \in [\vec{b}]$. It is not difficult to show that $\vec{p}$ and $\vec{q}$ are orthogonal. [Do so.] Under these conditions, it is customary to say that $[\vec{a}]$ is orthogonal to $[\vec{b}]$, or that $[\vec{a}]$ and $[\vec{b}]$ are orthogonal.

These notions suggest the following definitions for orthogonality of vectors and of directions of vectors.

> **Definition 12-6**
> (a) $\vec{a} \perp \vec{b} \longleftrightarrow \vec{a} \cdot \vec{b} = 0$
> (b) $[\vec{a}] \perp [\vec{b}] \longleftrightarrow \forall_{\vec{x}} \forall_{\vec{y}} [(\vec{x} \in [\vec{a}] \text{ and } \vec{y} \in [\vec{b}]) \longrightarrow \vec{x} \perp \vec{y}]$

[Read '$\vec{a} \perp \vec{b}$' as you would '$\vec{a}$ is orthogonal to $\vec{b}$'; read '$[\vec{a}] \perp [\vec{b}]$' as you would 'the direction of $\vec{a}$ is orthogonal to the direction of $\vec{b}$'.]

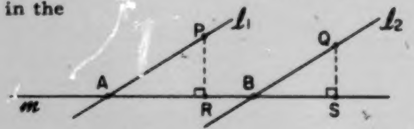
Answers for Part C [cont.]

4. $||P - O||^2 = (\vec{b}(\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})) \cdot (\vec{b}(\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b}))$
$= (\vec{a} \cdot \vec{b})^2/(\vec{b} \cdot \vec{b})$
5. $||A - P||^2 + ||P - O||^2 = \vec{a} \cdot \vec{a} - (\vec{a} \cdot \vec{b})^2/(\vec{b} \cdot \vec{b}) + (\vec{a} \cdot \vec{b})^2/(\vec{b} \cdot \vec{b})$
$= \vec{a} \cdot \vec{a}$. $||(A - P) + (P - O)||^2 = ||A - O||^2 = ||\vec{a}||^2 = \vec{a} \cdot \vec{a}$.
Hence, for the value of 'b' obtained in Exercise 2,
$||A - P||^2 + ||P - O||^2 = ||(A - P) + (P - O)||^2$.
6. Theorems 12-18 and 12-20.
7. $||A - P||^2 = (\vec{b} \cdot \vec{b})b^2 - 2(\vec{a} \cdot \vec{b})b + \vec{a} \cdot \vec{a}$. So, the least value of $||A - P||$ is the least value of $(\vec{b} \cdot \vec{b})b^2 - 2(\vec{a} \cdot \vec{b})b + \vec{a} \cdot \vec{a}$. Noting that
$$(\vec{b} \cdot \vec{b})b^2 - 2(\vec{a} \cdot \vec{b})b + \vec{a} \cdot \vec{a} = (\vec{b} \cdot \vec{b})\left(b^2 - 2b(\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})\right) + \vec{a} \cdot \vec{a}$$
$$= (\vec{b} \cdot \vec{b})\left(b - (\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})\right)^2 + \vec{a} \cdot \vec{a} - (\vec{a} \cdot \vec{b})^2/(\vec{b} \cdot \vec{b})$$
we see that $||A - P||$ is as small as it can be when $b = (\vec{a} \cdot \vec{b})/(\vec{b} \cdot \vec{b})$. [As a matter of fact, we can see that the least value for '$||A - P||$' is $\sqrt{\vec{a} \cdot \vec{a} - (\vec{a} \cdot \vec{b})^2/(\vec{b} \cdot \vec{b})}$.]
8. Exercises 5 and 6 suggest the Pythagorean theorem; Exercises 3 and 7 suggest that the shortest distance from a point to a line is along the perpendicular from the point to the line.

*

Sample Quiz

Suppose that $\ell_1 \parallel \ell_2$, $R = \text{proj}_m(P)$, and that $S = \text{proj}_m(Q)$, as shown in the picture at the right.



1. Give each of the following in terms of A, P, R, B, Q, S.
(a) $\text{proj}_{[m]}(P - A) =$ _____ (b) $\text{proj}_{[m]}(S - Q) =$ _____
(c) $\text{proj}_{[m]}(B - S) =$ _____ (d) $\text{proj}_{[m]}(B - Q) =$ _____
2. Given that $R - A = S - B$, what can you say about $P - A$ and $Q - B$?
3. Given that $P - A = (Q - B)2$, what can you say about $\text{proj}_{[m]}(P - A)$ and $\text{proj}_{[m]}(Q - B)$?
4. Given that $P - A = (Q - B)3$, what can you say about $(P - A) \cdot \text{proj}_{[m]}(P - A)$ and $(Q - B) \cdot \text{proj}_{[m]}(Q - B)$?

Key to Sample Quiz

1. (a) $R - A$ (b) $S - S$ [or: $\vec{0}$] (c) $B - S$ (d) $B - S$
2. They are equal.
3. The former is twice the latter. [I.e., $\text{proj}_{[m]}(P - A) = \text{proj}_{[m]}(Q - B) \cdot 2$.]
4. The former is 9 times the latter.

An immediate consequence of Definition 12–6 is:

‖ Theorem 12–25 $[\vec{a}] \perp [\vec{b}] \longleftrightarrow \vec{a} \perp \vec{b}$

[Prove this theorem.] Now that we are able to talk in formal terms about orthogonal directions, we can make use of what we have so far to obtain a useful criterion for perpendicularity of lines in terms of their directions. We do this as follows:

Given lines l and m, there are proper translations—say, $\vec{a}$ and $\vec{b}$—such that $[l] = [\vec{a}]$ and $[m] = [\vec{b}]$. Now, $l \perp m$ if and only if $[l] \subseteq [m]^{\perp}$. So, $l \perp m$ if and only if $[\vec{a}] \subseteq [\vec{b}]^{\perp}$. Since $[\vec{a}] \subseteq [\vec{b}]^{\perp}$ if and only if $\vec{a} \cdot \vec{b} = 0$, it follows that $[\vec{a}] \subseteq [\vec{b}]^{\perp}$ if and only if $[\vec{a}] \perp [\vec{b}]$. Thus, $l \perp m$ if and only if $[\vec{a}] \perp [\vec{b}]$. Since $[l] = [\vec{a}]$ and $[m] = [\vec{b}]$, we have:

‖ Theorem 12–26 $l \perp m \longleftrightarrow [l] \perp [m]$

In order to show that two proper directions—say, $[\vec{a}]$ and $[\vec{b}]$—are orthogonal, all we need do is to show that the dot product of some non-$\vec{0}$ member of $[\vec{a}]$ by some non-$\vec{0}$ member of $[\vec{b}]$ is 0. So, to determine whether two lines—say, l and m—are perpendicular, all we need do is to compute the dot product of any non-$\vec{0}$ member of $[l]$ by any non-$\vec{0}$ member of $[m]$. The lines are perpendicular if and only if this dot product is 0.

As an example, suppose that $(\vec{u}_1, \vec{u}_2)$ is linearly independent, that $\vec{u}_1$ and $\vec{u}_2$ are unit vectors such that $\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{3}$, and that A, B, C, and D are points such that $C - A = \vec{u}_1 5 + \vec{u}_2$ and $D - B = \vec{u}_1 2 - \vec{u}_2 4$, as shown in the picture at the right. To determine whether $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$, we compute:

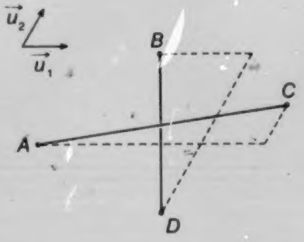


Fig. 12–5

$$\begin{aligned}(C - A) \cdot (D - B) &= (\vec{u}_1 5 + \vec{u}_2) \cdot (\vec{u}_1 2 - \vec{u}_2 4)\\ &= (\vec{u}_1 \cdot \vec{u}_1)10 + (\vec{u}_2 \cdot \vec{u}_2) \cdot -4 + (\vec{u}_1 \cdot \vec{u}_2) \cdot -18\\ &= 10 + -4 + \tfrac{1}{3} \cdot -18\\ &= 0\end{aligned}$$

So, $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$.

Given that $\vec{a}$ and $\vec{b}$ are orthogonal and that $\vec{p} \in [\vec{a}]$ and $\vec{q} \in [\vec{b}]$, it follows that $\vec{p} = \vec{a}p$ and $\vec{q} = \vec{b}q$ for some p and q so that $\vec{p} \cdot \vec{a} = (\vec{a}p) \cdot (\vec{b}q) = (\vec{a} \cdot \vec{b})pq = 0 \cdot (pq) = 0$. Thus, $\vec{p}$ and $\vec{q}$ are orthogonal.

Proof of Theorem 12-25. Only-if part: Suppose that $[\vec{a}] \perp [\vec{b}]$. Then, since $\vec{a} \in [\vec{a}]$ and $\vec{b} \in [\vec{b}]$ it follows, by Definition 12-6(b) that $\vec{a} \perp \vec{b}$.

If part: Suppose that $\vec{a} \perp \vec{b}$. We have just proved that if $\vec{p} \in [\vec{a}]$ and $\vec{q} \in [\vec{b}]$ then $\vec{p} \perp \vec{q}$. So, by Definition 12-6(b), $[\vec{a}] \perp [\vec{b}]$.

Note that in Theorem 12-25 both '⊥'s are read as 'is orthogonal to'. In Theorem 12-26, however, the left hand '⊥' is read as 'is perpendicular to' and the right hand '⊥' as 'is orthogonal to'.

Note that just as in TC 79(2), Definition 12-6 can be generalized to apply to any two subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathcal{T}$:

$$\mathcal{K}_1 \perp \mathcal{K}_2 \Longleftrightarrow \forall_{\vec{x}} \forall_{\vec{y}} \, [(\vec{x} \in \mathcal{K}_1 \text{ and } \vec{y} \in \mathcal{K}_2) \Longrightarrow \vec{x} \perp \vec{y}]$$

Alternatively,

$$\mathcal{K}_1 \perp \mathcal{K}_2 \Longleftrightarrow \mathcal{K}_1 \subseteq \mathcal{K}_2^{\perp}.$$

Exercises

Part A

Suppose that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a linearly independent sequence of unit vectors, and that $\vec{u}_1 \cdot \vec{u}_2 = -\frac{7}{10}$, $\vec{u}_2 \cdot \vec{u}_3 = \frac{4}{5}$ and $\vec{u}_1 \cdot \vec{u}_3 = \frac{3}{5}$. In each of the following you are given vectors $B - A$ and $D - C$. You are to determine whether $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

1. $B - A = \vec{u}_1 4 + \vec{u}_2 \cdot -3$
   $D - C = \vec{u}_3 5$
2. $B - A = \vec{u}_1 + \vec{u}_2$
   $D - C = \vec{u}_2 - \vec{u}_3$
3. $B - A = \vec{u}_1 2 - \vec{u}_3$
   $D - C = \vec{u}_1 2 + \vec{u}_3$
4. $B - A = \vec{u}_1 3 + \vec{u}_2 5$
   $D - C = \vec{u}_2 \cdot -4 + \vec{u}_3 2$

Part B

Suppose that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis for $\mathscr{T}$, that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are unit vectors, and that $\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{2}$, $\vec{u}_2 \cdot \vec{u}_3 = 0$, and $\vec{u}_3 \cdot \vec{u}_1 = -\frac{1}{2}$. Given the coordinates of points A, B, C, and D with respect to a coordinate system determined by the given basis:

$A(1, 3, 4)$ $B(3, -4, -1)$ $C(-4, 3, 2)$ $D(3, 2, 4)$

answer these questions.

1. Express the vectors $B - A$, $C - A$, $D - A$, and $D - C$ as linear combinations of the given basis vectors.
2. What are the orthogonal components of $B - A$, $C - A$, $D - A$, and $D - C$ with respect to $\vec{u}_1$? [*Hint*: $\text{comp}_{\vec{u}_1}(B - A) = (B - A) \cdot \vec{u}_1/(\vec{u}_1 \cdot \vec{u}_1) = (B - A) \cdot \vec{u}_1$, since $\vec{u}_1 \cdot \vec{u}_1 = 1$.]
3. Compute the following.
   (a) $(B - A) \cdot (C - A)$
   (b) $(D - A) \cdot (D - C)$
   (c) $(B - A) \cdot (D - C)$
4. Are any pairs of vectors whose dot products you computed in Exercise 3 orthogonal? Explain.
5. (a) What is the orthogonal component of $B - A$ with respect to $C - A$? [*Hint*: $\text{comp}_{\vec{a}}(\vec{b}) = (\vec{b} \cdot \vec{a})/(\vec{a} \cdot \vec{a})$]
   (b) Make use of the results in part (a) to compute the coordinates of the point P such that $P \in \overleftrightarrow{AC}$ and $\overleftrightarrow{BP} \perp \overleftrightarrow{AC}$.
6. (a) Give the coordinates of a point, say E, such that $E \neq B$ and $B - E$ is orthogonal to $C - A$. [*Hint*: See Exercise 5(b) or, better, Exercise 6(c).]
   (b) Describe, in "geometric" terms, the set of all points E which satisfy the conditions in part (a).
   (c) Describe, in algebraic terms, the set of all points E which satisfy the conditions in part (a). [*Hint*: Given that E has coordinates (e_1, e_2, e_3), obtain an equation in 'e_1', 'e_2', and 'e_3' from '$(C - A) \cdot (B - E) = 0$'.]
   (d) Give the coordinates of the point of intersection of $\overleftrightarrow{AC}$ with the plane which contains B and is perpendicular to $\overleftrightarrow{AC}$.

Parts A-C are adequate for one class and one homework assignment. In this case we recommend selecting exemplary exercises from each part for your class work, with the remaining exercises assigned as homework.

Answers for Part A

1. $(B - A) \cdot (D - C) = (\vec{u}_1 4 + \vec{u}_2 \cdot -3) \cdot \vec{u}_3 5 = (\vec{u}_1 \cdot \vec{u}_3)20 + (\vec{u}_2 \cdot \vec{u}_3) \cdot -15$
   $= 12 + -12 = 0$. So, $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.
2. $(B - A) \cdot (D - C) = (\vec{u}_1 + \vec{u}_2) \cdot (\vec{u}_2 - \vec{u}_3) = (\vec{u}_1 \cdot \vec{u}_2) + (\vec{u}_2 \cdot \vec{u}_2) - (\vec{u}_1 \cdot \vec{u}_3)$
   $- (\vec{u}_2 \cdot \vec{u}_3) = -7/10 + 1 - 3/5 - 4/5 = -11/10 \neq 0$. So, $\overleftrightarrow{AB} \not\perp \overleftrightarrow{CD}$.
3. $(B - A) \cdot (D - C) = (\vec{u}_1 2 - \vec{u}_3) \cdot (\vec{u}_1 2 + \vec{u}_3) = (\vec{u}_1 \cdot \vec{u}_1)4 - \vec{u}_3 \cdot \vec{u}_3 = 4 - 1$
   $= 3 \neq 0$. So, $\overleftrightarrow{AB} \not\perp \overleftrightarrow{CD}$.
4. $(B - A) \cdot (D - C) = (\vec{u}_1 3 + \vec{u}_2 5) \cdot (\vec{u}_2 \cdot -4 + \vec{u}_3 2) = (\vec{u}_1 \cdot \vec{u}_2) \cdot -12$
   $+ (\vec{u}_2 \cdot \vec{u}_2) \cdot -20 + (\vec{u}_1 \cdot \vec{u}_3)6 + (\vec{u}_2 \cdot \vec{u}_3)10 = 42/5 + -20 + 18/5 + 8 = 0$.
   So, $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

Answers for Part B

1. $B - A = \vec{u}_1 2 + \vec{u}_2 \cdot -7 + \vec{u}_3 \cdot -5$; $C - A = \vec{u}_1 \cdot -5 + \vec{u}_3 \cdot -2$;
   $D - A = \vec{u}_1 2 + \vec{u}_2 \cdot -1$; $D - C = \vec{u}_1 7 + \vec{u}_2 \cdot -1 + \vec{u}_3 2$
2. $\text{comp}_{\vec{u}_1}(B - A) = (B - A) \cdot \vec{u}_1 = (\vec{u}_1 \cdot \vec{u}_1)2 + (\vec{u}_2 \cdot \vec{u}_1) \cdot -7 + (\vec{u}_3 \cdot \vec{u}_1) \cdot -5$
   $= 1$.
   $\text{comp}_{\vec{u}_1}(C - A) = (C - A) \cdot \vec{u}_1 = (\vec{u}_1 \cdot \vec{u}_1) \cdot -5 + (\vec{u}_1 \cdot \vec{u}_3) \cdot -2 = -4$.
   $\text{comp}_{\vec{u}_1}(D - A) = (D - A) \cdot \vec{u}_1 = (\vec{u}_1 \cdot \vec{u}_1)2 + (\vec{u}_2 \cdot \vec{u}_1) \cdot -1 = 3/2$.
   $\text{comp}_{\vec{u}_1}(D - C) = (D - C) \cdot \vec{u}_1 = (\vec{u}_1 \cdot \vec{u}_1)7 + (\vec{u}_2 \cdot \vec{u}_1) \cdot -1 + (\vec{u}_3 \cdot \vec{u}_1)2$
   $= 11/2$.
3. (a) $(B - A) \cdot (C - A) = (\vec{u}_1 2 + \vec{u}_2 \cdot -7 + \vec{u}_3 \cdot -5) \cdot (\vec{u}_1 \cdot -5 + \vec{u}_3 \cdot -2) = 7$
   (b) $(D - A) \cdot (D - C) = (\vec{u}_1 2 + \vec{u}_2 \cdot -1) \cdot (\vec{u}_1 7 + \vec{u}_2 \cdot -1 + \vec{u}_3 2) = 17/2$
   (c) $(B - A) \cdot (D - C) = (\vec{u}_1 2 + \vec{u}_2 \cdot -7 + \vec{u}_3 \cdot -5) \cdot (\vec{u}_1 7 + \vec{u}_2 \cdot -1 + \vec{u}_3 2)$
   $= 1$
4. No, for none of the computed dot products are zero.
5. (a) $\text{comp}_{(C - A)}(B - A) = [(B - A) \cdot (C - A)]/[(C - A) \cdot (C - A)]$.
   Since $(B - A) \cdot (C - A) = 7$ and $(C - A) \cdot (C - A) = 19$,
   $\text{comp}_{(C - A)}(B - A) = 7/19$.
   (b) From our work in Part C on page 111, we see that
   $P = A + (C - A)p$, where $p = [(B - A) \cdot (C - A)]/[(C - A) \cdot (C - A)]$.
   From part (a), we have that $p = 7/19$. So, P has coordinates $(-16/19, 3, 62/19)$. [The computations are $(1 - 5 \cdot 7/19, 3 + 0 \cdot 7/19, 4 - 2 \cdot 7/19)$.]

Answers for Part B [cont.]

6. (a) Let (e_1, e_2, e_3) be the coordinates of E, where $E \neq B$ and $B - E$ is orthogonal to $C - A$. Then $B - E = \vec{u}_1(3 - e_1) + \vec{u}_2(-4 - e_2) + \vec{u}_3(-1 - e_3)$ so that

$$\begin{aligned}(B - E)\cdot(C - A) &= (\vec{u}_1\cdot\vec{u}_1)[5(e_1 - 3)] + (\vec{u}_1\cdot\vec{u}_2)[5(4 + e_2)]\\ &\quad + (\vec{u}_1\cdot\vec{u}_3)[5(1 + e_3)] + (\vec{u}_1\cdot\vec{u}_3)[2(e_1 - 3)]\\ &\quad + (\vec{u}_2\cdot\vec{u}_3)[2(4 + e_2)] + (\vec{u}_3\cdot\vec{u}_3)[2(1 + e_3)]\\ &= 5(e_1 - 3) + \frac{5}{2}(4 + e_2) - \frac{5}{2}(1 + e_3) - (e_1 - 3)\\ &\quad + 0 + 2(1 + e_3)\\ &= 4(e_1 - 3) + \frac{5}{2}(4 + e_2) - \frac{1}{2}(1 + e_3)\\ &= 4e_1 + \frac{5}{2}e_2 - \frac{1}{2}e_3 - \frac{5}{2}.\end{aligned}$$

Thus, $(B - E)\cdot(C - A) = 0$ if and only if $8e_1 + 5e_2 - e_3 = 5$. Hence, any point E whose coordinates (e_1, e_2, e_3) are such that $8e_1 + 5e_2 - e_3 = 5$ satisfies the condition that $(B - E) \perp (C - A)$.

(b) The points E such that $(B - E) \perp (C - A)$ are the points which are contained in the plane through B and perpendicular to $\overleftrightarrow{AC}$.

(c) See part (a).

(d) A point whose coordinates are (p_1, p_2, p_3) is on $\overleftrightarrow{AC}$ if and only if $p_1 = 1 - 5t$, $p_2 = 3$, and $p_3 = 4 - 2t$, for some t. Therefore, the coordinates of the point in question are such that, for some t, $8(1 - 5t) + 5\cdot 3 - (4 - 2t) = 5$. So, $t = 7/19$. Therefore, the coordinates of the point in question are $(-16/19, 3, 62/19)$. [This answer should agree with that in 5(b) — for we are talking about the same point!]

Part C

Suppose that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is an orthonormal basis for $\mathcal{T}$ and that A, B, C, and D have coordinates with respect to a coordinate system determined by this basis as follows:

$$A(12, 1, -2) \qquad B(1, 1, -1) \qquad C(5, 0, 1) \qquad D(6, -3, 8)$$

Answer these questions.

1. Give the components of the vectors $B - A$, $C - B$, $D - C$, and $A - D$ with respect to the given basis.
2. Compute the following.
   (a) $(B - A) \cdot (C - B)$ (b) $(C - B) \cdot (D - C)$
   (c) $(C - B) \cdot (A - D)$ (d) $(D - C) \cdot (B - A)$
3. Using the results in Exercise 2, tell which of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{AD}$ are perpendicular.
4. (a) What is the orthogonal component of $B - A$ with respect to $C - B$?
   (b) Compute the coordinates of the point P such that $P \in \overleftrightarrow{AB}$ and $\overleftrightarrow{CP} \perp \overleftrightarrow{AB}$.
   (c) Write an equation which describes the set of all points X such that $\overleftrightarrow{CX} \perp \overleftrightarrow{AB}$.
   (d) What are the coordinates of the point of intersection of the set described in part (c) and $\overleftrightarrow{AB}$?

12.05 Chapter Summary

Vocabulary Summary

perpendicularity
  of plane to line
  of line to line
  of plane to plane
oblique
orthogonal projection
  of a point on a line
  of a point on a plane
  of a vector on a direction
orthogonal vectors

Definitions

12-1. (a) $\pi \perp l \longleftrightarrow [\pi] = [l]^{\perp}$ (b) $l \perp \pi \longleftrightarrow \pi \perp l$
12-2. $[\vec{a}, \vec{b}]^{\perp} = \{\vec{x}\colon \forall_{\vec{y} \in [\vec{a}, \vec{b}]}\, \vec{x} \cdot \vec{y} = 0\}$
12-3. (a) $m \perp l \longleftrightarrow [m] \subseteq [l]^{\perp}$ (b) $\sigma \perp \pi \longleftrightarrow [\pi]^{\perp} \subseteq [\sigma]$
12-4. (a) $\text{proj}_l\,(P)$ = the point of intersection of l and $\overleftrightarrow{P[l]^{\perp}}$.
  (b) $\text{proj}_\pi\,(P)$ = the point of intersection of π and $\overleftrightarrow{P[\pi]^{\perp}}$.
12-5. $P + \text{proj}_{[\pi]}\,(\vec{b}) = Q \longleftrightarrow \Big(Q - P \in [\pi] \text{ and } \vec{b} - (Q - P) \in [\pi]^{\perp}\Big)$
12-6. (a) $\vec{a} \perp \vec{b} \longleftrightarrow \vec{a} \cdot \vec{b} = 0$
  (b) $[\vec{a}] \perp [\vec{b}] \longleftrightarrow \forall_{\vec{x}} \forall_{\vec{y}} [(\vec{x} \in [\vec{a}] \text{ and } \vec{y} \in [\vec{b}]) \longrightarrow \vec{x} \perp \vec{y}]$

Answers for Part C

1. $B - A$: $(-11, 0, 1)$ $C - B$: $(4, -1, 2)$
   $D - C$: $(1, -3, 7)$ $A - D$: $(6, 4, -10)$
2. (a) -42 $[-11 \cdot 4 + 0 \cdot -1 + 1 \cdot 2]$ (b) 21 $[4 \cdot 1 + -1 \cdot -3 + 2 \cdot 7]$
   (c) 0 $[4 \cdot 6 + -1 \cdot 4 + 2 \cdot -10]$ (d) -4 $[1 \cdot -11 + -3 \cdot 0 + 7 \cdot 1]$
3. $\overleftrightarrow{BC} \perp \overleftrightarrow{AD}$ [Exercise 2(c)]
4. (a) $\text{comp}_{(C - B)}(B - A) = (B - A) \cdot (C - B)/(C - B) \cdot (C - B)$
   $= -42/21 = -2$
   (b) The point P is such that $P = A + (B - A)[\text{comp}_{(B - A)}(C - A)]$. Since $\text{comp}_{(B - A)}(C - A) = (C - A) \cdot (B - A)/(B - A) \cdot (B - A) = 80/122 = 40/61$, $P = A + (B - A)\frac{40}{61}$. So, the coordinates of P are $(\frac{292}{61}, 1, -\frac{82}{61})$. [As a check, note that the components of $C - P$ are $(\frac{13}{61}, -1, \frac{143}{61})$ and the components of $B - A$ are $(-11, 0, 1)$. So, $(C - P) \cdot (B - A) = \frac{13}{61} \cdot -11 + -1 \cdot 0 + \frac{143}{61} \cdot 1 = 0$.]
   (c) Let P be any point whose coordinates are (p_1, p_2, p_3) and such that $\overleftrightarrow{CP} \perp \overleftrightarrow{AB}$. Then, $C - P$ has components $(5 - p_1, -p_2, 1 - p_3)$ so that
   $$(C - P) \cdot (B - A) = (5 - p_1) \cdot -11 + -p_2 \cdot 0 + (1 - p_3) \cdot 1$$
   $$= 11p_1 - p_3 - 54$$
   So, $(C - P) \cdot (B - A) = 0$ if and only if $11p_1 - p_3 = 54$. Hence, the set of all points X such that $\overleftrightarrow{CX} \perp \overleftrightarrow{AB}$ are those which have coordinates (x_1, x_2, x_3) which satisfy the equation '$11x_1 - x_3 = 54$'.
   (d) By part (b), the coordinates are $(\frac{292}{61}, 1, -\frac{82}{61})$. [As a check, note that parametric equations for $\overleftrightarrow{AB}$ are:
   $$x_1 = 12 - 11r, \quad x_2 = 1, \quad x_3 = -2 + r$$
   Using these and the equation from part (c), we see that the value of the parameter 'r' which yields the coordinates of the point in question is such that $11(12 - 11r) - (-2 + r) = 54$. So, $r = 40/61$. Hence, the point has coordinates $(\frac{292}{61}, 1, -\frac{82}{61})$.]

Other Theorems

12-1. $\pi \perp l \longrightarrow l$ is a transversal of π

12-2. $(\sigma \parallel \pi$ and $m \parallel l) \longrightarrow [\sigma \perp m \longleftrightarrow \pi \perp l]$

Corollary. Parallel planes are perpendicular to the same lines and parallel lines are perpendicular to the same planes.

12-3. $(\sigma \perp m$ and $\pi \perp l) \longrightarrow [\sigma \parallel \pi \longleftrightarrow m \parallel l]$

Corollary. Planes perpendicular to parallel lines are parallel and lines perpendicular to parallel planes are parallel.

12-4. $(m \perp \sigma$ and $l \perp \pi) \longrightarrow [l \parallel \sigma \longleftrightarrow m \parallel \pi]$

12-5. $\overline{P[l]}^{\perp}$ is the plane which contains P and is perpendicular to l.

Corollary. There is one and only one plane which contains a given point and is perpendicular to a given line.

12-6. Planes which are perpendicular to two intersecting lines intersect in lines which are perpendicular to the plane of the two lines.

12-7. $l \perp \pi \longleftrightarrow [l] = [\pi]^{\perp}$

12-8. $\overleftrightarrow{P[\pi]}^{\perp}$ is the line which contains P and is perpendicular to π.

Corollary. There is one and only one line which contains a given point and is perpendicular to a given plane.

12-9. (a) $l \perp m \longleftrightarrow m \perp l$ (b) $\pi \perp \sigma \longleftrightarrow \sigma \perp \pi$

12-10. (a) $\sigma \perp \pi \longrightarrow \sigma \cap \pi$ is a line

(b) Coplanar perpendicular lines intersect.

12-11. Parallel lines are perpendicular to the same lines and parallel planes are perpendicular to the same planes.

12-12. (a) $\pi \perp l \longrightarrow [m \perp l \longleftrightarrow m \parallel \pi]$

(b) $l \perp \pi \longrightarrow [\sigma \perp \pi \longleftrightarrow \sigma \parallel l]$

12-13. If $l \perp \pi$ then (a) l is perpendicular to each line contained in π, and (b) each plane which contains l is perpendicular to π.

Corollary. $(l \subseteq \pi$ and $\sigma \perp l) \longrightarrow (\sigma \cap \pi) \perp l$

12-14. (a) A line which is perpendicular to each of two intersecting lines is perpendicular to the plane containing them.

(b) A plane which is perpendicular to each of two intersecting planes is perpendicular to their line of intersection.

12-15. If $l \not\perp \pi$ then (a) there is one and only one line through P which is perpendicular to l and parallel to π, and (b) there is one and only one plane through P which is perpendicular to π and parallel to l.

Corollary 1. If $l \subseteq \pi$ and $P \in \pi$ then there is one and only one line in π which contains P and is perpendicular to l.

Corollary 2. If $P \notin l$ then there is one and only one line which contains P, intersects l, and is perpendicular to l.

Corollary 3. Coplanar lines which are perpendicular to a given line are parallel.

Corollary 4. If $l \not\perp \pi$ then there is one and only one plane which contains l and is perpendicular to π.

12-16. $\Big(\sigma \perp \pi$ and $l \perp (\pi \cap \sigma)\Big) \longrightarrow [l \parallel \pi \longleftrightarrow l \perp \sigma]$

12-17. If l, m, and n are not all parallel to the same plane then there is one and only one line which is parallel to n and intersects both l and m.

Corollary. If $l \nparallel m$ then there is one and only one line which is perpendicular to both l and m and intersects both l and m.

12-18. $\text{proj}_{[\pi]}(\vec{b}) = \vec{b} - \text{proj}_{[\pi]^{\perp}}(\vec{b})$

Corollary 1. (a) $\text{proj}_{[\pi]}(\vec{b}) \in \mathcal{T}$

(b) $\text{proj}_{[\pi]}(\vec{b}) = \vec{c} \longleftrightarrow \left(\vec{c} \in [\pi] \text{ and } \vec{b} - \vec{c} \in [\pi]^{\perp}\right)$

Corollary 2. $(\vec{a}, \vec{b})$ is orthogonal $\longrightarrow$

$\text{proj}_{[\vec{a}, \vec{b}]}(\vec{r}) = \text{proj}_{[\vec{a}]}(\vec{r}) + \text{proj}_{[\vec{b}]}(\vec{r})$

12-19. (a) $\text{proj}_{[\pi]}(\vec{b}) \in [\pi]$ and $\text{proj}_{[\pi]}(\vec{b}) = \vec{b} \longleftrightarrow \vec{b} \in [\pi]$

(b) $\text{proj}_{[\pi]}(\vec{b}) = \vec{0} \longleftrightarrow \vec{b} \in [\pi]^{\perp}$

(c) $\text{proj}_{[\pi]}(\vec{b} + \vec{c}) = \text{proj}_{[\pi]}(\vec{b}) + \text{proj}_{[\pi]}(\vec{c})$

(d) $\text{proj}_{\pi}(\vec{b}b) = \text{proj}_{\pi}(\vec{b})b$

12-20. $l \perp \pi \longrightarrow \vec{b} = \text{proj}_{[l]}(\vec{b}) + \text{proj}_{[\pi]}(\vec{b})$

12-21. (a) $\text{proj}_{l}(P + \vec{b}) = \text{proj}_{l}(P) + \text{proj}_{[l]}(\vec{b})$

(b) $\text{proj}_{\pi}(P + \vec{b}) = \text{proj}_{\pi}(P) + \text{proj}_{[\pi]}(\vec{b})$

12-22. (a) $\text{proj}_{\pi}(l)$ is a line if and only if $l \not\perp \pi$.

(b) If l and m are parallel lines which are not perpendicular to π then $\text{proj}_{\pi}(l) \parallel \text{proj}_{\pi}(m)$.

12-23. If $l \not\perp \pi$ then $\text{proj}_{\pi}(l)$ is the intersection with π of the plane which contains l and is perpendicular to π.

12-24. (a) If $\overline{AB}$ and $\overline{CD}$ are nondegenerate parallel intervals which are not perpendicular to π then $\text{proj}_{\pi}(\overline{CD}) : \text{proj}_{\pi}(\overline{AB}) = CD : AB$.

(b) If $\overline{AB}$ is a nondegenerate interval which is not perpendicular to π and $\sigma \parallel \pi$ then $\text{proj}_{\sigma}(\overline{AB}) : \text{proj}_{\pi}(\overline{AB}) = 1$.

12-25. $[\vec{a}] \perp [\vec{b}] \longleftrightarrow \vec{a} \perp \vec{b}$

12-26. $l \perp m \longleftrightarrow [l] \perp [m]$

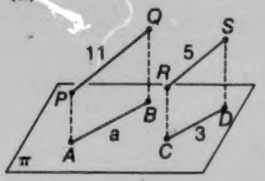
Chapter Test

1. Complete each of the following with either 'direction', 'bidirection', 'line', or 'plane'.
   (a) Given that l is a line, $[l]^{\perp}$ is a ________.
   (b) Given that π is a plane, $[\pi]^{\perp}$ is a ________.
   (c) Given that π and σ are nonparallel planes, $\pi \cap \sigma$ is a ________.
   (d) Given that $\vec{a}$ is a proper translation, $[\vec{a}]$ is a ________.
   (e) Given that $(\vec{a}, \vec{b})$ is linearly independent, $[\vec{a}, \vec{b}]^{\perp}$ is a ________.
2. True or false?
   (a) Lines which are perpendicular to parallel lines are parallel.
   (b) Lines which are perpendicular to parallel planes are parallel.

(c) Given that l is oblique to π, it follows that $[l] \subseteq [\pi]^{\perp}$.
(d) Lines which are parallel to the same plane are parallel lines.
(e) Planes which are parallel to the same line are parallel planes.
(f) Planes which are perpendicular to the same line are parallel planes.
(g) Planes which are perpendicular to the same plane are parallel planes.
(h) Given that $\pi \perp \sigma$, if $l \perp \pi$ and $m \perp \sigma$ then $l \perp m$.
(i) Given a line l and a plane π, there is a plane which is parallel to l and perpendicular to π.
(j) Given a point P and a plane π, there is a plane which contains P and is perpendicular to π.

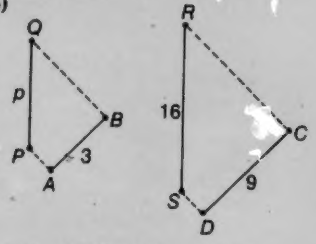
3. In each of the following, assume that $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$ and that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are the projections on π of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$, respectively. Make use of the information given in the pictures to answer the questions.

(a)



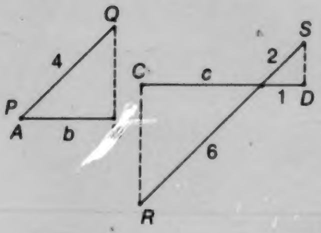
What is the value of 'a'?

(b)



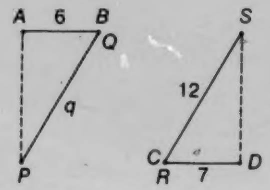
What is the value of 'p'?

(c)



What is the value of 'b'? Of 'c'?

(d)



What is the value of 'q'?

4. Given a coordinate system with respect to the orthonormal basis $(\vec{i}, \vec{j}, \vec{k})$, suppose that A, B, C, and D have coordinates as follows:

A: (1, −3, 5) B: (2, −4, 6)
C: (4, 0, −3) D: (0, 3, 4)

(a) Give the components of $B - A$, $D - C$, and $C - A$.
(b) Determine whether or not $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

Key to Chapter Test

1. (a) proper bidirection
   (b) proper direction
   (c) line
   (d) proper direction
   (e) proper direction

2. (a) False. (b) True.
   (c) False. (d) False.
   (e) False. (f) True.
   (g) False. (h) True.
   (i) True. (j) True.

3. (a) 33/5 [$a/3 = 11/5$] (b) 16/3 [$p/16 = 3/9$]
   (c) 2; 3 [$b/4 = 1/2 = c/6$] (d) 72/7 [$q/12 = 6/7$]

4. (a) $B - A$: (1, −1, 1); $D - C$: (−4, 3, 7); $C - A$: (3, 3, −8)
   (b) $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$, since $(B - A) \cdot (D - C) = 1 \cdot -4 + -1 \cdot 3 + 1 \cdot 7 = 0$.
   (c) $$\begin{aligned} \mathrm{comp}_{(B-A)}(C - A) &= (C - A) \cdot (B - A) / (B - A) \cdot (B - A) \\ &= (3 \cdot 1 + 3 \cdot -1 + -8 \cdot 1)/(1 \cdot 1 + -1 \cdot -1 + 1 \cdot 1) \\ &= -8/3 \end{aligned}$$
   (d) $3e_1 + 3e_2 - 8e_3 =$ [illegible] [or: $(e_1 - 4)3 + (e_2 - 0) \cdot 3 + (e_3 + 3) \cdot -8 = 0$]

TC 100 (1)

*

The background topic deals with quadratic functions. The most important point dealt with is the use of "completing the square" to obtain a "standard form" for the defining expression of such a function. [See (3) on page 103.] You will recognize this as a step toward obtaining the usual formula for solving a quadratic equation. It is, however, a step toward many other results — see, for example the theorem following (3) — and, so, is of much greater importance than is the quadratic formula.

(c) Compute $\text{comp}_{(B - A)}(C - A)$.

(d) Let (e_1, e_2, e_3) be the coordinates of a point E such that $(E - C) \perp (C - A)$. Write an equation in 'e_1', 'e_2', and 'e_3' which describes all such points E.

Background Topic

In Chapter 1 you reviewed the notion of function, paying special attention to linear mappings of $\mathscr{R}$. These are the mappings of $\mathscr{R}$ into $\mathscr{R}$ which can be described by equations of the form:

$$y = ax + b \qquad [a \neq 0]$$

Such mappings are also called *linear functions*. Precisely:

$$\begin{gathered} f \text{ is a linear function [on } \mathscr{R}] \\ \longleftrightarrow \\ \exists_{a \neq 0} \exists_b\, f = \{(x, y): y = ax + b\} \end{gathered}$$

From your earlier work in algebra you are also acquainted with quadratic functions—those mappings of $\mathscr{R}$ into itself which can be described by equations of the form:

$$y = ax^2 + bx + c \qquad [a \neq 0]$$

Precisely:

$$\begin{gathered} f \text{ is a quadratic function [on } \mathscr{R}] \\ \longleftrightarrow \\ \exists_{a \neq 0} \exists_b \exists_c\, f = \{(x, y): y = ax^2 + bx + c\} \end{gathered}$$

In the following exercises you will review, and extend, your knowledge of these functions.

Just as the simplest linear function is the identity mapping on $\mathscr{R}$ which is described by the equation '$y = x$', the simplest quadratic function is the squaring mapping which is described by the equation '$y = x^2$'.

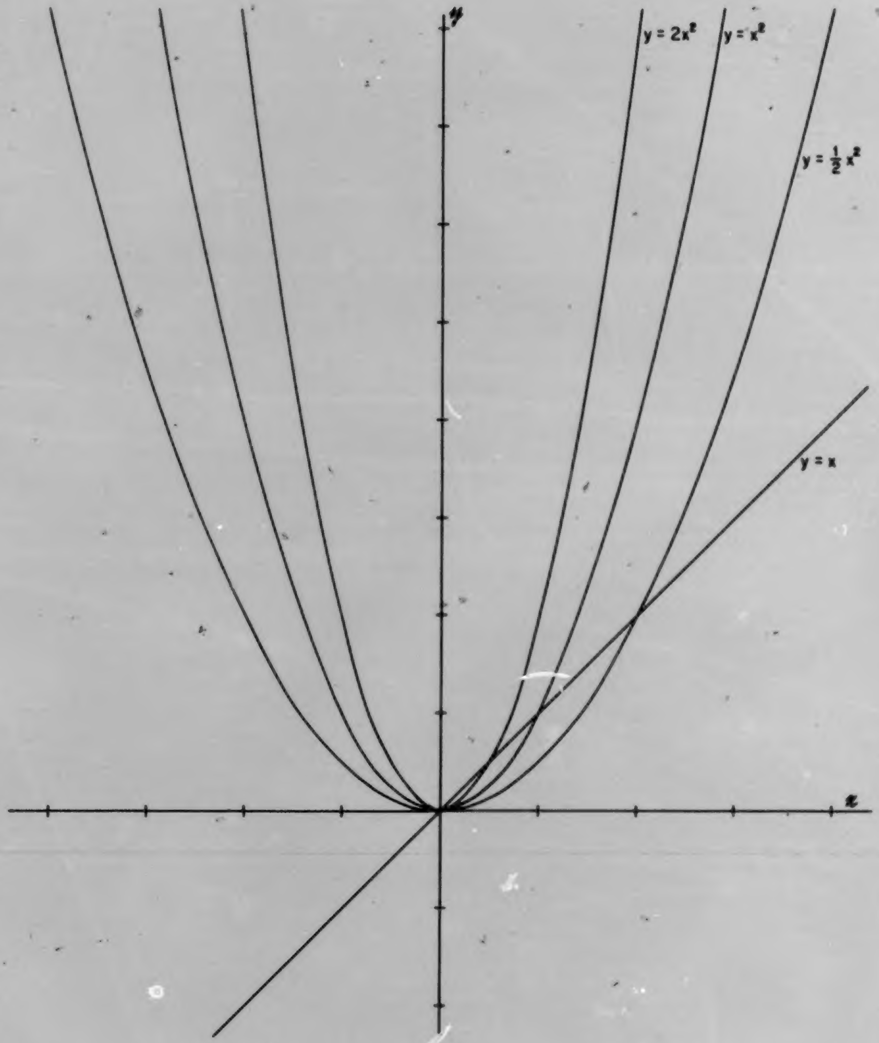
Part A

1. (a) Draw coordinate axes on a sheet of cross-section paper and draw graphs of the equations '$y = x$' and '$y = x^2$'. [Plot the origin, (0, 0), about one-third of the way up from the foot of your paper and use a scale of about 1 inch. Pay particular attention to the shape of the graph of '$y = x^2$' near the origin.]

   (b) Using the same coordinate axes as in part (a), draw graphs of '$y = 2x^2$' and '$y = \frac{1}{2}x^2$'.

Answers for Part A

1. (a), (b)



2. Tell which of the following equations describe linear functions and which describe quadratic functions.

(a) $x - y = 1$	(b) $x^2 - y = 1$
(c) $x^2 + y^2 = 1$	(d) $x + 2y = 3$
(e) $x^2 - y^2 = 0$	(f) $x^2 - y^2 = 1$
(g) $x^2 + 2xy + y^2 = 0$	(h) $y + x^2 = 4(x - 1)$
(i) $2(x + 1) = x(x + 1)$	(j) $2 - y = 2(x - 1)^2$
(k) $2(y + 1) = x(x + 1)$	(l) $y = \sqrt{x^2}$
(m) $y = (x - 3)(x + 2) - (x + 1)(x - 4)$	(n) $y = \sqrt{(x - 1)^2}$

3. (a) Although six of the sets referred to in Exercise 2 are neither linear nor quadratic functions, two of these six are, nevertheless, functions. Which six, and which two?
   (b) Sketch graphs of these two functions, using the same coordinate axes for both graphs.
   (c) After having sketched one of the graphs asked for in part (b), how could you have predicted what the other would be? [*Hint*: Could you have made some use of a translation of $\mathscr{R}$ and of function composition?]
4. (a) Draw graphs of the sets described in (e) and (i). [*Hint*: Factoring will help. Recall that $ab = 0 \leftrightarrow (a = 0 \text{ or } b = 0)$.]
   *(b) Draw graphs of the sets described in (c) and (f).

Part B

1. (a) On cross-section paper, draw a graph of the function f such that, for each x, $f(x) = 2x$.
   (b) Use your graph to estimate $f(\frac{1}{3})$, $f(-\frac{1}{2})$, and $f(2)$.
   (c) Suppose that, for each x, $g(x) = 2(x - 1)$. Use your graph of f to estimate $g(\frac{3}{2})$, $g(-\frac{1}{2})$, and $g(3)$.
   (d) Use your graph of f to construct a graph of g.
   (e) Suppose, that, for each x, $h(x) = 2(x + \frac{1}{2})$. Use your graph of f to construct a graph of h.
   (f) Suppose that you are given a number p. Tell how to use your graph of f to construct a graph of a function k, where, for each x, $k(x) = 2(x - p)$.
2. Repeat Exercise 1 for the functions f, g, h, and k such that, for each x, $f(x) = x^2$, $g(x) = (x - 1)^2$, $h(x) = (x + \frac{1}{2})^2$, and $k(x) = (x - p)^2$.
3. Suppose that you are given a function f and a number p. Suppose, also, that you have drawn a graph of f.
   (a) Tell how you could use this graph to construct a graph of g, where $g(x) = f(x - p)$.
   *(b) Instead of drawing a new graph for g, you could make a simple change in your drawing which would turn what was a graph of f into a graph of g. Explain.
4. (a) On cross-section paper draw a graph of f, where $f(x) = \frac{1}{2}x$.

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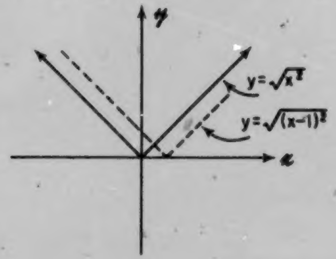
Answers for Part A [cont.]

2. (a), (d), (g), (m) are equations which describe linear functions; (b), (h), (j), (k) are equations which describe quadratic functions.

3. (a) The six equations which describe neither linear nor quadratic functions are (c), (e), (f), (i), (l), and (n). The two, of these six, which describe functions are (l) and (n).

   (b)



   (c) Consider the translation — say, t — of $\mathcal{R}$ such that, for any a, t maps a on a - 1. Thus, t is a linear function on $\mathcal{R}$ which maps 0 on −1 and has slope 1. That is, $t = \{(x, y):\ y = x - 1\}$. Let $f = \{(x, y):\ y = \sqrt{x^2}\}$. Then, for any a, $[f \circ t](a) = f(t(a)) = f(a - 1) = \sqrt{(a - 1)^2}$. So, $f \circ t = \{(x, y):\ y = \sqrt{(x - 1)^2}\}$. Thus, given that we have drawn the graph of $\{(x, y):\ y = \sqrt{x^2}\}$, all that you need to do to obtain the graph of $\{(x, y):\ y = \sqrt{(x - 1)}$ is to "shift" each point of the graph one unit to the right.

4. (a) The set of points described by (e) is

$$\{(x, y):\ y = \sqrt{x^2} \text{ or } y = -\sqrt{x^2}\},$$

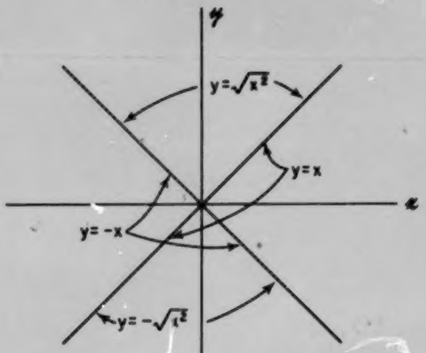
and by (i) is

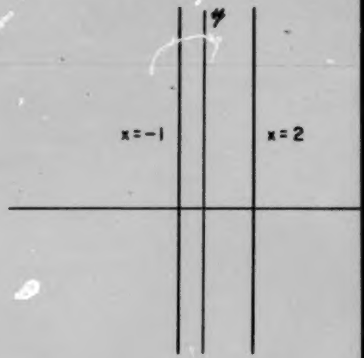
$$\{(x, y):\ x = 2 \text{ or } x = -1\}.$$

[In the case of (e), another set description is:

$$\{(x, y):\ y = x \text{ or } y = -x\}$$]

Here is a graph for (e). Here is a graph for (i).





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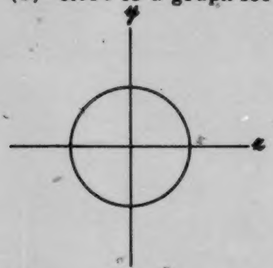
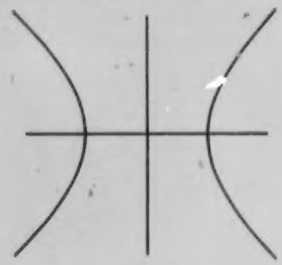
Answers for Part A [cont.]

4. (b) Here is a graph for (c).

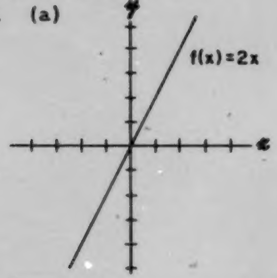


Here is a graph for (f).



Answers for Part B

1. (a)



(b) Estimates are:

$f(\frac{5}{3}) \doteq 3\frac{1}{3}$

$f(-\frac{3}{2}) \doteq -3$

$f(2) \doteq 4$

(c) As in the case of Exercise 3(c) of Part A, "sliding" each point of the graph of f one unit to the right will yield the graph of g. It is easy to show that the translation which maps a on a + 1 maps each point of f on a point of g. To do so, let $(a, b) \in f$. Then b = 2a. Now, $g(a + 1) = 2((a + 1) - 1) = 2a = b$, so that $(a + 1, b) \in g$. So, $g(\frac{8}{3}) = f(\frac{5}{3}) \doteq \frac{10}{3}$; $g(-\frac{1}{2}) = f(-\frac{3}{2}) \doteq -3$; and $g(3) = f(2) \doteq 4$.

(d) One obtains the graph of g from the graph of f by the "sliding" each point of f one unit to the right, as described in 1(c), above.

(e) The graph of h can be obtained from that of f by "sliding" each point of f 3/2 units to the left.

(f) The graph of k is obtained from that of f by "sliding" each point (a, b) of f to the point (a + p, b).

2. (a) The graph of $f(x) = x^2$ has been given in Exercise 1(a) of Part A.

(b) $f(\frac{5}{3}) \doteq 2.8$; $f(-\frac{3}{2}) \doteq 2.2$; $f(2) = 4$.

(c) $g(\frac{8}{3}) = f(\frac{5}{3}) \doteq 2.8$; $g(-\frac{1}{2}) = f(-\frac{3}{2}) \doteq 2.2$; $g(3) = f(2) = 4$.

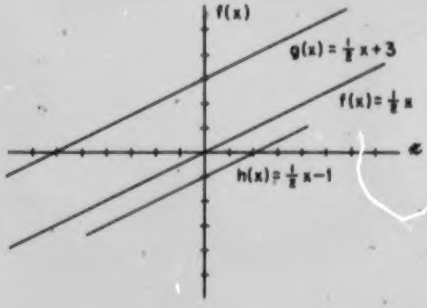
(d), (e), (f) The responses to these exercises are those given in Exercises 1(d), 1(e), and 1(f), respectively.

Answers for Part B [cont.]

3. (a) By "sliding" the graph of f p units in the sense of p. [By "sliding" each point (a, f(a)) of f to (a + p, f(a)).]

(b) In the exercises above, the graphs of g, h, and k were obtained from the graph of f by "sliding" the graph of f to the left or to the right p units in the sense of p. A simple way to change the graph of f to the graph of the others would be to erase the second coordinate axis and to redraw it p units to the left or right of its original position in the opposite sense of −p. Thus, for the graph of g, redraw the second coordinate axis one unit to the left of its original location. [Such an act has the effect of increasing the first coordinate of each point by one.]

4. (a)



(b) Use your graph of *f* to construct a graph of *g*, where $g(x) = \frac{1}{2}x + 3$. [*Hint*: You *could* note that $g(x) = \frac{1}{2}(x + 6)$ and proceed as in Exercise 1. Find a different way.]

(c) Use your graph of *f* to construct a graph of *h*, where $h(x) = \frac{1}{2}x - 1$.

(d) Suppose that you are given a number *q*. Tell how to use your graph of *f* to construct a graph of a function *k*, where $k(x) = f(x) + q$.

5. Repeat Exercise 4 for the functions *f*, *g*, *h*, and *k* such that, for each *x*,

$$f(x) = x^2,\ g(x) = x^2 + 3,\ h(x) = x^2 - 1, \text{ and } k(x) = x^2 + q.$$

6. Draw another graph of the squaring function [that is, the function *f* of Exercise 5], and use it to construct a graph of

(a) g_1, where $\forall_x\ g_1(x) = (x - 1)^2 + 2$
(b) g_2, where $\forall_x\ g_2(x) = (x + 2)^2 - 1$
(c) g_3, where $\forall_x\ g_3(x) = -x^2$
(d) g_4, where $\forall_x\ g_4(x) = -(x - 2)^2 + 1$

Part C

1. Your work in Part B has illustrated the fact that, for any *p* and *q*, a graph of the function *f* such that

$$(*) \qquad f(x) = (x - p)^2 + q, \text{ for all } x$$

has a lowest point, and that this lowest point is the graph of the ordered pair (p, q). Talking about the function *f*—rather than about its graph—this means that $f(p) = q$ and, for each *x*, $f(x) \geq q$.

(a) This result concerning a function *f* with a definition like (*) can be proved very easily by using two properties of the squaring operation. What properties?

(b) Tell how the discussion preceding part (a) should be changed if (*) is replaced by:

$$(**) \qquad f(x) = -(x - p)^2 + q, \text{ for all } x.$$

2. The property we have noted of functions with definitions like (*) or (**) can be stated as follows:

A function *f* which has a definition like (*) or (**) has *an extreme value* at the argument *p*. This extreme value is *q*. It is a *minimum* value of *f* in the case of (*) and is a *maximum* value of *f* in the case of (**).

(a) Using the preceding statement as a pattern, complete the following:

For any $a \neq 0$, and any *p* and *q*, the function *f* such that

$$f(x) = a(x - p)^2 + q$$

has an extreme value at . . . .

(b) Justify your answer for part (a).

Answers for Part B [cont.]

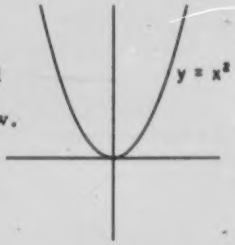
4. (b) A different way of obtaining the graph of g from the graph of f [from that of Exercise 1] is to note that $g(a) = f(a) + 3$. Thus, the translation which maps b on b + 3 may be composed with f so that a point (a, b) of f maps onto the point (a, b + 3). The effect of this composition is to "slide" the points of f upward 3 units. Thus to construct the graph of g from that of f this way, one need only construct a line parallel to the graph of f containing the point (0, 3).

(c) Since $h(x) = f(x) - 1$, by the procedure for construction suggested in 4(b), construct a line parallel to the graph of f and containing the point (0, −1).

(d) Construct a line parallel to the graph of f and containing the point (0, q).

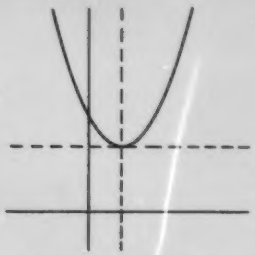
5. (a), (b), (c), (d) Given the graph of $f(x) = x^2$, as drawn for Exercise 1(a) in Part A, the graphs of each of g, h, and k, can be obtained (constructed) by "sliding" the graph of f up or down the second coordinate axis as many distance units and in the sense of q. [The students may wish to save time by cutting out a template for the graph of f and tracing the graphs of g, h, and k with the template. Perhaps they should be encouraged, instead, to plot a sufficient number of images of points on the graph of f to better understand what the translation at hand "does" to the graph of f.] Some student may suggest that he can "construct" the graph of k from the graph graph of f by erasing the first coordinate axis and redrawing it so many distance units above or below the original position according to the value of 'q'. If such a suggestion is made, the class should be given an opportunity to examine what change is made in the second coordinate of each point. Similarly, if the second coordinate axis is "moved" in such a manner, the students should be able to describe the change that occurs in the first coordinate of each point.

6. With this as the graph of '$y = x^2$', the related graphs of (a), (b), (c), and (d) are given below.

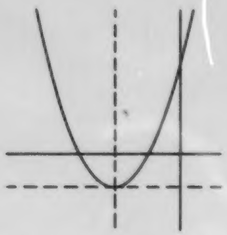


Answers for Part B [cont.]

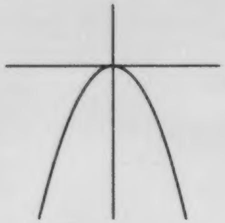
6. (a) graph of '$y = (x - 1)^2 + 2$'



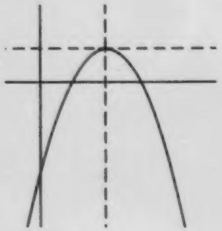
(b) graph of '$y = (x + 2)^2 - 1$'



(c) graph of '$y = -x^2$'



(d) graph of '$y = -(x - 2)^2 + 1$'



Answers for Part C

1. (a) The properties of squaring which are referred to are:

$$a^2 \geq 0 \text{ and } a^2 = 0 \text{ if and only if } a = 0$$

(b) Given (**), for each x, $f(x) \leq q$.

2. (a) Complete the statement as follows: the argument p, and this extreme value is q. This extreme value q is a minimum if $a > 0$ and is a maximum if $a < 0$.

(b) For $a > 0$ and $b \neq p$, $a(b - p)^2 > 0$. Since $f(p) = q$, we see that, for any b, $f(b) = a(b - p)^2 + q \geq 0 + q = f(p)$. So, in case $a > 0$, q is a minimum value of f.

For $a < 0$ and $b \neq p$, $a(b - p)^2 < 0$ so that, for any b, $f(b) = a(b - p)^2 + q \leq 0 + q = f(p)$. Thus, in case $a < 0$, q is a maximum value of f.

*

TC 103

Explanation called for in text: You may obtain a graph of f from a graph of '$y = ax^2$' by translating the latter parallel to the x-axis p units in the sense of p and then translating parallel to the y-axis q units in the sense of q.

Explanation called for in the text: Since $p = -b/(2a)$ and $ap^2 + q = c$, we have that $q = c - ap^2 = c - a[b^2/(4a^2)] = c - b^2/(4a) = (4ac - b^2)/(4a)$.

In the preceding exercises you have noted an important property of a quadratic function *f* such that, for each *x*,

$$(1) \qquad f(x) = a(x - p)^2 + q \qquad [a \neq 0].$$

You have also seen how to obtain a graph of *f* by, first, drawing a graph of '$y = ax^2$'. [Explain.] In fact, a description like (1) tells us several things about the function it describes which are not immediately obvious from a description like:

$$(2) \qquad f(x) = ax^2 + bx + c \qquad [a \neq 0]$$

So, it is natural to ask whether, when we are given a description like (2), we can find an equivalent one like (1). To answer this question, note that (1) is equivalent to:

$$(3) \qquad f(x) = ax^2 - 2apx + (ap^2 + q)$$

Comparing (2) and (3) we see that (1) will be equivalent to (2) if *p* and *q* are chosen so that

$$-2ap = b \text{ and } ap^2 + q = c.$$

Since $a \neq 0$ we can satisfy the first of these two requirements by choosing

$$p = -\frac{b}{2a}.$$

The second will be satisfied by taking *q* to be $c - ap^2$. With the choice already determined for *p* this means that

$$q = \frac{4ac - b^2}{4a}. \qquad \text{[Explain.]}$$

So, for any $a \neq 0$ and any *b* and *c*, (2) is equivalent to:

$$(4) \qquad f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

As an immediate consequence of this and Exercise 2 of Part C we have:

> The quadratic function *f* defined by:
> $f(x) = ax^2 + bx + c$, for all *x*
> has an extreme value at the argument $-b/(2a)$. This value is $(4ac - b^2)/(4a)$. It is a minimum if $a > 0$ and is a maximum if $a < 0$.

Part D

Each of the following expressions could be used to define a quadratic function. In each case, compute the extreme value of the function and tell whether it is a maximum or a minimum. [*Hint:* In any case, you can transform the given expression into an equivalent one of the form '$ax^2 + bx + c$'—if it is not already in this form—and then compute the corresponding value of '$(4ac - b^2)/(4a)$'. But, it will pay you to look for short cuts—for even the simplest looking ones.]

1. $3x^2 - 5x + 7$
2. $3x - 5 - 4x^2$
3. $4 - (x - 1)(x + 2)$
4. $x^2 - 2x + 1$
5. $x^2 + 4x + 4$
6. $x^2 - 2x + 3$
7. $x^2 + 4x + 5$
8. $x^2 - 2x$
9. $x(4 - x)$
10. $x^2 - 2x - 7$
11. $2x^2 - 4x + 5$
12. $7 - x^2 + 2x$
13. $x^2 + 3x + \frac{1}{4}$
14. $(x + \frac{3}{2})^2 - 2$

*

If you were asked to transform the quadratic expression in Exercise 11 of Part D into the form '$a(x - p)^2 + q$' you might reason as follows: In this case, $a = 2$, $b = -4$ and $c = 5$. So, $-b/(2a) = 1$ and $(4ac - b^2)/(4a) = 3$. Hence, the given expression is equivalent to '$2(x - 1)^2 + 3$'.

In working Exercise 11 you may, however, see another way of doing this problem: The given expression is equivalent to '$2(x^2 - 2x \quad) + 5$'. If I put in a '+1' as well as a '−2', I'll get the equivalent expression '$2(x^2 - 2x + 1) + 5 - 2$'. And, this is equivalent to '$2(x - 1)^2 + 3$'.

When using the first of these two methods you need to remember the new formulas:

$$-\frac{b}{2a} \text{ and: } \frac{4ac - b^2}{4a}$$

In the second method you use algebra that you already know—mainly that

$$x^2 - 2px + p^2 = (x - p)^2. \tag{5}$$

The object of this second method is to use the left side of (5) as a model for "completing" part of the given expression so as to obtain something which can be replaced by an expression like that on the right side of (5). For this reason, the method is called "completing the square". You will find many uses for this procedure.

Answers for Part D

1. Extreme value is 59/12 [$(4 \cdot 3 \cdot 7 - 5^2)/(4 \cdot 3)$], and it is a minimum [for $3 > 0$].
2. Extreme value is −71/16 [$(4 \cdot -4 \cdot -5 - 3^2)/(4 \cdot -4)$], and is a maximum [for $-4 < 0$].
3. Extreme value is 25/4 [$(4 \cdot -1 \cdot 6 - 1^2)/(4 \cdot -1)$], and is a maximum [for $-1 < 0$]. [Alternate solution: Note that 9/4 is the maximum value for '$-(x^2 + x - 2)$'. So, the maximum value for '$4 - (x^2 + x - 2)$'. is 4 + 9/4 or 25/4.]
4. Since $x^2 - 2x + 1 = (x - 1)^2 \geq 0$, the extreme value occurs when $x = 1$ and is clearly 0, and is a minimum. Of course, by "brute force", the extreme value is $(4 \cdot 1 \cdot 1 - 2^2)/(4 \cdot 1) = 0$ and is a minimum since $1 > 0$.
5. Since $x^2 + 4x + 4 = (x + 2)^2 \geq 0$, the extreme value occurs when $x = -2$ and is 0. The extreme value is a minimum.
6. The extreme value is 2 [$(4 \cdot 1 \cdot 3 - 2^2)/(4 \cdot 1)$], and is a minimum. [Alternate solution: Note that $x^2 - 2x + 3 = (x^2 - 2x + 1) + 2 = (x - 1)^2 + 2$.]
7. Since $x^2 + 4x + 5 = (x^2 + 4x + 4) + 1 = (x + 2)^2 + 1$, we know by Exercise 5 that the extreme value is 1 and is a minimum. [Alternately, since $(4 \cdot 1 \cdot 5 - 4^2)/(4 \cdot 1) = 1$ and $1 > 0$, the extreme value is 1 and is a minimum.]
8. Since $x^2 - 2x = (x^2 - 2x + 1) - 1 = (x - 1)^2 - 1$, we know, by Exercise 4, that the extreme value is −1 and is a minimum. [Alternately, since $(4 \cdot 1 \cdot 0 - 2^2)/(4 \cdot 1) = -1$ and $1 > 0$, the extreme value is −1 and is a minimum.]
9. Since $x(4 - x) = -x^2 + 4x$, the extreme value is 4 [$(4 \cdot -1 \cdot 0 - 4^2)/(4 \cdot -1)$] and is a maximum. [Alternately, $x(4 - x) = 4x - x^2 = -(x^2 - 4x + 4) + 4 = -(x - 2)^2 + 4$. So, the extreme value is 4 and is a maximum.]
10. The extreme value is −8 [$(4 \cdot 1 \cdot -7 - 2^2)/(4 \cdot 1)$] and is a minimum. [Alternately, $x^2 - 2x - 7 = (x^2 - 2x + 1) - 7 - 1 = (x - 1)^2 - 8$, so that the extreme value is −8 and is a minimum.]
11. The extreme value is 3 [$(4 \cdot 2 \cdot 5 - 4^2)/(4 \cdot 2)$] and is a minimum. [Alternately, $2x^2 - 4x + 5 = 2(x^2 - 2x) + 5 = 2(x^2 - 2x + 1) + 5 - 2 = 2(x - 1)^2 + 3$. So, the extreme value is 3 and is a minimum.]
12. Since $7 - x^2 + 2x = -(x^2 - 2x - 7)$, we know from Exercise 10 that the extreme value is 8 and is a maximum. [Alternately, since $-1 < 0$ and $(4 \cdot -1 \cdot 7 - 2^2)/(4 \cdot -1) = 8$, the extreme value is 8 and is a maximum.]
13. The extreme value is −2 [$(4 \cdot 1 \cdot \frac{1}{4} - 3^2)/(4 \cdot 1)$] and is a minimum. [Alternately, $x^2 + 3x + \frac{1}{4} = (x^2 + 3x + \frac{9}{4}) + \frac{1}{4} - \frac{9}{4} = (x + \frac{3}{2})^2 - 2$ so that the extreme value is −2 and is a minimum.]
14. Since $(x + \frac{3}{2})^2 - 2 = x^2 + 3x + \frac{1}{4}$, the answers are those of Exercise 13.

Part E

Practice completing the square. A "program" is given for each of the first six exercises. Before filling in the blanks, study the program to see how it was arrived at. You may find it helpful to write such programs for the later exercises; doing so will help you to avoid certain kinds of blunders. A solution is given for the first exercise. Be sure you understand how to find the appropriate value for '*p*' in (5), above.

1. $2x^2 - 3x + 1 = 2(x^2 - \frac{3}{2}x + \_\_\_\_) + 1 - 2 \cdot \_\_\_\_ = 2(x - \_\_\_\_)^2 + \_\_\_\_$ [*Solution*: $2x^2 - 3x + 1 = 2(x^2 - \frac{3}{2}x + (\frac{3}{4})^2) + 1 - 2 \cdot (\frac{3}{4})^2 = 2(x - \frac{3}{4})^2 + -\frac{1}{8}$]
2. $x^2 - 2x + 3 = (x^2 - 2x + \_\_\_\_) + 3 - \_\_\_\_ = (x - \_\_\_\_)^2 + \_\_\_\_$
3. $x^2 + 2x - 3 = (x^2 + 2x + \_\_\_\_) - 3 - \_\_\_\_ = (x + \_\_\_\_)^2 + \_\_\_\_$
4. $x^2 + x + 1 = (x^2 + x + \_\_\_\_) + 1 - \_\_\_\_ = (x + \_\_\_\_)^2 + \_\_\_\_$
5. $2x^2 - 2x + 1 = 2(x^2 - x + \_\_\_\_) + 1 - 2 \cdot \_\_\_\_$
   $= 2(x - \_\_\_\_)^2 + \_\_\_\_$
6. $-x^2 + 4x + 3 = -(x^2 - 4x + \_\_\_\_) + 3 + \_\_\_\_$
   $= -(x - \_\_\_\_)^2 + \_\_\_\_$

7. $x^2 - 8x - 9$
8. $4x^2 + 8x + 4$
9. $3x^2 - 18x + 13$
10. $x^2 + 5x + 4$
11. $x^2 + 7x - 3$
12. $-3x^2 + 21x - 37$
13. $\frac{1}{2}x^2 - x + 1$
14. $\frac{2}{3}x^2 + 2x - \frac{3}{2}$

Part F

1. Suppose that *f* is defined by:

$$(*) \qquad f(x) = ax^2 + bx + c \qquad [a \neq 0]$$

Show that, for each *t*,

$$f(-\frac{b}{2a} + t) = f(-\frac{b}{2a} - t).$$

[*Hint*: One way to do this is to substitute the indicated arguments for '*x*' in (*). We hope you know a simpler way.]

2. What does the result of Exercise 1 tell you about a graph of the function *f*?
3. Two of the easier ways of obtaining a graph of the function *f* which is defined by (*), when '*a*', '*b*', and '*c*' are given numerical values, both begin by getting (*) into the form:

$$f(x) = a(x - p)^2 + q$$

and then drawing a graph of '$y = ax^2$'.

(a) Tell how the graph just mentioned can be used in constructing a graph of *f*.

Answers for Part E

2. $x^2 - 2x + 3 = (x^2 - 2x + \underline{1}) + 3 - \underline{1} = (x - \underline{1})^2 + \underline{2}$
3. $x^2 + 2x - 3 = (x^2 + 2x + \underline{1}) - 3 - \underline{1} = (x + \underline{1})^2 + \underline{-4}$ [or: $(x + 1)^2 - 4$]
4. $x^2 + x + 1 = (x^2 + x + \underline{\frac{1}{4}}) + 1 - \underline{\frac{1}{4}} = (x + \underline{\frac{1}{2}})^2 + \underline{\frac{3}{4}}$
5. $2x^2 - 2x + 1 = 2(x^2 - x + \underline{\frac{1}{4}}) + 1 - 2 \cdot \underline{\frac{1}{4}} = 2(x - \underline{\frac{1}{2}})^2 + \underline{\frac{1}{2}}$
6. $-x^2 + 4x + 3 = -(x^2 - 4x + \underline{4}) + 3 + \underline{4} = -(x - \underline{2})^2 + \underline{7}$
7. $x^2 - 8x - 9 = (x^2 - 8x + 16) - 9 - 16 = \underline{(x - 4)^2 - 25}$
8. $4x^2 + 8x + 4 = 4(x^2 + 2x + 1) = \underline{4(x + 1)^2}$
9. $3x^2 - 18x + 13 = 3(x^2 - 6x + 9) + 13 - 27 = \underline{3(x - 3)^2 - 14}$
10. $x^2 + 5x + 4 = (x^2 + 5x + \frac{25}{4}) + 4 - \frac{25}{4} = \underline{(x + \frac{5}{2})^2 - \frac{9}{4}}$
11. $x^2 + 7x - 3 = (x^2 + 7x + \frac{49}{4}) - 3 - \frac{49}{4} = \underline{(x + \frac{7}{2})^2 - \frac{61}{4}}$
12. $-3x^2 + 21x - 37 = -3(x^2 - 7x + \frac{49}{4}) - 37 + \frac{147}{4} = \underline{-3(x - \frac{7}{2})^2 - \frac{1}{4}}$
13. $\frac{1}{2}x^2 - x + 1 = \frac{1}{2}(x^2 - 2x + 1) + 1 - \frac{1}{2} = \underline{\frac{1}{2}(x - 1)^2 + \frac{1}{2}}$
14. $\frac{2}{3}x^2 + 2x - \frac{3}{2} = \frac{2}{3}(x^2 + 3x + \frac{9}{4}) - \frac{3}{2} - \frac{3}{2} = \underline{\frac{2}{3}(x + \frac{3}{2})^2 - 3}$

Answers for Part F

1. By completing the square, we see that, for each x,

$$f(x) = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}.$$

So, for each t, $f(-\frac{b}{2a} + t) = at^2 + \frac{4ac - b^2}{4a}$ and $f(-\frac{b}{2a} - t) = a(-t)^2 + \frac{4ac - b^2}{4a}$. Since $t^2 = (-t)^2$, it follows that, for each t, $f(-\frac{b}{2a} + t) = f(-\frac{b}{2a} - t)$.

2. It tells us that the graph is symmetric about the graph of the line whose equation is '$x = -\frac{b}{2a}$'.
3. (a) Having drawn the graph of '$y = ax^2$', locate the points in the graph of f by "shifting" each point p units in the sense of p and q units in the sense of q.

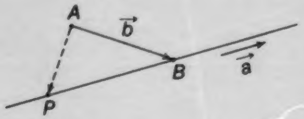
(b) Tell how to redraw the coordinate axes so that the graph of '$y = ax^2$' becomes a graph of f.

(c) Either method requires that you start with a graph of '$y = ax^2$'. Consider the case in which $a > 0$ and figure out how a graph of the simpler equation '$y = x^2$' can be used as a graph of '$y = ax^2$' by

- (i) recalibrating the y-axis, or
- ☆(ii) recalibrating the x-axis, or
- ☆☆(iii) recalibrating both axes.

Part G

Consider a point A and a line $\overleftrightarrow{B[\vec{a}]}$, and let $\vec{b} = B - A$.



1. Find the point P of $\overleftrightarrow{B[\vec{a}]}$ for which $\|P - A\|$ is as small as possible. [Intuitively, find the point of $\overleftrightarrow{B[\vec{a}]}$ nearest A.] [*Hint:* $P \in \overleftrightarrow{B[\vec{a}]}$ if and only if, for some t, $P = B + \vec{a}t$. Also, $P - A = (B + \vec{a}t) - A = \vec{b} + \vec{a}t$. Finally, $\|P - A\|$ is as small as possible exactly when $\|P - A\|^2$ is as small as possible. Obtain a quadratic function in 't'.]
2. Show that $\|P - A\|$ is as small as possible if and only if $(P - A) \perp \vec{a}$.
3. (a) What is the minimum value for $\|P - A\|$ given that $\vec{a} \cdot \vec{a} = 10$, $\vec{b} \cdot \vec{b} = 9$, and $\vec{a} \cdot \vec{b} = 6$?

   (b) What is the minimum value of $\|P - A\|$ given that $\vec{a} \cdot \vec{a} = 2$, $\vec{b} \cdot \vec{b} = 3$, and $\vec{a} \cdot \vec{b} = \frac{3}{2}$?

Answers for Part F [cont.]

3. (b) Having drawn the graph of '$y = ax^2$', "shift" the x-axis p units in the sense of $-p$ and "shift" the y-axis q units in the sense of $-q$. The graph drawn for '$y = ax^2$' is now, with respect to the newly located axes, a graph for f.

   (c) (i) Use $1/a$ times as long as the original unit on the y-axis.

   (ii) Use a unit $\sqrt{a}$ times as long as the original unit on the x-axis.

   (iii) Use units a times as long as the original units on both axes. [$y = ax^2$ if and only if $(ay) = (ax)^2$.]

   Note that it results from (iii) that the graphs of any two quadratic functions are similar [in the strict sense of 'similar' which you are accustomed to use in geometry]. Each can be obtained from any of them by a uniform stretching and a translation.

Answers for Part G

1. By the hint $P - A = \vec{b} + \vec{a}t$, for some t. So,

$$\begin{aligned}||P - A||^2 &= (\vec{a}\cdot\vec{a})t^2 + 2(\vec{a}\cdot\vec{b})t + \vec{b}\cdot\vec{b}\\ &= (\vec{a}\cdot\vec{a})\left(t^2 + \frac{2(\vec{a}\cdot\vec{b})}{\vec{a}\cdot\vec{a}}t + \frac{(\vec{a}\cdot\vec{b})^2}{(\vec{a}\cdot\vec{a})^2}\right) + \vec{b}\cdot\vec{b} - \frac{(\vec{a}\cdot\vec{b})^2}{\vec{a}\cdot\vec{a}}\\ &= (\vec{a}\cdot\vec{a})\left(t + \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)^2 + \frac{(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})^2}{\vec{a}\cdot\vec{a}}\end{aligned}$$

   From what we know about quadratic functions, it follows that $||P - A||$ is at a minimum when $t = -(\vec{a}\cdot\vec{b})/(\vec{a}\cdot\vec{a})$. So, the point P of $\overleftrightarrow{B[\vec{a}]}$ for which $||P - A||$ is as small as possible is such that $P = B - \vec{a}(\vec{a}\cdot\vec{b})/(\vec{a}\cdot\vec{a})$.

   [Note that, since the minimum value of $||P - A||^2$ is nonnegative, the calculations in answer to Exercise 1 give us another proof of the important Theorem 11-8(a).]

2. By Exercise 1, we have that $||P - A||$ is as small as possible if and only if $P - A = \vec{b} - \vec{a}(\vec{a}\cdot\vec{b})/(\vec{a}\cdot\vec{a})$. Now, since $(\vec{b} - \vec{a}(\vec{a}\cdot\vec{b})/(\vec{a}\cdot\vec{a}))\cdot\vec{a} = (\vec{b}\cdot\vec{a}) - (\vec{a}\cdot\vec{b}) = 0$, it follows that $||P - A||$ is as small as possible if and only if $(P - A)\cdot\vec{a} = 0$, that is, if and only if $(P - A) \perp \vec{a}$.

3. By the results in Exercise 1, the minimum value of $||P - A||$ is $\sqrt{(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})^2/(\vec{a}\cdot\vec{a})}$.

   (a) In this case the minimum value for $||P - A||$ is $\frac{3\sqrt{15}}{5}$. [or: $\sqrt{\frac{27}{5}}$. This is computed as follows: $\sqrt{9 - 36/10} = \sqrt{9 \cdot \frac{3}{5}} = \frac{3}{5}\sqrt{15}$.]

   (b) In this case, the minimum value is $\frac{\sqrt{30}}{4}$ [$\sqrt{3 - (\frac{9}{4})/2} = \sqrt{\frac{15}{8}}$].

Chapter Thirteen

Orthonormal Coordinate Systems

13.01 Coordinates and Perpendicularity

In the work on coordinate systems in Chapter 10 [of Volume 1], we have considered various problems concerning the description of planes and lines by means of equations which refer to a given coordinate system. In this section we shall see that the results we obtained appear in a new light in the special case in which the given coordinate system is an orthonormal one—that is, in case the terms of the basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ of the coordinate system are pairwise orthogonal unit vectors. Before discussing this special case it will be helpful to summarize the results which were obtained in Chapter 10, for a coordinate system with any origin $O \in \mathscr{E}$ and any basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ for $\mathscr{T}$.

We began by recalling that, for any point A and any vectors $\vec{p}$ and $\vec{q}$, the set $A[\vec{p}, \vec{q}]$ is described, with respect to the given coordinate system, by the parametric equations:

$$(1) \qquad \begin{cases} x_1 = a_1 + p_1 r + q_1 s \\ x_2 = a_2 + p_2 r + q_2 s \\ x_3 = a_3 + p_3 r + q_3 s \end{cases}$$

where (a_1, a_2, a_3) are the coordinates of A, and (p_1, p_2, p_3) and (q_1, q_2, q_3) are the components of $\vec{p}$ and $\vec{q}$. It follows that the system (1) describes a plane if and only if $(\vec{p}, \vec{q})$ is linearly independent. We then showed that $(\vec{p}, \vec{q})$ is linearly independent if and only if the vector $\vec{m}$ whose components (m_1, m_2, m_2) are given by:

$$(2) \qquad m_1 = \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \; m_2 = \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \text{ and } m_3 = \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}$$

is not $\vec{0}$. Moreover, we showed that if $\vec{m}$ is defined by (2) and is non-$\vec{0}$ then the equation:

In this chapter we shall draw upon many of the results derived in Chapter 10 of volume 1. Before beginning this chapter, however, it may be wise to quickly review the material presented in Chapter 10. One way to do this is to use the chapter test (pp. 472-473, volume 1) as a combined homework and class discussion assignment. During the discussion of the exercises, the teacher can quickly present those ideas which may have been lost in the meantime.

To expedite the review, you may wish to ditto or Xerox the test items for the Chapter 10 test.

Here are some suggested assignments for section 13.01

(i) Part A is best used for class discussion to continue and extend the review of Chapter 10. This work can alternate between group discussions and supervised seat work.

(ii) Parts B and C constitute one reasonable homework assignment. It may be more efficient, however, to permit students to do this assignment in teams.

(iii) Part D lends itself best to a teacher directed class project.

(iv) Part E and the discussion that follows should be directed by the teacher to insure that students do not miss the point we wish to make.

(v) Parts F and G are appropriate for homework, but together are a little unreasonable for a single assignment. It is recommended that you either break the material into two assignments, assign selected exercises to particular students, or permit students to work in teams. Another alternative would be to include some exercises of Part F in the discussion of Part E, with the remainder of Part E and Part F as a homework assignment.

(vi) Part H (if treated) should be teacher directed.

(3) $$(x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0$$

describes the same plane $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ as does the system (1). This showed that any plane can be described by an equation like (3). And, as we already know, for any vector $\vec{m}$ with components (m_1, m_2, m_3), whether these are obtained from equations like (2) or not, equation (3) describes a plane if and only if $\vec{m} \neq \vec{0}$. As we noted at the time, it follows that, for any $\vec{m}$ and c, the equation:

(4) $$x_1m_1 + x_2m_2 + x_3m_3 = c$$

describes a plane if and only if $\vec{m} \neq \vec{0}$, and that any plane can be described by an equation like (4).

Next, we showed that (4) and a similar equation:

(5) $$x_1n_1 + x_2n_2 + x_3n_3 = d$$

describe nonparallel planes if and only if $(\vec{m}, \vec{n})$ is linearly independent. As in the case of $(\vec{p}, \vec{q})$, $(\vec{m}, \vec{n})$ is linearly independent if and only if the vector $\vec{r}$ whose components (r_1, r_2, r_3) are given by:

(6) $$r_1 = \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix},\ r_2 = \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix},\ \text{and } r_3 = \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}$$

is not $\vec{0}$. Moreover, we saw that if $\vec{r}$ is defined by (6) and is non-$\vec{0}$ then the line of intersection of the planes described by (4) and (5) is, itself, described by the parametric equations:

(7) $$\begin{cases} x_1 = c_1 + r_1t \\ x_2 = c_2 + r_2t \\ x_3 = c_3 + r_3t \end{cases}$$

where (c_1, c_2, c_3) are the coordinates of any chosen point C which belongs to both planes.

Finally, we showed that the equation (4) and the system (7) describe a plane and a transversal to that plane if and only if

(8) $$r_1m_1 + r_2m_2 + r_3m_3 \neq 0;$$

and this led us to the discovery of third order determinants and to the result that $(\vec{p}, \vec{q}, \vec{r})$ is linearly independent if and only if

(9) $$\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \neq 0.$$

It is noteworthy that *none of these results depends in any way on perpendicularity.* Each of them was established in Volume 1 prior to our work on perpendicularity. Nevertheless, as mentioned at the beginning of this section, bringing in the notion of perpendicularity will add to our understanding of these results.

To see how this comes about, let's consider any plane π, a point $A \in \pi$, and a line $m \perp \pi$. Suppose, also, that $\vec{m}$ is a non-$\vec{0}$ vector in the direction of m. It follows that $\pi = \overleftrightarrow{A[\vec{m}]}^{\perp}$ and that

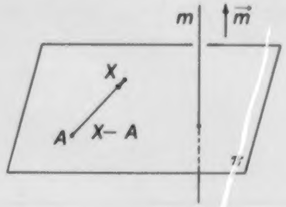


Fig. 13–1

$$\begin{aligned} X \in \pi &\longleftrightarrow \text{proj}_m (X) = \text{proj}_m (A) \\ &\longleftrightarrow \text{proj}_m (X) - \text{proj}_m (A) = \vec{0} \\ &\longleftrightarrow \text{proj}_{[\vec{m}]} (X - A) = \vec{0} \\ &\longleftrightarrow (X - A) \cdot \vec{m} = 0. \end{aligned}$$

Consequently, the plane $\overleftrightarrow{A[\vec{m}]}^{\perp}$ is described by the equation:

(10) $$(X - A) \cdot \vec{m} = 0$$

in the sense that a point X belongs to $\overleftrightarrow{A[\vec{m}]}^{\perp}$ if and only if it satisfies (10). If we now introduce an origin $O \in \mathscr{E}$ and, for each point X, let $\vec{x}$ be the position vector with respect to O of X we see that, since

$$X - A = (X - O) - (A - O) = \vec{x} - \vec{a},$$

equation (10) is equivalent to:

(11) $$(\vec{x} - \vec{a}) \cdot \vec{m} = 0$$

Equation (11) also describes $\overleftrightarrow{A[\vec{m}]}^{\perp}$ in the sense that a point X belongs to this plane if and only if its position vector $\vec{x}$ satisfies (11).

Although we have not as yet introduced a coordinate system [only an origin], equation (11) seems to bear some relation to equation (3). [If you recall a theorem from Chapter 11 you may see why it might.] Remember that a basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ for $\mathscr{T}$ is said to be orthonormal when its terms are pairwise orthogonal unit vectors—that is, when

$$\vec{u}_2 \cdot \vec{u}_3 = \vec{u}_3 \cdot \vec{u}_1 = \vec{u}_1 \cdot \vec{u}_2 = 0 \text{ and } \vec{u}_1 \cdot \vec{u}_1 = \vec{u}_2 \cdot \vec{u}_2 = \vec{u}_3 \cdot \vec{u}_3 = 1.$$

Recall also that one of the most useful properties of such a basis is that if

$$\vec{p} = \vec{u}_1 p_1 + \vec{u}_2 p_2 + \vec{u}_3 p_3 \text{ and } \vec{q} = \vec{u}_1 q_1 + \vec{u}_2 q_2 + \vec{u}_3 q_3$$

then

$$\vec{p} \cdot \vec{q} = p_1 q_1 + p_2 q_2 + p_3 q_3.$$

It follows that if we do introduce a coordinate system with origin O using an *orthonormal* basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ for $\mathscr{T}$ then, with respect to this coordinate system, equation (11) is equivalent to (3):

$$(x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0$$

So, we have a new proof that an equation like (3) for which $(m_1, m_2, m_3) \neq (0, 0, 0)$ describes a plane, and that any plane can be described by such an equation. Unlike our earlier proof [see Theorem 10–15] this one applies *only* to the case in which the coordinate system is an orthonormal one. But, in this case, we now have extra information about the numbers (m_1, m_2, m_3). They are the components of a non-$\vec{0}$ vector in the direction of a line m which is perpendicular to the plane described by the equation. For example, given that the equation:

$$6x_1 + 5x_2 - 2x_3 = -65$$

is an equation for a plane π with respect to an *orthonormal* coordinate system, we know that $(6, 5, -2)$ are the components of a vector in the direction of a line which is perpendicular to π. With this information, it is a simple task to write parametric equations for the line through a given point and perpendicular to π.

A line which is perpendicular to a plane is called *a normal* to the plane. Also, the components of a non-$\vec{0}$ vector in the direction of a line are often called *direction numbers* of the line. Using these terms, the results of this discussion are summarized in:

> Theorem 13–1 With respect to an orthonormal coordinate system, an equation:
>
> (i) $(x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0$
>
> or:
>
> (ii) $x_1 m_1 + x_2 m_2 + x_3 m_3 = c,$
>
> where $(m_1, m_2, m_3) \neq (0, 0, 0)$, describes a plane whose normals have the direction numbers (m_1, m_2, m_3); and any such plane can be described by equations like (i) and. also, by equations like (ii).

Notice that, since planes are parallel if and only if they are perpendicular to parallel lines, the corollary of Theorem 10–17 is, for orthonormal coordinate systems, a corollary of the present theorem.

Theorem 13–1 will supply new insights into the other results obtained in Chapter 10 and summarized above. More precisely, it will improve our understanding of the special cases of these results in which the coordinate system is assumed to be an orthonormal one. Thus, *we shall adopt this assumption throughout the remainder of this section.*

Exercises

Part A

Suppose that π and σ are the planes described by the equations:

$$x_1 m_1 + x_2 m_2 + x_3 m_3 = c$$

and:

$$x_1 n_1 + x_2 n_2 + x_3 n_3 = d,$$

where (m_1, m_2, m_3) and (n_1, n_2, n_3) are the components of non-$\vec{0}$ vectors $\vec{m}$ and $\vec{n}$.

1. According to Theorem 13–1, π is perpendicular to any line m whose direction is $[\vec{m}]$. What does this tell you about $[\pi]$? About $[\vec{m}]$?
   [*Hint*: Your answers should consist of two short equations.]
2. Show that $\vec{r} \in [\pi]$ if and only if $\vec{r} \cdot \vec{m} = 0$.
3. Show that, for any point C, the parametric equations (7) on page 108 describe a line which is parallel to π if and only if the vector $\vec{r}$ whose components are (r_1, r_2, r_3) is not $\vec{0}$ and $r_1 m_1 + r_2 m_2 + r_3 m_3 = 0$.
4. Show that [for any point C] the system (7) describes a transversal of π if and only if the condition (8) on page 108 is satisfied.
5. Assume that $(m_1, m_2, m_3) = (4, 9, -5)$ and that $c = 2$—that is, assume that π is described by the equation
   '$4x_1 + 9x_2 - 5x_3 = 2$'.
   (a) Let R and S be points whose coordinates are $(5, 2, -6)$ and $(8, 4, 0)$, respectively. Give the components of $S - R$ and determine whether or not $\overleftrightarrow{RS}$ is parallel to π.
   (b) Let T be the point with coordinates $(6, 5, -13)$. Give the components of $T - R$ and determine whether or not $\overleftrightarrow{RT}$ is parallel to π.

Answers for Part A

1. $[\pi] = [\vec{m}]^{\perp}$; $[\vec{m}] = [\pi]^{\perp}$

2. By Exercise 1, $[\pi] = [\vec{m}]^{\perp}$. So, $\vec{r} \in [\pi]$ if and only if $\vec{r} \in [\vec{m}]^{\perp}$. Since $\vec{r} \in [\vec{m}]^{\perp}$ if and only if $\vec{r} \cdot \vec{m} = 0$, it follows, by the replacement rule for biconditionals, that $\vec{r} \in [\pi]$ if and only if $\vec{r} \cdot \vec{m} = 0$.

3. If $\vec{r} \neq \vec{0}$ and $r_1m_1 + r_2m_2 + r_3m_3 = 0$ then (7) describes a line whose direction is that of $\vec{r}$ and $\vec{r} \in [\vec{m}]^{\perp} = [\pi]$. So, in this case, (7) describes a line which is parallel to π. On the other hand, suppose that (7) describes a line which is parallel to π. It follows that $\vec{r} \neq \vec{0}$ and that $\vec{r} \in [\pi] = [\vec{m}]^{\perp}$ — that is, $r_1m_1 + r_2m_2 + r_3m_3 = 0$.

4. If condition (8) is satisfied then $\vec{r} \neq \vec{0}$. It follows that (7) describes a line which, by the contrapositive of the only if-part of Exercise 3, is not parallel to π. Since it is not parallel to π, the line is a transversal of π. On the other hand, if (7) describes a transversal to π then $\vec{r} \neq \vec{0}$ and, by the contrapositive of the if-part of Exercise 3, (8) is satisfied.

   [Exercise 3 and 4 are, essentially, Theorem 10-17 and its corollary on page 457 of volume 1 and students may cite these as solutions. Note that in reproving the corollary we have, as in the commentary for page 457 of volume 1, taken note of the fact that it is not merely the contrapositive of the first part of the theorem. The phrase 'describe a plane and a line which are not parallel' is not equivalent to 'do not describe a plane and a line which are parallel'.]

5. (a) The components of S - R are (3, 2, 6). $(S - R) \cdot \vec{m} = 3 \cdot 4 + 2 \cdot 9 + 6 \cdot -5 = 0$. So, $\overleftrightarrow{RS} \parallel \pi$.

   (b) The components of T - R are (1, 3, -7). $(T - R) \cdot \vec{m} = 1 \cdot 4 + 3 \cdot 9 + -7 \cdot -5 = 66 \neq 0$. So, $\overleftrightarrow{RT} \not\parallel \pi$.

(c) Write parametric equations for the line m which is perpendicular to π and contains the point with coordinates (3, −1, 2). Determine the coordinates of the point at which m intersects π.

(d) Is $\overleftrightarrow{RS} \perp m$? How about $\overleftrightarrow{RT}$? Explain your answers.

(e) One of the lines $\overleftrightarrow{RS}$ and $\overleftrightarrow{RT}$ is a transversal of π. Determine the coordinates of that line's point of intersection with π.

6. To say that $\vec{r} \in [\pi] \cap [\sigma]$ amounts to saying that the components (r_1, r_2, r_3) of $\vec{r}$ satisfy a certain pair of equations. What equations?
7. We have seen that π and σ are nonparallel if and only if $(\vec{m}, \vec{n})$ is linearly independent. Apply what you know about determinants to your answer for Exercise 6 to obtain a description of the vectors which belong to $[\pi] \cap [\sigma]$ in case $\pi \not\parallel \sigma$.
8. Assuming that $(m_1, m_2, m_3) = (1, -2, 2)$, $(n_1, n_2, n_3) = (2, -1, 1)$, $c = 3$, and $d = 4$, show that $\pi \not\parallel \sigma$ and write parametric equations which describe $\pi \cap \sigma$.

Part B

Suppose that π is the plane described by the system:

$$\begin{cases} x_1 = a_1 + p_1r + q_1s \\ x_2 = a_2 + p_2r + q_2s \\ x_3 = a_3 + p_3r + q_3s \end{cases}$$

where (p_1, p_2, p_3) and (q_1, q_2, q_3) are the components of linearly independent vectors $\vec{p}$ and $\vec{q}$.

1. Since $[\pi] = [\vec{p}, \vec{q}]$ it follows that $\vec{m} \in [\pi]^{\perp}$ if and only if $\vec{m}$ [not its components] satisfies each of two equations. What equations?
2. To say that $\vec{m} \in [\pi]^{\perp}$ amounts to saying that the components (m_1, m_2, m_3) of $\vec{m}$ satisfy a certain pair of equations. What equations?
3. Apply your knowledge of determinants to your answer for Exercise 2 to obtain the components of a non-$\vec{0}$ vector $\vec{m}$ which belongs to $[\pi]^{\perp}$.
4. Use Theorem 13-1 and your answer for Exercise 3 to write a single equation for π.
5. (a) Describe in two ways the plane $A[\vec{p}, \vec{q}]$, where A has coordinates (3, 2, −5) and $\vec{p}$ and $\vec{q}$ have components (4, −6, 3) and (−2, 1, 5).

   (b) Describe the direction $[\vec{p}, \vec{q}]^{\perp}$.

   (c) Would any of your answers for parts (a) and (b) remain correct if the coordinate system were not orthonormal?

Part C

Suppose that the components of $\vec{p}$, $\vec{q}$, and $\vec{r}$ are (p_1, p_2, p_3), (q_1, q_2, q_3), and (r_1, r_2, r_3). Let's make use of the fact that we are dealing with an

Answers for Part A [cont.]

5. (c) Making direct use of (7) and the given equation '$4x_1 + 9x_2 - 5x_3 = 2$' for π, one set of parametric equations is:

$$\begin{cases} x_1 = 3 + 4t \\ x_2 = -1 + 9t \\ x_3 = 2 - 5t \end{cases}$$

The line m intersects π at the point where $t = 9/122$. $[4(3 + 4t) + 9(-1 + 9t) - 5(2 - 5t) = 2]$ So, the coordinates of the point of intersection of m and π are $(\frac{201}{61}, -\frac{41}{122}, \frac{199}{122})$.

(d) Yes, because $(S - R)\cdot\vec{m} = 0$.; No, because $(T - R)\cdot\vec{m} \neq 0$.

(e) $\overleftrightarrow{RT}$ is a transversal of π and has parametric equations:

$$\begin{cases} x_1 = 6 + t \\ x_2 = 5 + 3t \\ x_3 = -13 - 7t \end{cases}$$

So, $\overleftrightarrow{RT}$ intersects π at the point where $t = -2[4(6 + t) + 9(5 + 3t) - 5(-13 - 7t) = 2]$ — that is, at the point whose coordinates are $(4, -1, 1)$.

6. The equations are:

$$r_1m_1 + r_2m_2 + r_3m_3 = 0$$
$$r_1n_1 + r_2n_2 + r_3n_3 = 0$$

7. In case $\pi \nparallel \sigma$, the vectors which belong to $[\pi] \cap [\sigma]$ are in the direction of the vector whose components are

$$\left(\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}\right).$$

8. From the given information, equations for π and σ are:

$$x_1 - 2x_2 + 2x_3 = 3$$
$$2x_1 - x_2 + x_3 = 4$$

$\pi \nparallel \sigma$ since $\left(\begin{vmatrix} -2 & 2 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}\right) = (0, 3, 3) \neq (0, 0, 0)$.

Since $(5/3, -2/3, 0)$ are the coordinates of a point in $\pi \cap \sigma$, parametric equations for $\pi \cap \sigma$ are:

$$\begin{cases} x_1 = 5/3 \\ x_2 = -2/3 + 3t \\ x_3 = 3t \end{cases}$$

Answers for Part B

1. $\vec{m}\cdot\vec{p} = 0$ and $\vec{m}\cdot\vec{q} = 0$

2. $m_1p_1 + m_2p_2 + m_3p_3 = 0$ and $m_1q_1 + m_2q_2 + m_3q_3 = 0$

3. $\left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}\right)$. [This is clearly different from $(0, 0, 0)$ since $(\vec{p}, \vec{q})$ is linearly independent.]

4. $(x_1 - a_1)\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} + (x_2 - a_2)\begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} + (x_3 - a_3)\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = 0$

5. (a) Here are parametric equations for $\overline{A[\vec{p}, \vec{q}]}$:

$$\begin{cases} x_1 = 3 + 4r - 2s \\ x_2 = 2 - 6r + s \\ x_3 = -5 + 3r + 5s \end{cases}$$

Here is a single equation for $\overline{A[\vec{p}, \vec{q}]}$:

$$(x_1 - 3)\begin{vmatrix} -6 & 3 \\ 1 & 5 \end{vmatrix} + (x_2 - 2)\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} + (x_3 + 5)\begin{vmatrix} 4 & -6 \\ -2 & 1 \end{vmatrix} = 0$$

which simplifies to:

$$(x_1 - 3)\cdot -33 + (x_2 - 2)\cdot -26 + (x_3 + 5)\cdot -8 = 0$$

or:

$$33x_1 + 26x_2 + 8x_3 = 111$$

(b) $[\vec{p}, \vec{q}]^{\perp} = [\vec{m}]$, where $\vec{m}$ has components $(33, 26, 8)$.

(c) Part (a) would remain correct, but the answer for part (b) would not.

orthonormal basis to give an alternate proof for the condition (9).

1. Show that if $(\vec{p}, \vec{q}, \vec{r})$ is linearly independent then condition (9) on page 108 is satisfied. [*Hint*: Assuming that $(\vec{p}, \vec{q}, \vec{r})$ is linearly independent, what do you know about $\vec{p}$ and $[\vec{q}, \vec{r}]$? Supposing that $\vec{s}$ is a non-$\vec{0}$ vector in $[\vec{q}, \vec{r}]^{\perp}$, what follows concerning $\vec{p}$ and $\vec{s}$? Can you describe one such vector $\vec{s}$?]
2. Show that if $(\vec{p}, \vec{q}, \vec{r})$ is linearly dependent then condition (9) is not satisfied. [*Hint*: Consider two cases—that in which $(\vec{q}, \vec{r})$ is linearly dependent and that in which $(\vec{q}, \vec{r})$ is linearly independent. In the second case, what do you know about $\vec{p}$ and $[\vec{q}, \vec{r}]$?]
3. Explain why your results in Exercises 1 and 2 show that $(\vec{p}, \vec{q}, \vec{r})$ is linearly independent if and only if condition (9) is satisfied.

Part D

Prove:

> **Theorem 13-2** If, with respect to an orthonormal basis for $\mathscr{T}$, the components of the linearly independent vectors $\vec{p}$ and $\vec{q}$ are (p_1, p_2, p_3) and (q_1, q_2, q_3), respectively, then $[\vec{p}, \vec{q}]^{\perp} = [\vec{m}]$, where $\vec{m}$ is the vector whose components are
>
> $$\left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}\right).$$

Part E

It follows from Theorem 13-1 that, for $\vec{m} \neq \vec{0}$, the equation:

$$(*) \qquad \vec{x} \cdot \vec{m} = c$$

describes a plane which is perpendicular to the line m, where

$$m = \overleftrightarrow{O[\vec{m}]}.$$

In the following exercises you will establish this result in another way and, in doing so, discover a geometrical interpretation for the value of 'c' in (*).

1. Find the position vector $\vec{c}$ of a point C which belongs both to m and to the set described by (*). [*Hint*: Since O is the origin, $\vec{c} = C - O$. What does this tell you about $\vec{c}$ if $C \in m$?]
2. Show that $\vec{x} \cdot \vec{m} = c \longleftrightarrow \vec{m}\left(\dfrac{\vec{x} \cdot \vec{m}}{\vec{m} \cdot \vec{m}}\right) = \vec{c}$, where $\vec{c}$ is the vector described in Exercise 1.
3. Show that $\vec{x} \cdot \vec{m} = c \longleftrightarrow \text{proj}_m (X) = C$—that is, show that (*) describes the plane $\overleftrightarrow{C[\vec{m}]}^{\perp}$. [*Hint*: Since $X = O + \vec{x}$, what is $\text{proj}_m (X)$?]

Answers for Part C

1. Answering the questions in the hint, we have that $\vec{p} \notin [\vec{q}, \vec{r}]$ and that $\vec{p} \cdot \vec{s} \neq 0$. One such vector $\vec{s}$ has components

$$\left(\begin{vmatrix} q_2 & q_3 \\ r_2 & r_3 \end{vmatrix}, \begin{vmatrix} q_3 & q_1 \\ r_3 & r_1 \end{vmatrix}, \begin{vmatrix} q_1 & q_2 \\ r_1 & r_2 \end{vmatrix}\right).$$

Since $\vec{p} \cdot \vec{s} \neq 0$ and since we are dealing with an orthonormal basis, we have that

$$p_1 \begin{vmatrix} q_2 & q_3 \\ r_2 & r_3 \end{vmatrix} + p_2 \begin{vmatrix} q_3 & q_1 \\ r_3 & r_1 \end{vmatrix} + p_3 \begin{vmatrix} q_1 & q_2 \\ r_1 & r_2 \end{vmatrix} \neq 0.$$

But, this means that $\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \neq 0$, which is what we wished to show.

2. Suppose that $(\vec{p}, \vec{q}, \vec{r})$ is linearly dependent. Either $(\vec{q}, \vec{r})$ is linearly dependent or it is linearly independent. In case $(\vec{q}, \vec{r})$ is linearly dependent, then $\left(\begin{vmatrix} q_2 & q_3 \\ r_2 & r_3 \end{vmatrix}, \begin{vmatrix} q_3 & q_1 \\ r_3 & r_1 \end{vmatrix}, \begin{vmatrix} q_1 & q_2 \\ r_1 & r_2 \end{vmatrix}\right) = (0, 0, 0)$.

Since we are dealing with an orthonormal basis and since $\vec{p} \cdot \vec{0} = 0$, for any $\vec{p}$, it follows that

$$(*) \qquad p_1 \begin{vmatrix} q_2 & q_3 \\ r_2 & r_3 \end{vmatrix} + p_2 \begin{vmatrix} q_3 & q_1 \\ r_3 & r_1 \end{vmatrix} + p_3 \begin{vmatrix} q_1 & q_2 \\ r_1 & r_2 \end{vmatrix} = 0.$$

In case $(\vec{q}, \vec{r})$ is linearly independent, it follows that $\vec{p} \in [\vec{q}, \vec{r}]$ and that $\left(\begin{vmatrix} q_2 & q_3 \\ r_2 & r_3 \end{vmatrix}, \begin{vmatrix} q_3 & q_1 \\ r_3 & r_1 \end{vmatrix}, \begin{vmatrix} q_1 & q_2 \\ r_1 & r_2 \end{vmatrix}\right)$ are the components of a non-$\vec{0}$ vector — say, $\vec{s}$ — in $[\vec{q}, \vec{r}]^{\perp}$. Since $\vec{p} \in [\vec{q}, \vec{r}]$, $\vec{p} \cdot \vec{s} = 0$. Thus, we have (*) again. So, in either case,

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = 0$$

for the left hand side of (*) is equal to the given third order determinant.

3. The result in Exercise 2 is equivalent, by contraposition, to the converse of the result in Exercise 1. Hence, we have established the required biconditional.

Answer for Part D

Since $(\vec{p}, \vec{q})$ is linearly independent, we know that $[\vec{p}, \vec{q}]^{\perp}$ is a proper direction and that $\left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}\right)$ are the components of a non-$\vec{0}$ vector — say, $\vec{m}$ — in $[\vec{p}, \vec{q}]^{\perp}$. Hence $[\vec{p}, \vec{q}]^{\perp} = [\vec{m}]$.

Answers for Part E

1. Given that $\vec{c}$ belongs to both m and the set described by (*), it follows that $\vec{c} = \vec{m}t$, for some t, and $\vec{c} \cdot \vec{m} = c$. So, $c = \vec{m} \cdot \vec{c} = \vec{m} \cdot \vec{m}t$, so that $t = c/(\vec{m} \cdot \vec{m})$. Thus, $\vec{c} = \vec{m}(c/(\vec{m} \cdot \vec{m}))$.
2. Since $\vec{m} \neq \vec{0}$, $\vec{m} \cdot \vec{m} \neq 0$. So, $\vec{x} \cdot \vec{m} = c$ if and only if $(\vec{x} \cdot \vec{m})/(\vec{m} \cdot \vec{m}) = c/(\vec{m} \cdot \vec{m})$, and the latter is the case if and only if $\vec{m}(\vec{x} \cdot \vec{m})/(\vec{m} \cdot \vec{m}) = \vec{m}(c/(\vec{m} \cdot \vec{m})) = \vec{c}$.
3. By Exercise 2, we know that $\vec{x} \cdot \vec{m} = c$ if and only if $\vec{m}\ \mathrm{comp}_{\vec{m}}(\vec{x}) = \vec{c}$. Now, $\vec{m}\ \mathrm{comp}_{\vec{m}}(\vec{x}) = \mathrm{proj}_{[\vec{m}]}(\vec{x}) = \mathrm{proj}_m(X) - \mathrm{proj}_m(O) = \mathrm{proj}_m(X) - O$, so that $\vec{x} \cdot \vec{m} = c$ if and only if $\mathrm{proj}_m(X) - O = \vec{c}$. Since $\mathrm{proj}_m(X) - O = \vec{c}$ if and only if $\mathrm{proj}_m(X) = O + \vec{c} = C$, we have that $\vec{x} \cdot \vec{m} = c$ if and only if $\mathrm{proj}_m(X) = C$.

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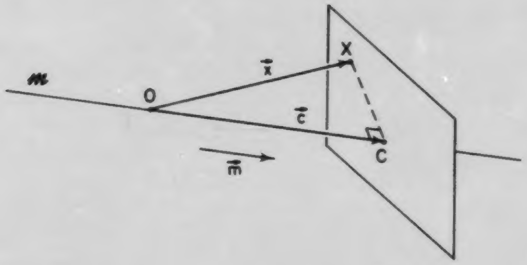
4.



5. (a) Since $\vec{c} = \vec{m}(c/(\vec{m} \cdot \vec{m}))$, $\vec{c} \cdot \vec{c} = (\vec{m} \cdot \vec{m})\left(\frac{c}{\vec{m} \cdot \vec{m}}\right)^2 = c^2/(\vec{m} \cdot \vec{m})$. Now, $\vec{c} \cdot \vec{c} = \|\vec{c}\|^2$ and $\vec{m} \cdot \vec{m} = \|\vec{m}\|^2$, so that $\|\vec{c}\| = |c|/\|\vec{m}\|$.
   (b) Since $\vec{c} = \vec{m}(c/(\vec{m} \cdot \vec{m})) = \vec{m}(c/\|\vec{m}\|^2) = (\vec{m}/\|\vec{m}\|)(c/\|\vec{m}\|)$, it follows that $c/\|\vec{m}\| = \vec{c} : (\vec{m}/\|\vec{m}\|)$. Note that $\vec{m}/\|\vec{m}\|$ is a unit vector.
6. From Exercise 5, the number $|c|/\|\vec{m}\|$ is the norm of the vector from O to the foot of the perpendicular from O to the given plane. So, this number is [intuitively] the distance from O to the given plane. [The notion of the distance between a point and a plane is developed formally in section 14.03.]

4. Draw a picture which illustrates the result obtained in Exercise 3. [Choose a case in which $\vec{c} \neq \vec{0}$.]

5. (a) Show that $\|\vec{c}\| = |c|/\|\vec{m}\|$.

(b) Express '$c/\|\vec{m}\|$' as a ratio of vectors, one of which is a unit vector.

6. On the basis of your answers for Exercise 5, interpret the numbers $|c|/\|\vec{m}\|$ and $c/\|\vec{m}\|$. [*Hint*: Recall that $\vec{c} = C - O$ and that C is the intersection of the plane described by (*) with the line through O which is perpendicular to this plane.]

*

In the preceding exercises you have seen that the results concerning planes and lines which were obtained in Sections 10.09 and 10.10 can, in the case of orthonormal coordinates, be derived by using Theorems 13-1 and 13-2. And you have seen that, in this case, you can tell quite a bit about a plane from a single equation for it. Specifically, for $(m_1, m_2, m_3) \neq (0, 0, 0)$, the equation:

$$(x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0$$

describes a plane which contains the point with coordinates (a_1, a_2, a_3) and has for its normals the lines whose direction numbers are (m_1, m_2, m_3). And the equation:

$$x_1m_1 + x_2m_2 + x_3m_3 = c$$

describes a plane with the same normals which is at the distance

$$|c|/\sqrt{m_1^2 + m_2^2 + m_3^2}$$

from the origin of the coordinate system. [There are two such planes and, by Exercise 2 of Part E, the one which is described by the equation can be determined by observing the sense of $\overrightarrow{mc}$.]

As is summarized in Part F of the following exercises, descriptions of planes and of lines by equations show very easily whether two planes, two lines, or a plane and a line are parallel and whether they are perpendicular.

Part F

Consider planes π and σ and lines k and l which are described by:

$$x_1m_1 + x_2m_2 + x_3m_3 = c \quad [\text{for } \pi],$$
$$x_1n_1 + x_2n_2 + x_3n_3 = d \quad [\text{for } \sigma],$$
$$\begin{cases} x_1 = a_1 + r_1t \\ x_2 = a_2 + r_2t \ [\text{for } k], \\ x_3 = a_3 + r_3t \end{cases} \qquad \begin{cases} x_1 = b_1 + s_1t \\ x_2 = b_2 + s_2t \ [\text{for } l] \\ x_3 = b_3 + s_3t \end{cases}$$

[It follows, of course, that $\vec{m}$, $\vec{n}$, $\vec{r}$, and $\vec{s}$ are non-$\vec{0}$, where $\vec{m}$ is the vector whose components are (m_1, m_2, m_3), etc. Explain.]

Justify each of the following criteria for parallelism or perpendicularity and tell which of them remain valid in case the coordinate system is not orthonormal. [Don't make a big deal of this.]

1. $\pi \parallel \sigma \longleftrightarrow \left(\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}\right) = (0, 0, 0)$
2. $k \parallel l \longleftrightarrow \left(\begin{vmatrix} r_2 & r_3 \\ s_2 & s_3 \end{vmatrix}, \begin{vmatrix} r_3 & r_1 \\ s_3 & s_1 \end{vmatrix}, \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix}\right) = (0, 0, 0)$
3. $k \parallel \pi \longleftrightarrow r_1m_1 + r_2m_2 + r_3m_3 = 0$
4. $k \perp l \longleftrightarrow r_1s_1 + r_2s_2 + r_3s_3 = 0$
5. $\pi \perp \sigma \longleftrightarrow m_1n_1 + m_2n_2 + m_3n_3 = 0$
6. $k \perp \pi \longleftrightarrow \left(\begin{vmatrix} r_2 & r_3 \\ m_2 & m_3 \end{vmatrix}, \begin{vmatrix} r_3 & r_1 \\ m_3 & m_1 \end{vmatrix}, \begin{vmatrix} r_1 & r_2 \\ m_1 & m_2 \end{vmatrix}\right) = (0, 0, 0)$
7. As you probably recalled, the criterion given in Exercise 3 is valid whether or not the coordinate system is orthonormal. In case the coordinate system is orthonormal there is a quick justification of this criterion based on Theorem 13-1 and a theorem from Chapter 12. What theorem?

Part G

Given any origin $O \in \mathscr{E}$ and any basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ for $\mathscr{T}$, the resulting coordinate system has $\overleftrightarrow{O[\vec{u}_1]}$ as its first coordinate axis and $\overline{O[\vec{u}_2, \vec{u}_3]}$ as its first coordinate plane. Since the coordinates of O are $(0, 0, 0)$ and the components of $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are, respectively, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, the first coordinate axis can be described by the parametric equations:

$$x_1 = t, x_2 = 0, x_3 = 0$$

and the first coordinate plane can be described by the parametric equations:

$$x_1 = 0, x_2 = r, x_3 = s$$

[Explain.] What single equation describes the first coordinate plane? [The answer should be obvious, but, for practice, check by using Theorem 10-15.]

1. Give parametric equations for (a) the second and (b) the third coordinate axis.
2. Give a single equation for (a) the second and (b) the third coordinate plane.

Answers for Part F

[If any of $\vec{m}$, $\vec{n}$, $\vec{r}$, and $\vec{s}$ is $\vec{0}$, then the corresponding set is not, as given, a plane or line.]

1. The criterion is judtified in general by Corollary 1 to Theorem 10-16 and Theorem 10-14. In the particular case of an orthonormal coordinate system the criterion is justified by Theorems 13-1 and 10-14 and the fact that planes perpendicular to parallel lines are parallel.
2. The criterion is justified, for any cartesian coordinate system, by Theorem 10-14.
3. The criterion is justified for any cartesian coordinate system, by Theorem 10-17. For an orthonormal coordinate system it is justified by Theorem 13-1, Theorem 11-12, and Theorem 12-12(a).
4. The criterion is justified by Theorem 11-12, but only for orthonormal coordinate systems.
5. The criterion is justified — but only for orthonormal coordinate systems — by Theorem 13-1, Exercise 4, and the fact that planes are perpendicular if and only if their normals are perpendicualr.
6. The criterion is valid, but only for orthonormal coordinate systems. Use Theorem 13-1 and Theorem 10-14. [A line is perpendicular to a plane if and only if it is parallel to a normal to the plane.]
7. Theorem 12-12(a). [See answer for Exercise 3.]

Answers for Part G

A single equation for the first coordinate plane is '$x_1 = 0$'. This may be obtained by simplifying the following equation:

$$(x_1 - 0)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + (x_2 - 0)\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + (x_3 - 0)\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

which results from the use of Theorem 10-15.

1. (a) $x_1 = 0$, $x_2 = r$, $x_3 = 0$ (b) $x_1 = 0$, $x_2 = 0$, $x_3 = r$
2. (a) $x_2 = 0$ (b) $x_3 = 0$

3. Consider the plane π described by:

$$x_1m_1 + x_2m_2 + x_3m_3 = c \qquad [(m_1, m_2, m_3) \neq (0, 0, 0)]$$

Use the results of Part F to determine what conditions (m_1, m_2, m_3) satisfy if and only if
(a) π is parallel to the first coordinate axis.
(b) π is parallel to the first coordinate plane.
(c) π is parallel to both the second and the third coordinate axes.
(d) π is perpendicular to the first coordinate plane. [In parts (d), (e), and (f), assume that the coordinate system is orthonormal.]
(e) π is perpendicular to the first coordinate axis.
(f) π is perpendicular to the second and third coordinate planes.

4. Draw six pictures of an orthonormal coordinate system and complete them to illustrate the six parts of Exercise 3.

5. You have seen that, with respect to an orthonormal coordinate system, equations of the form:

(i) $$x_2m_2 + x_3m_3 = c \qquad [(m_2, m_3) \neq (0, 0)]$$

describe planes perpendicular to the first coordinate plane [for short: *the* (x_2, x_3)*-plane*], and that equations of the form:

(ii) $$x_1 = d$$

describe planes perpendicular to the first coordinate axis [for short: *the* x_1*-axis*].
(a) Draw a picture showing two intersecting planes with equations like (i) and showing their line of intersection. How is their intersection related to the coordinate planes? To the coordinate axes? [*Hint*: To picture a plane, picture its intersections with the coordinate planes. When picturing two planes, use dashed lines to represent lines in one plane which are hidden by the other.]
(b) Repeat part (a) for a first plane with an equation like (i) and a second with an equation like (ii).

6. Consider the plane π whose equation with respect to a certain orthonormal coordinate system is:

$$3x_2 + 2x_3 = 6$$

(a) Draw a picture showing the coordinate axes and the lines in which π intersects the coordinate planes. [These lines are called *the traces* of π in the coordinate planes.]
(b) On your picture locate the point P with coordinates $(0, 4/3, 1)$ and draw the line l through this point parallel to the x_1-axis.

Answers for Part G [cont.]

3. (a) $m_1 = 0$. [The first coordinate axis has direction numbers $(1, 0, 0)$. Using Exercise 3 of Part F, the condition (m_1, m_2, m_3) must satisfy is: $1 \cdot m_1 + 0 \cdot m_2 + 0 \cdot m_3 = 0$.]
(b) $m_2 = 0$ and $m_3 = 0$ [By Exercise 1 of Part F, the condition which must be satisfied is:

$$\left(\begin{vmatrix} m_2 & m_3 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ 1 & 0 \end{vmatrix}\right) = (0, 0, 0)]$$

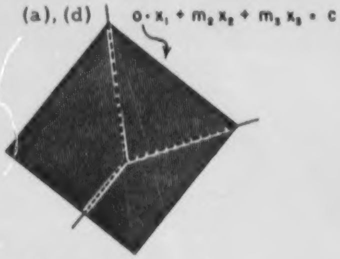
(c) $m_2 = 0$ and $m_3 = 0$. [For π to be parallel to the second coordinate axis, $m_2 = 0$. For π to be parallel to the third coordinate axis, $m_3 = 0$. Thus, for π to be parallel to both the second and third coordinate axes, $m_2 = 0$ and $m_3 = 0$.]
(d) $m_1 = 0$ [By Exercise 5 in Part F, $m_1 \cdot 1 + m_2 \cdot 0 + m_3 \cdot 0 = 0$.]
(e) $m_2 = m_3 = 0$ [By Exercise 6 in Part F, the condition that must be satisfied is:

$$\left(\begin{vmatrix} 0 & 0 \\ m_2 & m_3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ m_3 & m_1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ m_1 & m_2 \end{vmatrix}\right) = (0, 0, 0)$$

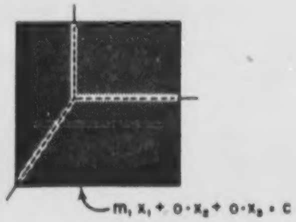
Hence, $(0, m_3, m_2) = (0, 0, 0)$.]
(f) $m_2 = m_3 = 0$ [By Exercise 5 in Part F, π is perpendicular to the second coordinate plane when $0 \cdot m_1 + 1 \cdot m_2 + 0 \cdot m_3 = 0$—that is, when $m_2 = 0$. Similarly, π is perpendicular to the third coordinate plane when $m_3 = 0$. So, π is perpendicular to both the second and third coordinate planes when $m_2 = 0 = m_3$.]

4. Here are appropriate pictures for the parts of Exercise 3:

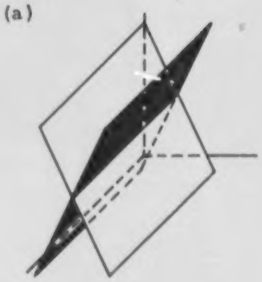
(a), (d)



(b), (c)
(e), (f)



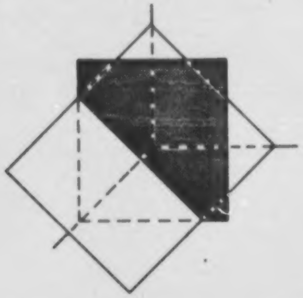
5. (a)



The intersection of the given planes is perpendicular to the first coordinate plane and parallel to both the second and third coordinate planes. It is parallel to the first coordinate axis and is perpendicular to both the second and third coordinate axes.

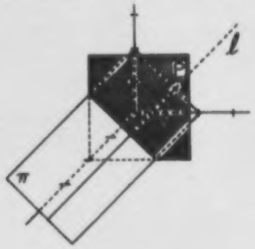
Answers for Part G [cont.]

5 (b)



The intersection of the given planes is parallel to the first coordinate plane. It is perpendicular to the first coordinate axis.

6. Here is a picture for parts (a), (b), and (c).



(b) $P \in \pi$, since $3 \cdot \frac{4}{3} + 2 \cdot 1 = 6$. Each point of ℓ belongs to π.

7. (a) third coordinate plane (b) second coordinate plane
(c) first and third (d) first and second
(e) first (f) third

Answers for Part H

Since $[\pi] = [m]^{\perp}$, if we can compute the bidirection $[\pi]$ in terms of the orthonormal components of $\vec{m}$, we will have calculated $[\vec{m}]^{\perp}$.

The value of 'c' is irrelevant since m_1, m_2, and m_3 are the direction numbers of all lines which are normals to each plane parallel to π; and each value of 'c' determines one of these planes.

Any two linearly independent members of $[\pi]$ provide us with a basis for $[\pi]$.

1. Parametric equations for π are:

$$\begin{cases} x_1 = \frac{5}{2} + \frac{3}{2}s - 2t \\ x_2 = s \\ x_3 = t \end{cases}$$

Components of $\vec{p}$ and $\vec{q}$ such that $[\pi] = [\vec{p}, \vec{q}]$ are $(3/2, 1, 0)$ and $(-2, 0, 1)$, respectively.

Does the point P belong to π? Is there any point on l which does not belong to π?

(c) Represent on your picture the plane whose equation is '$x_1 = 3$', and draw the line of intersection of this plane with π.

7. Each of the following equations describes a plane which is perpendicular to at least one of the coordinate planes of some orthonormal coordinate system. Which coordinate planes?
(a) $x_1 - 5x_2 = 3$ (b) $x_1 - x_3 = 0$ (c) $x_2 = 5$
(d) $x_3 = 0$ (e) $2x_2 + 4x_3 = 7$ (f) $x_1 + 7x_2 = 15$

*Part H

Theorem 13-2 gives an easy way of calculating $[\vec{p}, \vec{q}]^{\perp}$ in case $(\vec{p}, \vec{q})$ is linearly independent and one knows the components of $\vec{p}$ and $\vec{q}$ with respect to some orthonormal basis. It is sometimes convenient to be able to calculate $[\vec{m}]^{\perp}$ in case $\vec{m} \neq \vec{0}$ in terms of given orthonormal components of $\vec{m}$. According to Theorem 13-1, we can do this if we are able to compute the bidirection $[\pi]$ of the plane π described by the equation:

$$(*) \qquad x_1m_1 + x_2m_2 + x_3m_3 = c \qquad [(m_1, m_2, m_3) \neq (0, 0, 0)]$$

[Explain.] To compute $[\pi]$ means, in this case, to find a way of describing the components of all members of $[\pi]$ in terms of the numbers m_1, m_2, m_3, and c which are given to us by (*). [We can expect that the value of 'c' is irrelevant to this problem. Why?] This we can do if we can find the components of two linearly independent members of $[\pi]$. [Explain.]

1. In case (*) is such that, say, $m_1 \neq 0$ you already know how to find the components of vectors $\vec{p}$ and $\vec{q}$ such that $[\pi] = [\vec{p}, \vec{q}]$. Do so in case the equation for π is '$2x_1 - 3x_2 + 4x_3 = 5$'. [*Hint*: Find parametric equations for π.]

*

If you followed the hint for Exercise 1, you may have found that $[\pi] = [\vec{p}, \vec{q}]$ where the components of $\vec{p}$ and $\vec{q}$ are $(\frac{3}{2}, 1, 0)$ and $(-2, 0, 1)$. [To obtain this result, solve the given equation for 'x_1' and obtain parametric equations for π with 'x_2' and 'x_3' as parameters, by adjoining the trivial equations '$x_2 = x_2$' and '$x_3 = x_3$'.] The same method works for any equation like (*) in case $m_1 \neq 0$; and similar methods work in case $m_2 \neq 0$ and in case $m_3 \neq 0$. So, the problem of computing $[\pi]$ when π is given by an equation like (*) can always be solved. There are, however, advantages to having a solution of this problem which does not require us to decide among three cases.

*

2. If π is described by (*) then we know that $\vec{r} \in [\pi]$ if and only if its components (r_1, r_2, r_3) satisfy the equation:

$$(**) \qquad r_1m_1 + r_2m_2 + r_3m_3 = 0$$

   The simplest—but useless—solution of this equation is (0, 0, 0). Another solution which is almost as simple is $(0, m_3, -m_2)$. Give two other solutions of (**) which are as simple as this one.

3. Each of the three nontrivial solutions of (**) from Exercise 2 gives the components of a vector in $[\pi]$. Why must these three vectors be linearly dependent?

4. Let $\vec{p}_1, \vec{p}_2$, and $\vec{p}_3$ be the vectors whose components are $(0, m_3, -m_2)$, $(-m_3, 0, m_1)$ and $(m_2, -m_1, 0)$. Use determinants to show that the sequence $(\vec{p}_1, \vec{p}_2, \vec{p}_3)$ is linearly dependent but [since $\vec{m} \neq \vec{0}$] has a 2-termed linearly independent subsequence.

5. Show that, for $\vec{m} \neq \vec{0}$, (r_1, r_2, r_3) satisfies (**) if and only if there are numbers r, s, and t such that

$$\begin{cases} r_1 = - m_3s + m_2t \\ r_2 = m_3r - m_1t \\ r_3 = -m_2r + m_1s \end{cases}$$

6. Prove:

> **Theorem 13-3** If, with respect to an orthonormal basis for $\mathscr{T}$, the components of the non-$\vec{0}$ vector $\vec{m}$ are (m_1, m_2, m_3) then $[\vec{m}]^\perp = [\vec{p}_1, \vec{p}_2, \vec{p}_3]$, where the components of $\vec{p}_1, \vec{p}_2$, and $\vec{p}_3$ are $(0, m_3, -m_2)$, $(-m_3, 0, m_1)$, and $(m_2, -m_1, 0)$, respectively.

7. In each of the following you are given orthonormal components of a non-$\vec{0}$ vector $\vec{m}$. Use Theorem 13-3 to find two vectors $\vec{p}$ and $\vec{q}$ such that $[\vec{m}]^\perp = [\vec{p}, \vec{q}]$.

   (a) (−2, 3, 5) (b) (1, 6, 0) (c) (4, 0, 0)

13.02 Lines and Planes in Orthonormal Coordinates

[Although many of the results in this section hold without the assumption that the coordinate systems referred to are orthonormal, we shall make this assumption throughout the section. You should, however, try to be aware of the places where it is not needed.]

Given a coordinate system, we have two ways of describing a given plane—we can describe it either by giving parametric equations or by

---

Answers for Part H [cont.]

2. Two other simple solutions of (**) are $(m_3, 0, -m_1)$ and $(m_2, -m_1, 0)$.

3. Any three members of $[\pi]$ are linearly dependent since $[\pi]$ is a two-dimensional vector space.

4. $$\begin{vmatrix} 0 & m_3 & -m_2 \\ -m_3 & 0 & m_1 \\ m_2 & -m_1 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} 0 & m_1 \\ -m_1 & 0 \end{vmatrix} + m_3 \cdot \begin{vmatrix} m_1 & -m_3 \\ 0 & m_2 \end{vmatrix} - m_2 \cdot \begin{vmatrix} -m_3 & 0 \\ m_2 & -m_1 \end{vmatrix}$$

$$= m_3(m_1m_2) - m_2(m_3m_1) = 0$$

   Since this determinant is 0, the sequence $(\vec{p}_1, \vec{p}_2, \vec{p}_3)$ is linearly dependent.

   Consider the subsequences $(\vec{p}_1, \vec{p}_2)$, $(\vec{p}_2, \vec{p}_3)$, and $(\vec{p}_3, \vec{p}_1)$ of the sequence $(\vec{p}_1, \vec{p}_2, \vec{p}_3)$. The determinant triples associated with these subsequences are

$$\left(\begin{vmatrix} m_3 & -m_2 \\ 0 & m_1 \end{vmatrix}, \begin{vmatrix} -m_2 & 0 \\ m_1 & -m_3 \end{vmatrix}, \begin{vmatrix} 0 & m_3 \\ -m_3 & 0 \end{vmatrix}\right) = (m_3m_1, m_2m_3, m_3^2)$$

$$\left(\begin{vmatrix} 0 & m_1 \\ -m_1 & 0 \end{vmatrix}, \begin{vmatrix} m_1 & -m_3 \\ 0 & m_2 \end{vmatrix}, \begin{vmatrix} -m_3 & 0 \\ m_2 & -m_1 \end{vmatrix}\right) = (m_1^2, m_1m_2, m_3m_1)$$

$$\text{and} \left(\begin{vmatrix} -m_1 & 0 \\ m_3 & -m_2 \end{vmatrix}, \begin{vmatrix} 0 & m_2 \\ -m_2 & 0 \end{vmatrix}, \begin{vmatrix} m_2 & -m_1 \\ 0 & m_3 \end{vmatrix}\right) = (m_1m_2, m_2^2, m_2m_3).$$

   If each of these is (0, 0, 0) then $m_3^2 = m_1^2 = m_2^2 = 0$ and, so, $\vec{m} = \vec{0}$. Since $\vec{m} \neq \vec{0}$ it follows that at least one of these determinent triples is not (0, 0, 0) and, so, that one of the two-termed subsequences of $(\vec{p}_1, \vec{p}_2, \vec{p}_3)$ is linearly independent.

5. (r_1, r_2, r_3) satisfies (**) if and only if the vector with these components is in $[\vec{p}_1, \vec{p}_2, \vec{p}_3]$. But, this is the case if and only if there are numbers r, s, and t such that $(r_1, r_2, r_3) = (-m_3s + m_2t,\ m_3r - m_1t,\ -m_2r + m_1s)$.

6. Suppose that $\vec{m}$ is a non-$\vec{0}$ vector with orthonormal components (m_1, m_2, m_3). A vector $\vec{r} \in [\vec{m}]^\perp$ if and only if its components (r_1, r_2, r_3) satisfy (**). By Exercise 5, the latter is the case if and only if $\vec{r} \in [\vec{p}_1, \vec{p}_2, \vec{p}_3]$ where $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ have the components $(0, m_3, -m_2)$, $(-m_3, 0, m_1)$, and $(m_2, -m_1, 0)$, respectively. So, $[\vec{m}]^\perp = [\vec{p}_1, \vec{p}_2, \vec{p}_3]$.

7. (a) Let $\vec{p}$ and $\vec{q}$ have components (−5, 0, −2) and (3, 2, 0), respectively. [These correspond to $\vec{p}_2$ and $\vec{p}_3$, respectively. From the solution to Exercise 4, we know that when $m_1 \neq 0$, $(\vec{p}_2, \vec{p}_3)$ is linearly independent.]

   (b) Let $\vec{p}$ and $\vec{q}$ have components (0, 0, 1) and (6, −1, 0), respectively.

   (c) Let $\vec{p}$ and $\vec{q}$ have components (0, 0, 4) and (0, −4, 0), respectively.

a single equation. You should by now have no difficulty either in passing from one kind of description to the other or in determining whether or not two such descriptions describe the same plane. Similarly, there are two ways of describing a given line—we can describe it by parametric equations or by a system of two equations, each of which describes one of two nonparallel planes through the given line.

Exercises

Part A

1. (a) What, by definition, is the set $\overleftrightarrow{A[\vec{r}]}$? Under what conditions is this set a line?

   (b) Under what conditions are $\overleftrightarrow{A[\vec{r}]}$ and $\overleftrightarrow{B[\vec{s}]}$ nonparallel lines?

   (c) Under the assumption that $\overleftrightarrow{A[\vec{r}]}$ and $\overleftrightarrow{B[\vec{s}]}$ are skew lines, find the components of a non-$\vec{0}$ vector which is orthogonal to both $\vec{r}$ and $\vec{s}$. [*Hint*: Assume, as usual, that $\vec{r}$ and $\vec{s}$ have components (r_1, r_2, r_3) and (s_1, s_2, s_3), respectively.]

   (d) Write an equation for the plane which contains a given point C and is parallel to each of the lines of part (c). [*Hint*: Assume, as usual, that C has coordinates (c_1, c_2, c_3).]

*2. Consider the lines l and m described by:

$$\begin{cases} x_1 = 2 - t \\ x_2 = 3 + 2t \text{ [for } l\text{]} \\ x_3 = 1 + 3t \end{cases} \qquad \begin{cases} x_1 = -3 + t \\ x_2 = 3 - 3t \text{ [for } m\text{]} \\ x_3 = -1 - 2t \end{cases}$$

Find parametric equations for the line n which is perpendicular to both l and m and which intersects both l and m. [*Hint*: Recall that n is a subset of the plane which contains l and whose bidirection contains that of n. So, this plane intersects m at a point of n. (i) As in Exercise 1(c), use the given direction numbers of l and m to find direction numbers of n. (ii) Use these and the direction numbers of l to find direction numbers of a normal to the plane containing l and n. (iii) Use these and the coordinates of some point on l to find an equation of this plane. (iv) Use this and the equations of m to find the coordinates of a point on n. (v) Write parametric equations for n. (vi) Check to see that the line you have described is the one you were asked to find.]

Part B

Consider the system:

$$(*) \qquad \begin{cases} x_1m_1 + x_2m_2 + x_3m_3 = c \\ x_1n_1 + x_2n_2 + x_3n_3 = d \end{cases}$$

Here are suggestions for using the exercises of section 13.02

(i) Parts A and B may be used in class to illustrate the statements in the second paragraph of the section.

(ii) Part C can be a homework assignment.

(iii) Part D and the discussion following should be presented under teacher direction. The exercises of Part E provide examples.

(iv) Part F can be a second homework assignment.

Answers for Part A

1. (a) $\overleftrightarrow{A[\vec{r}]} = \{X: X - A \in [\vec{r}]\}$ [Definition 7-5(a)]
   This set is a line if and only if $\vec{r} \neq \vec{0}$.

   (b) $\overleftrightarrow{A[\vec{r}]}$ and $\overleftrightarrow{B[\vec{s}]}$ are nonparallel lines if and only if $(\vec{r}, \vec{s})$ is linearly independent.

   (c) Let $\vec{t}$ be any non-$\vec{0}$ vector which is orthogonal to both $\vec{r}$ and $\vec{s}$, and assume that $\vec{t}$ has components (t_1, t_2, t_3). Then, since we are assuming that $\vec{t}$ is orthogonal to both $\vec{r}$ and $\vec{s}$, we have that

   $$t_1r_1 + t_2r_2 + t_3r_3 = 0$$

   and $$t_1s_1 + t_2s_2 + t_3s_3 = 0.$$

   So,

   $$(t_1, t_2, t_3) = \left(\begin{vmatrix} r_2 & r_3 \\ s_2 & s_3 \end{vmatrix} t,\ \begin{vmatrix} r_3 & r_1 \\ s_3 & s_1 \end{vmatrix} t,\ \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix} t\right), \text{ for some } t \neq 0.$$

   (d) Since the required plane is parallel to both $\overleftrightarrow{A[\vec{r}]}$ and $\overleftrightarrow{B[\vec{s}]}$, its bidirection is $[\vec{r}, \vec{s}]$ and the line described in part (c) is perpendicular to this plane. Thus, an equation for this plane is:

   $$(x_1 - c_1)\begin{vmatrix} r_2 & r_3 \\ s_2 & s_3 \end{vmatrix} + (x_2 - c_2)\begin{vmatrix} r_3 & r_1 \\ s_3 & s_1 \end{vmatrix} + (x_3 - c_3)\begin{vmatrix} r_1 & r_3 \\ s_1 & s_3 \end{vmatrix} = 0$$

2. (i) Direction numbers for ℓ are $(-1, 2, 3)$ and for m are $(1, -3, -2)$. So, if direction numbers for n are (n_1, n_2, n_3) then

   $$-n_1 + 2n_2 + 3n_3 = 0$$

   and $$n_1 - 3n_2 - 2n_3 = 0.$$

   So,

   $$(n_1, n_2, n_3) = \left(\begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix} t,\ \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} t,\ \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} t\right)$$

   $$= (5t, t, t), \text{ for some } t.$$

   Hence, n has direction numbers $(5, 1, 1)$.

   (ii) Since direction numbers for ℓ are $(-1, 2, 3)$ and for n are $(5, 1, 1)$, direction numbers for a normal to the plane containing ℓ and n are

   $$\left(\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix},\ \begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix},\ \begin{vmatrix} 5 & 1 \\ -1 & 2 \end{vmatrix}\right) = (1, -16, 11).$$

Answers for Part A [cont.]

2. (iii) Any plane whose normals have direction numbers (1, −16, 11) has an equation of the form '$x_1 - 16x_2 + 11x_3 = t$'. Since the plane in question contains ℓ and the point whose coordinates are (2, 3, 1) is in ℓ, $t = -35$. So, an equation for the required plane is:

$$x_1 - 16x_2 + 11x_3 = -35$$

(iv) Using the equation from (iii) and the parametric equations for m, we have that $(-3 + t) - 16(3 - 3t) + 11(-1 - 2t) = -35$. So, $t = 1$. Thus, the coordinates of the point of intersection of m with the plane containing ℓ and n are (−2, 0, −3).

(v) Since n contains the point of m whose coordinates are (−2, 0, −3) and has direction numbers (5, 1, 1), parametric equations for n are:

$$\begin{cases} x_1 = -2 + 5t \\ x_2 = t \\ x_3 = -3 + t \end{cases}$$

(vi) To check that the line described by these parametric equations is perpendicular to both ℓ and m, we note that the vectors whose components are (−1, 2, 3), (1, −3, −2), and (5, 1, 1) are non-$\vec{0}$ vectors in the directions of ℓ, m, and n, respectively, and that $-1 \cdot 5 + 2 \cdot 1 + 3 \cdot 1 = 0$ and $1 \cdot 5 + -3 \cdot 1 + -2 \cdot 1 = 0$. So, $\ell \perp n$ and $m \perp n$. The point common to m and n has coordinates (−2, 0, −3). The point common to ℓ and n is such that, for some s and t,

$$\begin{cases} 2 - t = -2 + 5s \\ 3 + 2t = s \\ 1 + 3t = -3 + s. \end{cases}$$

For this system, $t = -1$ and $s = 1$. So, the coordinates of the point common to ℓ and n are (3, 1, −2). [Note that in finding a solution of this last system one solves two of the three equations and then checks to see that their solution satisfies the remaining equation.]

Answers for Part B

1. (a) The system (*) describes a line when

$$\left(\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \right) \neq (0, 0, 0).$$

(b) The system (*) describes a line when $(\vec{m}, \vec{n})$ is linearly independent, where $\vec{m}$ and $\vec{n}$ are vectors whose components are (m_1, m_2, m_3) and (n_1, n_2, n_3), respectively.

2. The triple of determinants given in 1(a) give direction numbers of the line.

3. Suppose that X has coordinates (x_1, x_2, x_3) and that these coordinates satisfy (*). Then, for any a and b,

$$(x_1m_1 + x_2m_2 + x_3m_3)a = ca$$

and

$$(x_1n_1 + x_2n_2 + x_3n_3)b = db$$

so that $(x_1m_1 + x_2m_2 + x_3m_3)a + (x_1n_1 + x_2n_2 + x_3n_3)b = ca + db$. But, this means that the coordinates of X satisfy (**). Hence, for any a and b, they satisfy (**).

4. $\vec{x} \cdot (\vec{m}a + \vec{n}b) = ca + db$

5. From Exercise 4, $\vec{p} = \vec{m}a + \vec{n}b$. To say that (*) describes a line amounts to saying that $(\vec{m}, \vec{n})$ is linearly independent. So, assuming this, $\vec{p} = \vec{0}$ if and only if a and b are both 0. Hence, (**) describes a plane if and only if a and b are not both 0.

6. Equation (**) will describe a plane containing C if and only if a and b are not both 0 and $(\vec{c} \cdot \vec{m} - c)a + (\vec{c} \cdot \vec{n} - d)b = 0$. An obvious nontrivial solution for this equation is obtained by taking $a = \vec{c} \cdot \vec{n} - d$ and $b = c - \vec{c} \cdot \vec{m}$. [The solution is nontrivial because C does not belong to ℓ.]

7. Any plane containing ℓ can be described by specifying one of its points which is not on ℓ. Once this is done an equation for the plane can be found as in Exercise 6.

and the equation:

$$(**)\quad (x_1m_1 + x_2m_2 + x_3m_3)a + (x_1n_1 + x_2n_2 + x_3n_3)b = ca + db$$

1. Under what condition does the system (*) describe a line? State this condition (a) in terms of determinants and (b) in terms of the vectors $\overrightarrow{m}$ and $\overrightarrow{n}$ whose components are (m_1, m_2, m_3) and (n_1, n_2, n_3), respectively.
2. Assuming that (*) describes a line, what is the significance of the determinants in your answer for Exercise 1?
3. Show that, for any values of '*a*' and '*b*', if the coordinates of a point *X* satisfy (*) then they also satisfy (**).
4. The system (*) can be rewritten conveniently as:

   $$\overrightarrow{x} \cdot \overrightarrow{m} = c, \qquad \overrightarrow{x} \cdot \overrightarrow{n} = d$$

   Rewrite the equation (**) in the form '$\overrightarrow{x} \cdot (\_\_\_\_) = \_\_\_\_$'.
5. Your answer for Exercise 4 should be an equation of the form '$\overrightarrow{x} \cdot \overrightarrow{p} = k$'. As you know, such an equation describes a plane if and only if $\overrightarrow{p} \neq \overrightarrow{0}$. Use your answers for Exercises 4 and 1(b) to show that if (*) describes a line then, for given values of '*a*' and '*b*', (**) describes a plane if and only if these given values are not both 0.
6. Assume that (*) describes a line *l*. Given the position vector $\overrightarrow{c}$ of any point *C* not on *l*, show how to find values of '*a*' and '*b*' such that (**) describes a plane containing *C*. [*Hint*: The equation (**) can be rewritten as '$(\overrightarrow{x} \cdot \overrightarrow{m} - c)a + (\overrightarrow{x} \cdot \overrightarrow{n} - d)b = 0$'.]
7. On the basis of the results you have obtained show that if (*) describes the line *l* then any plane π through *l* can be described by the equation obtained by choosing appropriate values for '*a*' and '*b*' in (**).

Part C

Let *l* be the line described by:

$$\begin{cases} 2x_1 - 3x_2 + x_3 = 8 \\ x_1 + 5x_2 - 2x_3 = -5 \end{cases}$$

1. Find direction numbers for *l*.
2. Find an equation for the plane through *l* which contains the origin.
3. Find an equation for the plane through *l* which is parallel to a line with direction numbers (2, −4, 3).
4. Find an equation for the plane π which contains *l* and the common perpendicular to *l* and the line *m* described by the equations:

   $$x_1 = 2 + t,\ x_2 = -1 + t,\ x_3 = 5 + 3t$$

Answers for Part C

1. By Exercise 2 of Part B, direction numbers of *l* can be found by evaluating appropriate determinants. The numbers obtained in this way are (1, 5, 13).
2. Using the procedure from Exercise 6 of Part B, we must choose a and b, not both 0, so that 8a - 5b = 0. Taking a = 5 and b = 8 will do. The resulting equation is '$18x_1 + 25x_2 - 11x_3 = 0$'.
3. In order that '$(2a + b)x_1 + (-3a + 5b)x_2 + (a - 2b)x_3 = 8a - 5b$' describe a plane parallel to a line with direction numbers (2, −4, 3) it is, by Exercise 3 of Part F on page 180, necessary and sufficient that $2(2a + b) - 4(-3a + 5b) + 3(a - 2b) = 0$ [and that $(a, b) \neq (0, 0)$]. Simplifying we obtain the equation '$19a - 24b = 0$' whose most obvious solution is given by '$a = 24$ and $b = 19$'. The required equation is, then '$67x_1 + 23x_2 - 14x_3 = 97$'. [There are other ways of obtaining an equation of the plane in question, but the purpose of these exercises is to explore the use of equations like (**) of Part B.]
4. Using the hint and the equation, mentioned in the solution of Exercise 3, for "the family of planes containing *l*", and the solution for Exercise 1, it follows that we need numbers a and b such that

   $$\begin{vmatrix} 2a + b & -3a + 5b & a - 2b \\ 1 & 5 & 13 \\ 1 & 1 & 3 \end{vmatrix} = 0.$$

   Since this equation reduces to '$a - 2b = 0$', a = 2 and b = 1 will do. Substituting in the equation for the family of planes we obtain '$5x_1 - x_2 = 11$'.

[*Hint*: A vector in the direction of a normal to π must be a linear combination of vectors in the directions of l and m. Recall that one way of saying that three vectors are linearly dependent is to say that a certain third order determinant is 0.]

*5. Find parametric equations for the line which is perpendicular to both l and m and intersects both lines.

6. Describe the line parallel to l which contains the point whose coordinates are (1, 2, 3)
   (a) by using Exercise 1, and
   (b) by using the original description of l to find a similar description of the desired line.

7. (a) Find an equation of the plane through l which is perpendicular to the plane described by:

$$2x_1 - 4x_2 + 3x_3 = 6$$

   (b) Compare your answers for part (a) and Exercise 3.

Part D

Consider the line l described by:

$$\begin{cases} x_1 = a_1 + r_1 t \\ x_2 = a_2 + r_2 t \\ x_3 = a_3 + r_3 t \end{cases} \qquad [\vec{r} \neq \vec{0}]$$

1. The given equations may be rewritten in vector form as:

$$\vec{x} - \vec{a} = \vec{r}t$$

[Explain.] Use this to show that $\vec{x}$ is the position vector of a point of l if and only if $(\vec{x} - \vec{a}, \vec{r})$ is linearly dependent.

2. Use Theorem 10–14 to express the linear dependence of $\vec{x} - \vec{a}$ and $\vec{r}$ in terms of determinants involving the components of these vectors.

3. From your answer for Exercise 2 you can obtain three equations, each of which is satisfied by the coordinates of any point on l. How many of these equations describe planes
   (i) if no component of $\vec{r}$ is zero?
   (ii) if just one component of $\vec{r}$ is zero?
   (iii) if just two components of $\vec{r}$ are zero?

4. (a) In each of the three cases of Exercise 3, tell how many different planes are described by the equations.
   (b) How are the planes in case (i) related to the coordinate planes?

*

Answers for Part C [cont.]

5. Proceeding as in Exercise 2 of Part A we begin by finding direction numbers of the line in question. These [which will have been found in solving Exercise 4] are

$$\left(\begin{vmatrix} 5 & 13 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 13 & 1 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} \right)$$

— that is, (2, 10, −4) or, more simply, (1, 5, −2). We next find a point of the line in question by finding the point of intersection of m and the plane of Exercise 4. This requires us to solve '$5(2 + t) - (-1 + t) = 11$' which we find is satisfied if $t = 0$. So, the point of intersection has coordinates (2, −1, 5). Hence, the common perpendicular is given by the parametric equations:

$$x_1 = 2 + t, \quad x_2 = -1 + 5t, \quad x_3 = 5 - 2t$$

As a check, this line intersects ℓ at the point at which $t = 4/15$ - that is, at the point whose coordinates are (34/15, 1/3, 67/15). [The value of 't' is most easily found by substituting from the parametric equations into each of the two equations which were given to describe ℓ. Finding that the same value of 't' satisfies both equations shows that the reputed common perpendicular does, indeed, intersect ℓ.]

6. (a) Parametric equations for the desired line are:

$$x_1 = 1 + t, \quad x_2 = 2 + 5t, \quad x_3 = 3 + 13t$$

   (b) The line in question is the intersection of planes which contain (1, 2, 3) and are parallel to those given to describe ℓ. So, the line in question is described by:

$$\begin{cases} 2x_1 - 3x_2 + x_3 = -1 \\ x_1 + 5x_2 - 2x_3 = 5 \end{cases}$$

7. (a) Using the equation of the family of planes through ℓ, which is referred to in the answer for Exercise 2, we see that $2(2a + b) - 4(-3a + 5b) + 3(a - 2b) = 0$ in the case of the plane through ℓ which is perpendicular to the one given in the present exercise. Solving, we find $a = 24$ and $b = 19$. Substituting in the equation of the family we obtain '$67x_1 + 23x_2 - 14x_3 = 97$' as an equation of the desired plane.

   (b) The equations describe the same plane. They should, since the plane through ℓ parallel to the normals of the given plane is the plane through ℓ perpendicular to the given plane.

In Part D you have seen how to obtain equations for at least two planes through any line which has been described by parametric equations. In fact, for $\overrightarrow{r} \neq \overrightarrow{0}$, the system of parametric equations:

$$(1) \qquad \begin{cases} x_1 = a_1 + r_1 t \\ x_2 = a_2 + r_2 t \\ x_3 = a_3 + r_3 t \end{cases}$$

is equivalent to the system:

$$(2) \qquad \begin{cases} (x_2 - a_2)r_3 = (x_3 - a_3)r_2 \\ (x_3 - a_3)r_1 = (x_1 - a_1)r_3 \\ (x_1 - a_1)r_2 = (x_2 - a_2)r_1 \end{cases}$$

In case none of r_1, r_2, and r_3 is zero, each of these equations describes a plane which contains l and is perpendicular to one of the three coordinate planes. In case r_3, say, is 0—but $r_1 \neq 0 \neq r_2$—the first two equations describe the same plane. [What is the simplest equation for this plane, and what can you say about l in this case?] In case, say, $r_1 = r_2 = 0$—but $r_3 \neq 0$—only two of the three equations describe planes. [What are the simplest equations for these planes, and what can you say about l?] Whenever one of the equations in (2) describes a plane it is the plane which contains l and is perpendicular to the first, second, or third coordinate plane, respectively. These planes are called *projecting planes* for l—the (x_2, x_3)-projecting plane, the (x_3, x_1)-projecting plane, and the (x_1, x_2)-projecting plane, respectively. In case l is perpendicular to a given one of the coordinate planes then so is any plane which contains l, and none of these planes is called a projecting plane for l into the given coordinate plane. [However, two of them are the projecting planes for l into the other two coordinate planes.]

In case none of the components of $\overrightarrow{r}$ is 0, system (2) can be summarized in the form:

$$(3) \qquad \frac{x_1 - a_1}{r_1} = \frac{x_2 - a_2}{r_2} = \frac{x_3 - a_3}{r_3}$$

If, say, $r_1 = 0$, but $r_2 \neq 0 \neq r_3$, (2) is equivalent to:

$$x_1 = a_1, \frac{x_2 - a_2}{r_2} = \frac{x_3 - a_3}{r_3};$$

if $r_1 = r_2 = 0$ and $r_3 \neq 0$, the system is equivalent to:

TC 121 (2)

Answers for Part D

1. [A first vector is the same as a second vector if and only if it has the same components as the second vector.] Since $\vec{r} \neq \vec{0}$, $(\vec{x} - \vec{a}, \vec{r})$ is linearly dependent if and only if $\vec{x} - \vec{a} \in [\vec{r}]$.
2. By Theorem 10-14, $(\vec{x} - \vec{a}, \vec{r})$ is linearly dependent if and only if

$$\left(\begin{vmatrix} x_2 - a_2 & x_3 - a_3 \\ r_2 & r_3 \end{vmatrix}, \begin{vmatrix} x_3 - a_3 & x_1 - a_1 \\ r_3 & r_1 \end{vmatrix}, \begin{vmatrix} x_1 - a_1 & x_2 - a_2 \\ r_1 & r_2 \end{vmatrix} \right) = (0, 0, 0).$$

3. The three equations referred to are:

$$\begin{aligned} (x_2 - a_2)r_3 - (x_3 - a_3)r_2 &= 0 \\ (x_3 - a_3)r_1 - (x_1 - a_1)r_3 &= 0 \\ (x_1 - a_1)r_2 - (x_2 - a_2)r_1 &= 0 \end{aligned}$$

   (i) All three equations refer to planes if no component of $\vec{r}$ is zero.

   (ii) All three equations refer to planes if just one component of $\vec{r}$ is zero.

   (iii) Exactly two of the equations refer to planes if just two components of $\vec{r}$ are zero.

4. (a) In case (i) there are three planes; in cases (ii) and (iii) there are two.

   (b) They are perpendicular to the first, second, and third coordinate planes, respectively.

TC 122

In case $r_3 = 0$ and $r_1 \neq 0 \neq r_2$ each of the first two equations in (2) describes a plane whose simplest equation is '$x_3 = a_3$'. In this case l is parallel to the third coordinate plane and, so, is perpendicular to the third coordinate axis.

In case $r_1 = r_2 = 0$ and $r_3 \neq 0$ the first two equations describe planes whose simplest equations are '$x_2 = a_2$' and '$x_1 = a_1$', respectively. In this case l is perpendicular to the third coordinate plane and parallel to the third coordinate axis.

$$x_1 = a_1, x_2 = a_2$$

It is customary in such cases to continue to use the form (3) with one or two of the denominators 'r_1', 'r_2', and 'r_3' replaced by '0'. So, for example, the "system":

$$\frac{x_1 - 1}{2} = \frac{x_2 - 2}{0} = \frac{x_3 - 3}{-1}$$

is to be interpreted as:

$$\frac{x_1 - 1}{2} = \frac{x_3 - 3}{-1}, x_2 - 2 = 0$$

and, so, describes the line with direction numbers (2, 0, −1) which contains the point with coordinates (1, 2, 3). With this convention any line can be described by equations like (3). Such equations are said to be *in symmetric form*.

It is easy to pass from parametric equations like (1) to equations in symmetric form like (3), and equally easy to go in the reverse direction. And, from equations in symmetric form it is easy to find equations of projecting planes. In the following exercises you will see how, when a line is described by giving the equations of two planes through it, one can easily find equations of the projecting planes and, so, describe the line by equations like (1) or (3).

Part E

Consider the line l described by:

$$\begin{cases} 3x_1 - 2x_2 + 4x_3 = 5 \\ 2x_1 + 2x_2 - x_3 = 4 \end{cases}$$

Recall that any plane through l can be described by choosing appropriate values for 'a' and 'b' in:

$$(*) \qquad (3a + 2b)x_1 - (2a - 2b)x_2 + (4a - b)x_3 = 5a + 4b$$

1. Find values for 'a' and 'b' in (*) which will give an equation for the (x_3, x_1)-projecting plane for l.
2. Find equations for the (x_1, x_2)-projecting plane and for the (x_2, x_3)-projecting plane for l.
3. Could you have found the projecting planes for l without writing down (*)? [*Hint*: For any projecting plane, you wish to "eliminate one of the variables" from the given equations for l.]

Answers for Part E

1. To obtain an equation for the (x_3, x_1)-projecting plane for l, values for 'a' and 'b' must be chosen so that $2a - 2b = 0$ — that is, so that $a = b$. Choosing $a = b = 1$, we obtain '$5x_1 + 3x_3 = 9$'.
2. To obtain an equation for the (x_1, x_2)-projecting plane, choose values for 'a' and 'b' such that $4a - b = 0$. Choosing $a = 1$ and $b = 4$, we obtain '$11x_1 + 6x_2 = 21$'.

   To obtain an equation for the (x_2, x_3)-projecting plane, choose values for 'a' and 'b' such that $3a + 2b = 0$. Choosing $a = -2$ and $b = 3$, we obtain '$10x_2 - 11x_3 = 2$'.
3. Yes. For example, to find an equation for the (x_1, x_2)-projecting plane "eliminate" 'x_3' from the given system. Multiplying on both sides of the second equation by 4 and "adding" the resulting equation to the first, we obtain:

   $$11x_1 + 6x_2 = 21$$

   In a similar manner, 'x_2' can be eliminated to yield an equation for the (x_1, x_3)-projecting plane and 'x_1' can be eliminated to yield an equation for the (x_2, x_3)-projecting plane.

TC 124 (1)

Answers for Part F

1. $5x_2 - 9x_3 = 13$ [Multiply by 3 on both sides of the second equation and subtract the first equation from the resulting equation.]
2. $5x_1 - x_3 = 17$; $9x_1 - x_2 = 28$
3. Solving '$5x_1 - x_3 = 17$' for 'x_1', we obtain '$x_1 = \frac{x_3 + 17}{5}$'. Solving '$9x_1 - x_2 = 28$' for 'x_1', we obtain '$x_1 = \frac{x_2 + 28}{9}$'. So, an equation for m in symmetric form is:

   $$\frac{x_1 - 0}{1} = \frac{x_2 + 28}{9} = \frac{x_3 + 17}{5}$$
4. From Exercise 3, we see that (0, −28, −17) are the coordinates of a point of m. Making use of the given equations, we see that this is the case, for $3 \cdot 0 - 2 \cdot -28 + 3 \cdot -17 = 0 + 56 - 51 = 5$ and $0 + -28 - 2 \cdot -17 = -28 + 34 = 6$.

Part F

Consider the line m described by:

$$\begin{cases} 3x_1 - 2x_2 + 3x_3 = 5 \\ x_1 + x_2 - 2x_3 = 6 \end{cases}$$

1. Without writing an equation like (*) of Part E, find an equation for the (x_2, x_3)-projecting plane for m. [*Hint*: With a little practice, you can write down such equations "at sight". Try to do so now.]
2. Find equations for the other projecting planes for m.
3. Solve each of the equations you found in Exercise 2 for 'x_1' and use the results to write equations for m in symmetric form. [*Hint*: $x_1 = \dfrac{x_1 - 0}{1}$]
4. From your answer for Exercise 3 you can read off the coordinates of a point of m. Check this much of your answer by substituting in the given equations for m.
5. From your answer for Exercise 3 you can read off direction numbers for m. You can also use determinants to find direction numbers for m directly from the given equations. Do so as a further check on your answer for Exercise 3.
6. Repeat Exercise 3 by using equations of two other projecting planes for m.
7. Find a system of equations in symmetric form for each line described below.

   (a) $\begin{cases} x - x_2 + x_3 = 4 \\ 2x_1 - x_2 + 4x_3 = 11 \end{cases}$ (b) $\begin{cases} 3x_1 - 6x_2 - 3x_3 = -13 \\ 3x_1 \qquad\quad + 9x_3 = 1 \end{cases}$

   (c) $\begin{cases} x_1 + x_2 - 2x_3 = 3 \\ 5x_1 + 2x_2 + 5x_2 = -9 \end{cases}$ (d) $\begin{cases} 3x_1 + x_2 - x_3 = -1 \\ 6x_1 + x_2 - 2x_3 = 1 \end{cases}$
8. Which of the lines in Exercise 7 are parallel? Which are perpendicular?
9. Are any of the lines in Exercise 7 parallel or perpendicular to a coordinate plane or a coordinate axis?

13.03 Finding Orthonormal Coordinate Systems

Since, as we have seen, some computations with coordinates become simpler when the coordinates are orthonormal, it is worthwhile to find out how to obtain orthonormal bases for $\mathcal{T}$. In doing this we shall, incidentally, show that the intuitively obvious result that there are orthonormal bases for $\mathcal{T}$ can be deduced from our postulates. What we shall do is to show that, given any basis $(\vec{p}, \vec{q}, \vec{r})$ for $\mathcal{T}$, there is an orthonormal basis—say, $(\vec{i}, \vec{j}, \vec{k})$—such that

$$[\vec{i}] = [\vec{p}] \text{ and } [\vec{i}, \vec{j}] = [\vec{p}, \vec{q}].$$

Answers for Part F [cont.]

5. From Exercise 3, we see that direction numbers for m are (1, 9, 5). Making use of determinants and the given equations, we obtain the same direction numbers for m, since

$$\left(\begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix}, \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \right) = (1, 9, 5).$$

6. From Exercise 1, $x_2 = (9x_3 + 13)/5$ and from Exercise 2, $x_2 = 9x_1 - 28$. So:

$$\frac{x_1 - (28/9)}{1/9} = \frac{x_2 - 0}{1} = \frac{x_3 + (13/9)}{5/9}$$

are symmetric equations for m. [As a partial check note that the denominators in this are proportional to those in the answer for Exercise 3. The former may, of course, be replaced by the latter.]

7. (a) Using the equations '$x_1 + 3x_3 = 7$' and '$x_2 + 2x_3 = 3$, we have:

$$\frac{x_1 - 7}{-3} = \frac{x_2 - 3}{-2} = \frac{x_3 - 0}{1}$$

   (b) Using the equations '$6x_2 + 12x_3 = 14$' and '$12x_1 - 18x_2 = -38$', we have:

$$\frac{x_1 + 19/6}{9} = \frac{x_2 - 0}{6} = \frac{x_3 - 7/6}{-3}$$

   (c) Using the equations '$x_1 + 3x_3 = -5$' and '$x_2 - 5x_3 = 8$', we have:

$$\frac{x_1 + 5}{-3} = \frac{x_2 - 8}{5} = \frac{x_3 - 0}{1}$$

   (d) Using the equations '$3x_1 - x_3 = 2$' and '$x_2 = -3$', we have '$\dfrac{x_1 - 0}{1} = \dfrac{x_3 + 2}{3}$ and $x_2 + 3 = 0$', or '$\dfrac{x_1 - 0}{1} = \dfrac{x_2 + 3}{0} = \dfrac{x_3 + 2}{3}$'.

8. The lines of parts (a) and (b) are parallel and each is perpendicular to each of the lines of parts (c) and (d).

9. The line of part (d) is parallel to the second coordinate plane and, so, perpendicular to the second coordinate axis.

To begin with, the most obvious way of satisfying the first condition is to take

$$\vec{i} = \vec{p}/\|\vec{p}\|.$$

[What other choice is there?] Evidently, $[\vec{p}, \vec{q}] = [\vec{i}, \vec{q}]$. So, to satisfy the second condition we need to find a unit vector $\vec{j}$ which is a linear combination of $\vec{i}$ and $\vec{q}$ and is orthogonal to $\vec{i}$. We have seen earlier that if

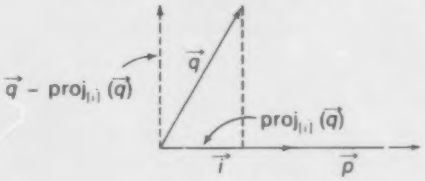

$\vec{q} - \text{proj}_{[\vec{i}]}(\vec{q})$
$\vec{q}$
$\text{proj}_{[\vec{i}]}(\vec{q})$
$\vec{i}$
$\vec{p}$

Fig. 13–2

$$\begin{aligned}\vec{q}_1 &= \vec{q} - \text{proj}_{[\vec{i}]}(\vec{q})\\ &= \vec{q} - \vec{i}(\vec{q} \cdot \vec{i})\end{aligned}$$

then $\vec{q}_1$ is a linear combination of $\vec{i}$ and $\vec{q}$ which is orthogonal to $\vec{i}$. [Check to make sure that $\vec{q}_1$ is orthogonal to $\vec{i}$.] Although $\vec{q}_1$ is not likely to be a unit vector, it is not $\vec{0}$. [Why?] So, with

$$\vec{j} = \vec{q}_1/\|\vec{q}_1\|,$$

we have orthogonal unit vectors $\vec{i}$ and $\vec{j}$ such that $[\vec{i}] = [\vec{p}]$ and $[\vec{i}, \vec{j}] = [\vec{p}, \vec{q}]$. [Explain. *Hint*: We have seen that $[\vec{i}, \vec{q}] = [\vec{p}, \vec{q}]$. Show that $[\vec{i}, \vec{q}_1] = [\vec{i}, \vec{q}]$ and that $[\vec{i}, \vec{j}] = [\vec{i}, \vec{q}_1]$.] All that remains is to find a unit vector $\vec{k}$ such that $\vec{k}$ is orthogonal to both $\vec{i}$ and $\vec{j}$. To do so we can copy the procedure used to find $\vec{j}$.

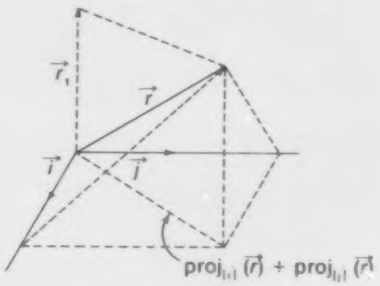

$\vec{r}_1$
$\vec{r}$
$\vec{i}$
$\vec{j}$
$\text{proj}_{[\vec{i}]}(\vec{r}) + \text{proj}_{[\vec{j}]}(\vec{r})$

Fig. 13–3

In addition to $\vec{i}$ the vector $-\vec{p}/\|\vec{p}\|$ is the only other unit vector in $[\vec{p}]$.

If $\vec{q}_1$ were $\vec{0}$ then $(\vec{i}, \vec{q})$ — and, so, $(\vec{p}, \vec{q})$ — would be linearly dependent.

Since $\vec{i} \in [\vec{p}]$ and $\vec{p} \in [\vec{i}]$, $[\vec{i}] = [\vec{p}]$ and $[\vec{i}, \vec{q}] = [\vec{p}, \vec{q}]$. Since by definition, $\vec{q}_1 \in [\vec{i}, \vec{q}]$ and $\vec{q} \in [\vec{i}, \vec{q}_1]$ it follows that $[\vec{i}, \vec{q}] = [\vec{i}, \vec{q}_1]$ and, so, $[\vec{i}, \vec{q}] = [\vec{p}, \vec{q}]$. Since, by definition, $\vec{j} \in [\vec{q}_1]$ and $\vec{q}_1 \in [\vec{j}]$ it follows that $[\vec{i}, \vec{j}] = [\vec{i}, \vec{q}_1]$. Hence, $[\vec{i}, \vec{j}] = [\vec{p}, \vec{q}]$.

In this section, only Part A is recommended for homework. Parts B and C should be treated under teacher direction, or may be used as extra-credit work.

Suppose that

$$\begin{aligned}\vec{r}_1 &= \vec{r} - \operatorname{proj}_{[\vec{i}]}(\vec{r}) - \operatorname{proj}_{[\vec{j}]}(\vec{r})\\ &= \vec{r} - \vec{i}(\vec{r}\cdot\vec{i}) - \vec{j}(\vec{r}\cdot\vec{j}).\end{aligned}$$

Using the fact that $\vec{i}$ and $\vec{j}$ are orthogonal unit vectors it is easy to show that $\vec{r}_1$ is orthogonal to both $\vec{i}$ and $\vec{j}$. [Do so.] Also, since $[\vec{p}, \vec{q}] = [\vec{i}, \vec{j}]$ and $\vec{r} \in [\vec{i}, \vec{j}, \vec{r}_1]$ it follows that $[\vec{p}, \vec{q}, \vec{r}] \subseteq [\vec{i}, \vec{j}, \vec{r}_1]$ and, so, that $(\vec{i}, \vec{j}, \vec{r}_1)$ is a basis for $\mathscr{T}$. Hence, with

$$\vec{k} = \vec{r}_1/\|\vec{r}_1\|,$$

$(\vec{i}, \vec{j}, \vec{k})$ is an orthonormal basis for $\mathscr{T}$.

The preceding is worth summarizing [and completing] in:

> **Theorem 13–4** If $(\vec{p}, \vec{q}, \vec{r})$ is any basis for $\mathscr{T}$ and
> $$\vec{i} = \vec{p}/\|\vec{p}\|,\quad \vec{j} = \vec{q}_1/\|\vec{q}_1\|,\quad \vec{k} = \vec{r}_1/\|\vec{r}_1\|,$$
> where
> $$\vec{q}_1 = \vec{q} - \vec{i}(\vec{q}\cdot\vec{i}) \text{ and } \vec{r}_1 = \vec{r} - \vec{i}(\vec{r}\cdot\vec{i}) - \vec{j}(\vec{r}\cdot\vec{j}),$$
> then $(\vec{i}, \vec{j}, \vec{k})$ is an orthonormal basis for $\mathscr{T}$ such that $[\vec{i}] = [\vec{p}]$ and $[\vec{i}, \vec{j}] = [\vec{p}, \vec{q}]$. Moreover, $\vec{p}\cdot\vec{i}$, $\vec{q}\cdot\vec{j}$, are $\vec{r}\cdot\vec{k}$ are positive.

Theorem 13–4 has a corollary which we state without proof:

> **Corollary**
> $$\|\vec{q}_1\|^2 = \begin{vmatrix}\vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q}\\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q}\end{vmatrix} / (\vec{p}\cdot\vec{p}), \text{ and}$$
> $$\|\vec{r}_1\|^2 = \begin{vmatrix}\vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{r}\\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{r}\\ \vec{r}\cdot\vec{p} & \vec{r}\cdot\vec{q} & \vec{r}\cdot\vec{r}\end{vmatrix} \Big/ \begin{vmatrix}\vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q}\\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q}\end{vmatrix}$$

The proof is a matter of straight-forward algebra but that of the second part is rather long. Our reason for calling attention to the corollary is that it can be used to obtain a new criterion for linear independence. To appreciate this result, you need to recall two others. The first is trivial:

(1) $\vec{p}\cdot\vec{p} \geq 0$ and is positive if and only if $\vec{p} \neq \vec{0}$.

The second has been proved in Chapter 11:

Proof of Corollary:
$$\begin{aligned}\vec{q}_1\cdot\vec{q}_1 &= [\vec{q} - \vec{i}(\vec{q}\cdot\vec{i})]\cdot[\vec{q} - \vec{i}(\vec{q}\cdot\vec{i})]\\ &= \vec{q}\cdot\vec{q} - (\vec{q}\cdot\vec{i})^2\\ &= \vec{q}\cdot\vec{q} - (\vec{q}\cdot\vec{p})^2/\|\vec{p}\|^2\\ &= [(\vec{q}\cdot\vec{q})(\vec{p}\cdot\vec{p}) - (\vec{q}\cdot\vec{p})^2]/(\vec{p}\cdot\vec{p})\end{aligned}$$

Note, in particular, that

$$(\vec{p}\cdot\vec{p})(\vec{q}_1\cdot\vec{q}_1) = \begin{vmatrix}\vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q}\\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q}\end{vmatrix}.$$

$$\begin{aligned}\vec{r}_1\cdot\vec{r}_1 &= [\vec{r} - \vec{i}(\vec{r}\cdot\vec{i}) - \vec{j}(\vec{r}\cdot\vec{j})]\cdot[\vec{r} - \vec{i}(\vec{r}\cdot\vec{i}) - \vec{j}(\vec{r}\cdot\vec{j})]\\ &= \vec{r}\cdot\vec{r} - (\vec{r}\cdot\vec{i})^2 - (\vec{r}\cdot\vec{j})^2\\ &= \vec{r}\cdot\vec{r} - (\vec{r}\cdot\vec{i})^2 - (\vec{r}\cdot\vec{q}_1)^2/(\vec{q}_1\cdot\vec{q}_1)\\ &= \vec{r}\cdot\vec{r} - (\vec{r}\cdot\vec{i})^2 - [\vec{r}\cdot\vec{q} - (\vec{r}\cdot\vec{i})(\vec{q}\cdot\vec{i})]^2/(\vec{q}_1\cdot\vec{q}_1)\\ &= \vec{r}\cdot\vec{r} - (\vec{r}\cdot\vec{p})^2/(\vec{p}\cdot\vec{p}) - [\vec{r}\cdot\vec{q} - (\vec{r}\cdot\vec{p})(\vec{q}\cdot\vec{p})/(\vec{p}\cdot\vec{p})]^2/(\vec{q}_1\cdot\vec{q}_1)\\ &= \{(\vec{r}\cdot\vec{r})(\vec{p}\cdot\vec{p})(\vec{q}_1\cdot\vec{q}_1) - (\vec{r}\cdot\vec{p})^2(\vec{q}_1\cdot\vec{q}_1) - [(\vec{r}\cdot\vec{q})(\vec{p}\cdot\vec{p})\\ &\quad - (\vec{r}\cdot\vec{p})(\vec{q}\cdot\vec{p})]^2/(\vec{p}\cdot\vec{p})\}/[(\vec{p}\cdot\vec{p})(\vec{q}_1\cdot\vec{q}_1)]\end{aligned}$$

The denominator reduces to the desired form and the numerator is equivalent to:

$$\begin{aligned}&\{[(\vec{r}\cdot\vec{r})(\vec{p}\cdot\vec{p}) - (\vec{r}\cdot\vec{p})^2][(\vec{q}\cdot\vec{q})(\vec{p}\cdot\vec{p}) - (\vec{q}\cdot\vec{p})^2] - [(\vec{r}\cdot\vec{q})(\vec{p}\cdot\vec{p})\\ &\quad - (\vec{r}\cdot\vec{p})(\vec{q}\cdot\vec{p})]^2\}/(\vec{p}\cdot\vec{p})\\ &= (\vec{r}\cdot\vec{r})[(\vec{q}\cdot\vec{q})(\vec{p}\cdot\vec{p}) - (\vec{q}\cdot\vec{p})^2] - (\vec{r}\cdot\vec{p})^2(\vec{q}\cdot\vec{q}) - (\vec{r}\cdot\vec{q})^2(\vec{p}\cdot\vec{p})\\ &\quad + 2(\vec{r}\cdot\vec{q})(\vec{r}\cdot\vec{p})(\vec{q}\cdot\vec{p})\\ &= (\vec{r}\cdot\vec{r})[(\vec{q}\cdot\vec{q})(\vec{p}\cdot\vec{p}) - (\vec{q}\cdot\vec{p})^2] - (\vec{r}\cdot\vec{q})[(\vec{r}\cdot\vec{q})(\vec{p}\cdot\vec{p}) - (\vec{r}\cdot\vec{p})(\vec{q}\cdot\vec{p})]\\ &\quad + (\vec{r}\cdot\vec{p})[(\vec{r}\cdot\vec{q})(\vec{q}\cdot\vec{p}) - (\vec{r}\cdot\vec{p})(\vec{q}\cdot\vec{q})]\\ &= \begin{vmatrix}\vec{r}\cdot\vec{r} & \vec{r}\cdot\vec{q} & \vec{r}\cdot\vec{p}\\ \vec{q}\cdot\vec{r} & \vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{p}\\ \vec{p}\cdot\vec{r} & \vec{p}\cdot\vec{q} & \vec{p}\cdot\vec{p}\end{vmatrix} = \begin{vmatrix}\vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q} & \vec{p}\cdot\vec{r}\\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{r}\\ \vec{r}\cdot\vec{p} & \vec{r}\cdot\vec{q} & \vec{r}\cdot\vec{r}\end{vmatrix}\end{aligned}$$

*

In the case of orthonormal coordinates the determinant just arrived at is the square of the determinant in (9) on page 108. Had we developed the procedure for multiplying determinants the result concerning (9) would give us an alternative proof for (3) in the orthonormal case. We know, however, that (3) is valid for any coordinate system and so, by its very form, is (9).

$$(2) \qquad \begin{vmatrix} \vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q} \\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q} \end{vmatrix} \geq 0 \text{ and is positive if and only if}$$

$$(\vec{p}, \vec{q}) \text{ is linearly independent}$$

This result can also be derived by using the first part of the corollary. In a similar manner, using the second part of the corollary, one can prove:

$$(3) \qquad \begin{vmatrix} \vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q} & \vec{p}\cdot\vec{r} \\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{r} \\ \vec{r}\cdot\vec{p} & \vec{r}\cdot\vec{q} & \vec{r}\cdot\vec{r} \end{vmatrix} \geq 0 \text{ and is positive if and only if}$$

$$(\vec{p}, \vec{q}, \vec{r}) \text{ is linearly independent.}$$

The corollary has another important use which we shall consider when we discuss distance in the next chapter. For, looking at Fig. 13-3, you should be able to see that if $\vec{r}$ is the position vector of a point R with respect to a point O in the plane $O\overline{[\vec{p}, \vec{q}]}$ then the distance between R and this plane is precisely $\|\vec{r}_1\|$. Similarly, $\|\vec{q}_1\|$ is the distance between a point Q, where $Q = O + \vec{q}$, and the line $O\overline{[\vec{p}]}$.

Part A

Suppose that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is an orthonormal basis for $\mathscr{T}$ and that, with respect to this basis, the components of $\vec{p}$, $\vec{q}$, and $\vec{r}$ are (1, 2, 2), (−1, 4, 1), and (2, 5, 7), respectively.

1. Show that $(\vec{p}, \vec{q}, \vec{r})$ is linearly independent. [*Hint*: You could use (3), but there is an easier way.]
2. Compute the components, with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$, of the vectors $\vec{i}, \vec{j}$, and $\vec{k}$ of Theorem 13-4. [*Hint*: First, compute the components of $\vec{i}$, noting that $\|\vec{p}\| = 3$. Use these to compute $\vec{q}\cdot\vec{i}$, $\vec{i}(\vec{q}\cdot\vec{i})$ and, then, the components of $\vec{q}_1$. Now, find the components of $\vec{j}$.]

☆Part B

Suppose that $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ and $(\vec{i}, \vec{j}, \vec{k})$ are two orthonormal bases [as, for example, in Part A]. Suppose that O and A are points of $\mathscr{E}$. One sometimes needs to be able to compute the coordinates (x, y, z) of a point X with respect to the coordinate system with origin A and basis $(\vec{i}, \vec{j}, \vec{k})$ when one knows the coordinates (x_1, x_2, x_3) of the same point with respect to the coordinate system with origin O and basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$. This can be done easily if one knows the coordinates (a_1, a_2, a_3) of A with respect to the first coordinate system and the components

Answers for Part A

1. $\begin{vmatrix} 1 & 2 & 2 \\ -1 & 4 & 1 \\ 2 & 5 & 7 \end{vmatrix} = 1\cdot 23 - 2\cdot -9 + 2\cdot -13 = 15 \neq 0$. So $(\vec{p}, \vec{q}, \vec{r})$ is linearly independent.
2. $\vec{i} = \vec{p}/\|\vec{p}\|$ and $\|\vec{p}\| = \sqrt{1 + 4 + 4} = 3$. So the components of $\vec{i}$ are $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, $\vec{j} = \vec{q}_1/\|\vec{q}_1\|$ where $\vec{q}_1 = \vec{q} - \vec{i}(\vec{q}\cdot\vec{i})$ $= \vec{q} - \vec{i}(-1\cdot\frac{1}{3} + 4\cdot\frac{2}{3} + 1\cdot\frac{2}{3}) = \vec{q} - \vec{i}3$. Therefore, the components of $\vec{q}_1$ are $(-1 - \frac{1}{3}\cdot 3,\ 4 - \frac{2}{3}\cdot 3,\ 1 - \frac{2}{3}\cdot 3)$ or $(-2, 2, -1)$. Since $\|\vec{q}_1\| = \sqrt{4 + 4 + 1} = 3$, it follows that the components of $\vec{j}$ are $(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. [As a check, note that $\vec{i}\cdot\vec{j} = \frac{1}{3}\cdot -\frac{2}{3} + \frac{2}{3}\cdot\frac{2}{3} + \frac{2}{3}\cdot -\frac{1}{3} = 0$.] $\vec{k} = \vec{r}_1/\|\vec{r}_1\|$ where $\vec{r}_1 = \vec{r} - \vec{i}(\vec{r}\cdot\vec{i}) - \vec{j}(\vec{r}\cdot\vec{j}) = \vec{r} - \vec{i}(26/3) + \vec{j}(1/3)$. So the components of $\vec{r}_1$ are $(-\frac{10}{9}, -\frac{5}{9}, \frac{10}{9})$]. Since $\|\vec{r}_1\| = \sqrt{100 + 25 + 100}/9 = \frac{5}{3}$, the components of $\vec{k}$ are $(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$. [As a check, one could compute $\vec{i}\cdot\vec{k}$ and $\vec{j}\cdot\vec{k}$ and note that both are 0.]

* * *

Since triples of numbers the sum of whose squares is a square may not be familiar, we give a list as an aid for making out additional exercises which "come out even": (1, 2, 2), (2, 4, 4), (2, 3, 6), (1, 4, 8), (3, 6, 6), (4, 4, 7), (2, 6, 9), (6, 6, 7), (4, 8, 8), (3, 4 12), (4, 6, 12), (2, 5, 14), (2, 10, 11), (5, 10, 10), (1 12, 12) (8, 9, 12), (2, 8, 16), (6, 12, 12), (8, 8, 14), (1, 6, 18), (6, 6, 17), (6, 10, 15).

Parts B and C deal with transformation of coordinates with respect to two orthonormal coordinate systems. Some students may find this interesting but we shall make no essential later use of the results. By doing these exercises [especially Part C] students may gain some confidence in using orthonormal coordinates.

(i_1, i_2, i_3), (j_1, j_2, j_3), and (k_1, k_2, k_3) of $\vec{i}$, $\vec{j}$, and $\vec{k}$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$. And, as we shall see, it is equally easy to compute the coordinates (x_1, x_2, x_3) when one is given the coordinates (x, y, z).

1. By assumption (x_1, x_2, x_3) are the components of $X - O$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$. With this in mind, what are (a_1, a_2, a_3)? What are (x, y, z)?
2. The components of a vector $\vec{r}$ with respect to the orthonormal coordinate system $(\vec{i}, \vec{j}, \vec{k})$ are $(\vec{r} \cdot \vec{i}, \vec{r} \cdot \vec{j}, \vec{r} \cdot \vec{k})$. [Explain.] Express the coordinates (x, y, z) of the point X as dot products.
3. The dot product of two vectors is easy to compute if you know the components of the vectors with respect to some orthonormal basis. [Explain.] Express the coordinates (x, y, z) in terms of the components of appropriate vectors with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$.

*

In the notation introduced in Part B, the coordinates of X with respect to the two orthonormal coordinate systems are related by:

$$(4) \qquad \begin{cases} x = (x_1 - a_1)i_1 + (x_2 - a_2)i_2 + (x_3 - a_3)i_3 \\ y = (x_1 - a_1)j_1 + (x_2 - a_2)j_2 + (x_3 - a_3)j_3 \\ z = (x_1 - a_1)k_1 + (x_2 - a_2)k_2 + (x_3 - a_3)k_3 \end{cases}$$

*

4. The components (i_1, i_2, i_3) of $\vec{i}$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ are $(\vec{i} \cdot \vec{u}_1, \vec{i} \cdot \vec{u}_2, \vec{i} \cdot \vec{u}_3)$. With this hint, show that the components of $\vec{u}_1$ with respect to the basis $(\vec{i}, \vec{j}, \vec{k})$ are (i_1, j_1, k_1).
5. Suppose that the coordinates of O with respect to the coordinate system with origin A and basis $(\vec{i}, \vec{j}, \vec{k})$ are (a, b, c). Use the results of Exercise 4 to write equations like (4) for computing (x_1, x_2, x_3) when (x, y, z) are known. [*Hint*: This should, now, be very easy.]
6. Use (4) to compute (a, b, c).

*

In view of Exercise 4 the coordinates of X with respect to the two orthonormal coordinate systems are related by:

$$(5) \qquad \begin{cases} x_1 = (x - a)i_1 + (y - b)j_1 + (z - c)k_1 \\ x_2 = (x - a)i_2 + (y - b)j_2 + (z - c)k_2 \\ x_3 = (x - a)i_3 + (y - b)j_3 + (z - c)k_3 \end{cases}$$

☆Part C

1. In Part A you were given an orthonormal basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ and computed the components with respect to this basis of the terms $\vec{i}$, $\vec{j}$, and $\vec{k}$ of another orthonormal basis. If your work was correct you found that these components are $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, $(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$, and $(-\frac{2}{3},$

Answers for Part B

1. (a_1, a_2, a_3) are the components of the vector $A - O$ with respect to the basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$, and (x, y, z) are the components of the vector $X - A$ with respect to $(\vec{i}, \vec{j}, \vec{k})$.
2. Recall that $\vec{r} = \vec{i}\,\mathrm{comp}_{\vec{i}}(\vec{r}) + \vec{j}\,\mathrm{comp}_{\vec{j}}(\vec{r}) + \vec{k}\,\mathrm{comp}_{\vec{k}}(\vec{r})$, so that $\vec{r} \cdot \vec{i} = \vec{i} \cdot \vec{i}\,\mathrm{comp}_{\vec{i}}(\vec{r}) + 0 + 0 = \mathrm{comp}_{\vec{i}}(\vec{r})$. Similarly one can show that $\vec{r} \cdot \vec{j} = \mathrm{comp}_{\vec{j}}(\vec{r})$ and $\vec{r} \cdot \vec{k} = \mathrm{comp}_{\vec{k}}(\vec{r})$. Therefore, the components of $\vec{r}$ with respect to $(\vec{i}, \vec{j}, \vec{k})$ are $(\vec{r} \cdot \vec{i}, \vec{r} \cdot \vec{j}, \vec{r} \cdot \vec{k})$. By similar reasoning, the components of $X - A$ with respect to $(\vec{i}, \vec{j}, \vec{k})$ are $((X - A) \cdot \vec{i}, (X - A) \cdot \vec{j}, (X - A) \cdot \vec{k})$, so that the coordinates of X with respect to the coordinate system with origin A, and basis $(\vec{i}, \vec{j}, \vec{k})$ are $((X - A) \cdot \vec{i}, (X - A) \cdot \vec{j}, (X - A) \cdot \vec{k})$.
3. [If the components of $\vec{r}$ and $\vec{s}$ with respect to some orthonormal basis are (r_1, r_2, r_3) and (s_1, s_2, s_3), respectively, then $\vec{r} \cdot \vec{s} = r_1s_1 + r_2s_2 + r_3s_3$.] By Exercise 2, $x = (X - A) \cdot \vec{i}$. Since the components of $X - A$, and $\vec{i}$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ are $((x_1 - a_1), (x_2 - a_2), (x_3 - a_3))$ and (i_1, i_2, i_3), respectively, it follows that $x = (x_1 - a_1)i_1 + (x_2 - a_2)i_2 + (x_3 - a_3)i_3$. Similarly, by considering the components of the vectors $\vec{j}$ and $\vec{k}$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$, one can show that $y = (x_1 - a_1)j_1 + (x_2 - a_2)j_2 + (x_3 - a_3)j_3$ and $z = (x_1 - a_1)k_1 + (x_2 - a_2)k_2 + (x_3 - a_3)k_3$. [See (4) on this page.]
4. The components of $\vec{u}_1$ with respect to $(\vec{i}, \vec{j}, \vec{k})$ are $(\vec{u}_1 \cdot \vec{i}, \vec{u}_1 \cdot \vec{j}, \vec{u}_1 \cdot \vec{k})$. By the hint, we see that $\vec{i} \cdot \vec{u}_1 = i_1$, i.e., $\vec{u}_1 \cdot \vec{i} = i_1$. Similarly, since the components (j_1, j_2, j_3) of $\vec{j}$ and (k_1, k_2, k_3) of $\vec{k}$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ are $(\vec{j} \cdot \vec{u}_1, \vec{j} \cdot \vec{u}_2, \vec{j} \cdot \vec{u}_3)$ and $(\vec{k} \cdot \vec{u}_1, \vec{k} \cdot \vec{u}_2, \vec{k} \cdot \vec{u}_3)$, respectively, we have that $\vec{u}_1 \cdot \vec{j} = \vec{j} \cdot \vec{u}_1 = j_1$, and $\vec{u}_1 \cdot \vec{k} = \vec{k} \cdot \vec{u}_1 = k_1$. Therefore, the components of $\vec{u}_1$ with respect to $(\vec{i}, \vec{j}, \vec{k})$ are (i_1, j_1, k_1). [By looking at the second terms of the appropriate ordered triples, one gets the components of $\vec{u}_2$ with respect to $(\vec{i}, \vec{j}, \vec{k})$, and by looking at the third terms, one gets the components of $\vec{u}_3$
5. By Exercise 4, the components of $\vec{u}_1$ with respect to $(\vec{i}, \vec{j}, \vec{k})$ are (i_1, j_1, k_1). Similarly one could show that the components of $\vec{u}_2$ and $\vec{u}_3$ with respect to $(\vec{i}, \vec{j}, \vec{k})$ are (i_2, j_2, k_2) and (i_3, j_3, k_3), respectively. If the coordinates of O with respect to $(\vec{i}, \vec{j}, \vec{k})$ are (a, b, c), then by precisely the same type of reasoning we used in Exercises 2 and 3, $(x_1, x_2, x_3) = ((X - O) \cdot \vec{u}_1, (X - O) \cdot \vec{u}_2, (X - O) \cdot \vec{u}_3)$. So $x_1 = (x - a)i_1 + (y - b)j_1 + (z - c)k_1$, $x_2 = (x - a)i_2 + (y - b)j_2 + (z - c)k_2$, and $x_3 = (x - a)i_3 + (y - b)j_3 + (z - c)k_3$. [See (5).]
6. Replacing 'X' by 'O', and therefore '(x_1, x_2, x_3)' by '$(0, 0, 0)$' and '(x, y, z)' by '(a, b, c)' we have that $a = -a_1i_1 - a_2i_2 - a_3i_3$, $b = -a_1j_1 - a_2j_2 - a_3j_3$, and $c = -a_1k_1 - a_2k_2 - a_3k_3$.

$-\frac{1}{3}, \frac{2}{3}$), respectively. Suppose, now, that you are given point O and that the coordinates of A with respect to the coordinate system with origin O and basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ are $(1, -1, 3)$. Write equations like (4) and (5) relating the coordinates of any point with respect to this coordinate system to its coordinates with respect to the coordinate system with origin A and basis $(\vec{i}, \vec{j}, \vec{k})$.

2. You are given, below, the (x_1, x_2, x_3)-coordinates of various points. Find their (x, y, z)-coordinates.
   (a) $(3, 2, -5)$ (b) $(6, 1, 0)$ (c) $(9, -4, 5)$ (d) $(0, 0, 1)$
3. In Exercise 2 you found the (x, y, z)-coordinates of several points. Check your results by using equations like (5) to find the (x_1, x_2, x_3)-coordinates of these points.
4. Equations (5) are equivalent to:

$$\begin{cases} x_1 = a_1 + xi_1 + yj_1 + zk_1 \\ x_2 = a_2 + xi_2 + yj_2 + zk_2 \\ x_3 = a_3 + xi_3 + yj_3 + zk_3 \end{cases}$$

   [Explain.] Write the corresponding equations for the point A and the vectors $\vec{i}, \vec{j}$, and $\vec{k}$ of the present exercises.
5. Recall that, in Part A, $\vec{i}$ and $\vec{j}$ were chosen so that $[\vec{i}, \vec{j}] = [\vec{p}, \vec{q}]$. Consequently, the plane $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ is the plane $\overleftrightarrow{A[\vec{i}, \vec{j}]}$. Do you see a quick way to obtain parametric equations for this plane from the equations you obtained in Exercise 4? Explain your [affirmative] answer.
6. How can you obtain an equation for $\overleftrightarrow{A[\vec{p}, \vec{q}]}$ from the equations like (4) which you gave in answer to Exercise 1?

13.04 Coordinates in a Given Plane

In many geometrical problems all the points involved belong to a single plane—say, the plane π. When coordinates are used in solving such a problem there is some advantage in using a coordinate system in which π is one of the coordinate planes. You may have already seen how to do this. [See the Background Topic on page 66.] However π may be described we shall know a point $A \epsilon \pi$ and vectors $\vec{p}$ and $\vec{q}$ such that $[\pi] = [\vec{p}, \vec{q}]$. Then, for any point $X \epsilon \pi$ there are uniquely determined numbers—say, r and s—such that

$$X = A + \vec{p}r + \vec{q}s. \tag{1}$$

The numbers (r, s) are the coordinates of X with respect to a coordinate system for π with origin A and basis $(\vec{p}, \vec{q})$ for $[\pi]$. Since it is often

Answers for Part C

1. Given that $(a_1, a_2, a_3) = (1, -1, 3)$, $(i_1, i_2, i_3) = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, $(j_1, j_2, j_3) = (-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$, and $(k_1, k_2, k_3) = (-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$, and substituting into (4), we obtain:

$$\begin{cases} x = (x_1 - 1)\frac{1}{3} + (x_2 + 1)\frac{2}{3} + (x_3 - 3)\frac{2}{3} \\ y = (x_1 - 1)\cdot -\frac{2}{3} + (x_2 + 1)\frac{2}{3} + (x_3 - 3)\cdot -\frac{1}{3} \\ z = (x_1 - 1)\cdot -\frac{2}{3} + (x_2 + 1)\cdot -\frac{1}{3} + (x_3 - 3)\cdot \frac{2}{3} \end{cases}$$

   To write equations like (5), we must first find the values of 'a', 'b', and 'c'. Using the results of Exercise 6, we have $a = -\frac{1}{3} + \frac{2}{3} - \frac{6}{3} = -\frac{5}{3}$, $b = \frac{2}{3} + \frac{2}{3} + \frac{3}{3} = \frac{7}{3}$, and $c = \frac{2}{3} - \frac{1}{3} - \frac{6}{3} = -\frac{5}{3}$. So, by (5),

$$\begin{cases} x_1 = (x + 5/3)\frac{1}{3} + (y - 7/3)\cdot -\frac{2}{3} + (z + 5/3)\cdot -\frac{2}{3} \\ x_2 = (x + 5/3)\frac{2}{3} + (y - 7/3)\frac{2}{3} + (z + 5/3)\cdot -\frac{1}{3} \\ x_3 = (x + 5/3)\frac{2}{3} + (y - 7/3)\cdot -\frac{1}{3} + (z + 5/3)\cdot \frac{2}{3} \end{cases}$$

2. (a) $(-\frac{8}{3}, \frac{10}{3}, -\frac{23}{3})$ (b) $(1, -1, -6)$
   (c) $(2, -8, -3)$ (d) $(-1, 2, -1)$
3. [Answer as given in Exercise 2.]
4. [From (5), $x_1 = xi_1 + yj_1 + zk_1 - (ai_1 + bj_1 + ck_1)$ and, also by (5), $a_1 = -(ai_1 + bj_1 + ck_1)$. So, $x_1 = a_1 + xi_1 + yj_1 + zk_1$. The other two equations can be obtained in a similar manner.] Comparing (4) and (5) it is evident that if the given equations are equivalent to (5) then the following are equivalent to (4):

$$\begin{cases} x = a + x_1i_1 + x_2i_2 + x_3i_3 \\ y = b + x_1j_1 + x_2j_2 + x_3j_3 \\ z = c + x_1k_1 + x_2k_2 + x_3k_3 \end{cases}$$

5.
$$\begin{cases} x_1 = a_1 + xi_1 + yj_1 \\ x_2 = a_2 + xi_2 + yj_2 \\ x_3 = a_3 + xi_3 + yj_3 \end{cases}$$

   are parametric equations for the plane $\overline{A[\vec{i}, \vec{j}]}$, since (a_1, a_2, a_3) are coordinates of A, and (i_1, i_2, i_3) and (j_1, j_2, j_3) are the components of the vectors $\vec{i}$ and $\vec{j}$, respectively, with respect to the coordinate system with origin O and basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$. Substituting $(1/3, 2/3, 2/3)$ for (i_1, i_2, i_3), $(-2/3, 2/3, -1/3)$ for (j_1, j_2, j_3), and $(1, -1, 3)$ for (a_1, a_2, a_3), we have the following parametric equations for $\overline{A[\vec{i}, \vec{j}]}$:

$$\begin{cases} x_1 = 1 + \frac{1}{3}x - \frac{2}{3}y \\ x_2 = -1 + \frac{2}{3}x + \frac{2}{3}y \\ x_3 = 3 + \frac{2}{3}x - \frac{1}{3}y \end{cases}$$

6. A point with coordinates (x, y, z) will be on $\overline{A[\vec{i}, \vec{j}]}$ if and only if $z = 0$, that is, if and only if

$$0 = (x_1 - 1)\cdot -\frac{2}{3} + (x_2 + 1)\cdot -\frac{1}{3} + (x_3 - 3)\cdot \frac{2}{3}.$$

   Simplifying this, we see that '$2x_1 + x_2 - 2x_3 = -5$' is an equation for $\overline{A[\vec{i}, \vec{j}]}$.

simpler to use orthonormal coordinates one frequently uses Theorem 13-4 to compute an orthonormal basis $(\vec{i}, \vec{j})$ for $[\pi]$. Then, for $X \in \pi$,

$$X = A + \vec{i}x + \vec{j}y, \tag{2}$$

where (x, y) are the coordinates of X with respect to the coordinate system for π with origin A and orthonormal basis $(\vec{i}, \vec{j})$. In this case, $x = (X - A) \cdot \vec{i}$ and $y = (X - A) \cdot \vec{j}$.

Exercises

Part A

Suppose given an orthonormal coordinate system with origin O and basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$. Let A, B, and C have coordinates $(1, -1, 3)$, $(2, 1, 5)$, and $(0, 3, 4)$, respectively.

1. Find an orthonormal basis $(\vec{i}, \vec{j})$ for $[\overleftrightarrow{ABC}]$. [*Hint*: You have done the work in Exercise 2 of Part A on page 127. Explain.]
2. Knowing the components of $\vec{i}$ and $\vec{j}$ with respect to the orthonormal basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ there is an easy method [using determinants] for finding the components of a vector $\vec{k}$ such that $(\vec{i}, \vec{j}, \vec{k})$ is an orthonormal basis for $\mathscr{T}$. Apply this method.
3. Is the vector $\vec{k}$ you found in the preceding exercise the same as the vector $\vec{k}$ you found earlier by using Theorem 13-4? How many unit vectors are there in $[\vec{i}, \vec{j}]^{\perp}$?
4. Explain how to find the (x_1, x_2, x_3)-coordinates of a point of π when you know its coordinates (x, y) with respect to the origin A and basis $(\vec{i}, \vec{j})$. [*Hint*: Consider parametric equations for $A[\vec{i}, \vec{j}]$.]

*5. Given the (x_1, x_2, x_3)-coordinates of a point of $\mathscr{E}$, how can you tell whether this point belongs to π and, if it does, find its (x, y)-coordinates? [*Hint*: Equations like (4) on page 128 can be useful but are not needed here.]

*

The exercises in Section 13.03 and in Part A, above, show that, given enough information about π in terms of some given orthonormal coordinate system for $\mathscr{E}$, you can actually introduce a more convenient coordinate system for π and, after using it to obtain results concerning figures in π, describe the results you obtain in terms of the given coordinate system. Fortunately, it is not often necessary to do all this. More usually we shall choose three noncollinear points A, B, and C of π, which bear some relation to the problem we are concerned with, let $\vec{p} = B - A$ and $\vec{q} = C - A$, and merely refer to Theorem 13-4 to

Here are suggestions for use of the exercises of section 13.04

(i) Part A and the discussion which follows should be presented under teacher direction. Part B may then be used to illustrate these results.

(ii) Part C is appropriate for homework. Students may be encouraged to check results among themselves or to work in teams.

(iii) The discussion on pages 135-136 and Part D should be teacher directed.

(iv) Part E is a reasonable homework assignment.

(v) Part F, and the discussion preceding and following it, is recommended for class discussion.

(vi) Part G is a good homework assignment.

Answers for Part A

1. $(\vec{i}, \vec{j})$ is an orthonormal basis for $[\overleftrightarrow{ABC}]$, where the components of $\vec{i}$ and $\vec{j}$ with respect to $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ are $(1/3, 2/3, 2/3)$ and $(-2/3, 2/3, -1/3)$. [Note that $\vec{p}$ and $\vec{q}$ of Part A on page 127 are $B - A$ and $C - A$, respectively, so that $(\vec{p}, \vec{q})$ is a basis for $[ABC]$. But by Part A, $[\vec{p}, \vec{q}] = [\vec{i}, \vec{j}]$, so $(\vec{i}, \vec{j})$ is also a basis for $[ABC]$.]

2. The vector with components

$$\left(\begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix}, \begin{vmatrix} i_3 & i_1 \\ j_3 & j_1 \end{vmatrix}, \begin{vmatrix} i_1 & i_2 \\ j_1 & j_2 \end{vmatrix} \right)$$

— that is, $(-2/3, -1/3, 2/3)$ — is orthogonal to both $\vec{i}$ and $\vec{j}$ and has norm 1. So, it is a suitable choice for $\vec{k}$. [It can be proved that the vector obtained in this way from orthonormal vectors $\vec{i}$ and $\vec{j}$ always has norm 1.]

3. Yes.; There are two, $\vec{k}$ and $-\vec{k}$.

4. Suppose the coordinates of a point X with respect to the origin A and basis $(\vec{i}, \vec{j})$ are (x, y). Then $X = A + \vec{i}x + \vec{j}y$. Therefore, the coordinates (x_1, x_2, x_3) of X are given by:

$$\begin{cases} x_1 = 1 + \frac{1}{3}x - \frac{2}{3}y \\ x_2 = -1 + \frac{2}{3}x + \frac{2}{3}y \\ x_3 = 3 + \frac{2}{3}x - \frac{1}{3}y \end{cases}$$

5. One may solve two of the equations for Exercise 4 for values of 'x' and 'y' and see whether these satisfy the third equation. [Given equations like (4) on page 128, the $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$-equation for π can be obtained by substituting '0' for 'z' in the third of them. The (x, y)-coordinates of the point are given by the first two equations.]

justify introducing an orthonormal basis $(\vec{i}, \vec{j})$ for π such that $\vec{i} \in [\vec{p}]$ and $\vec{q} \cdot \vec{j} > 0$. [Occasionally we shall also need the expressions for $\vec{i}$ and $\vec{j}$ in terms of $\vec{p}$ and $\vec{q}$ which are given in the theorem.] So, in the remainder of this section we shall assume that $\pi = O[\vec{i}, \vec{j}]$, where $(\vec{i}, \vec{j})$ is orthonormal and consider what use can be made of coordinates (x, y) for points of π with respect to the origin O and basis $(\vec{i}, \vec{j})$.

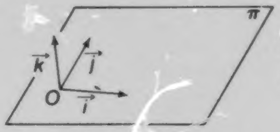


Fig. 13–4

Part B

Given a plane π we shall assume that O is some point of π and that $(\vec{i}, \vec{j})$ is some orthonormal basis for $[\pi]$. For each point $X \in \pi$ we shall speak of the numbers (x, y) such that $X = O + \vec{i}x + \vec{j}y$ as the *π-coordinates of X*. For the following exercises we shall assume that $\vec{k}$ is a vector such that $(\vec{i}, \vec{j}, \vec{k})$ is an orthonormal basis for $\mathscr{T}$ and, for each point $X \in \mathscr{E}$ we shall speak of the numbers (x, y, z) such that $X = O + \vec{i}x + \vec{j}y + \vec{k}z$ as the *$\mathscr{E}$-coordinates of X*. A point whose $\mathscr{E}$-coordinates are (x, y, z) belongs to π if and only if $z = 0$; and, in this case, the π-coordinates of the point are (x, y).

1. Suppose that $\vec{n}$ is a vector whose components with respect to $(\vec{i}, \vec{j}, \vec{k})$ are (n_1, n_2, n_3).
   (a) What is the condition on these components in order that $\vec{n} \in [\pi]$?
   (b) What further condition must be satisfied in order that $\vec{n}$ be a non-$\vec{0}$ member of $[\pi]$?
2. (a) For given values of 'a', 'b', and 'c', the equation:

$$ax + by + 0z = c$$

   describes, with respect to the $\mathscr{E}$-coordinate system, some subset of $\mathscr{E}$. Tell, as precisely as you can, what kind of subset. [*Hint*: How is this subset related to π?]
   (b) What kind of subset of $\mathscr{E}$ is described by the system consisting of the equation of part (a) and the equation '$z = 0$'?
   (c) Write an equation which describes the set of part (b) with reference to the π-coordinate system.
3. Suppose that l is a line contained in π.
   (a) How many planes are there whose intersection with π is l?
   (b) How many of the planes referred to in part (a) are perpendicular to π?

Answers for Part B

1. (a) $n_3 = 0$ (b) $(n_1, n_2) \neq (0, 0)$

2. (a) The given equation [with or without the term '0z'] describes a plane σ which is perpendicular to π. [This has been proved before, but it is worth noting now that if (x_0, y_0, z_0) are the coordinates of some point in σ then so are (x_0, y_0, z), for any real number z. So, for any point of σ the entire line through this point and parallel to the third coordinate axis is a subset of σ.]
   (b) The equation '$z = 0$' describes the plane π. So, the system in question describes the line of intersection of σ and π.
   (c) $ax + by = c$
3. (a) Infinitely many. (b) One.

4. Suppose that l_1 and l_2 are lines contained in π and that σ_1 and σ_2 are planes perpendicular to π and such that $\sigma_1 \cap \pi = l_1$ and $\sigma_2 \cap \pi = l_2$.
   (a) What can you say about σ_1 and σ_2 if $l_1 \parallel l_2$? If $l_1 \perp l_2$? Give reasons for your answers.
   (b) What can you say about l_1 and l_2 if $\sigma_1 \parallel \sigma_2$? If $\sigma_1 \perp \sigma_2$? Explain.

*

By an earlier agreement, the components (a, b, c) [with respect to the orthonormal basis $(\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$] of any non-$\overrightarrow{0}$ vector in the direction of a line l are direction numbers of l with respect to the $\mathscr{E}$-coordinate system of Part B. In case $l \subseteq \pi$ the third term of any sequence of direction numbers of l is 0. For such a line, we shall call the numbers (a, b) *direction numbers of* l *with respect to the* π*-coordinate system.* [When, as in this section, we are concerned mainly with points of π, and are dealing with a single coordinate system for π, we may refer to (a, b) merely as direction numbers of l.]

We also agreed to speak of a line n which is perpendicular to a given plane σ *as a normal to* σ. If $l \subseteq \pi$, we shall speak of a line n in π which is perpendicular to l as *a normal to* l [in π].

Some of the results obtained in Part B can now be summarized by saying that, with respect to the π-coordinate system, an equation:

$$ax + by = c \quad [(a, b) \neq (0, 0)] \tag{3}$$

describes a line l in π whose normals do have (a, b) as direction numbers; and that any such line may be described by an equation like (3).

When l is described by an equation like (3), the corresponding equation:

$$ax + by + 0z = c$$

describes, with respect to the $\mathscr{E}$-coordinate system, the plane σ perpendicular to π which intersects π in l. So, as is suggested in Exercise 4, it is not difficult to apply what we know about equations of planes in $\mathscr{E}$ to the problem of understanding equations of lines in π.

Part C

1. Picture an orthonormal coordinate system for π and draw the lines whose equations are given. Indicate which is which by writing its equation nearby. [It is customary to picture such a coordinate system so that $\overrightarrow{i}$ is pictured by a horizontal arrow pointing to

Answers for Part B [cont.]

4. (a) $\sigma_1 \parallel \sigma_2$ [For both $[\sigma_1]$ and $[\sigma_2]$ contain $[\pi]^{\perp}$ and $[l_1]$.]; $\sigma_1 \perp \sigma_2$ [By Theorem 12-14(b) $\sigma_2 \perp \sigma_1 \cap \pi$ and, so, by Theorem 12-13(b), $\sigma_1 \perp \sigma_2$.]
   (b) $l_1 \parallel l_2$ [By Theorem 9-16 — the intersections of a plane with parallel planes are parallel lines.]

   $l_1 \perp l_2$ [Since $\pi \perp \sigma_1$ and $\sigma_2 \perp \sigma_1$ it follows by Theorem 12-14(b) that $\pi \perp l_1$. Since $l_2 \subseteq \pi$ it follows by Theorem 12-12(a) that $l_1 \perp l_2$.]

the right and $\vec{j}$ by a vertical arrow pointing upward. Often, the first coordinate axis is called 'the x-axis' and the second is called 'the y-axis'.]

(a) $x - y = 1$
(b) $2x + 3y = 6$
(c) $x + y = 1$
(d) $3x - 2y = 6$
(e) $x + y = 2$
(f) $3x - 2y = 0$

2. Suppose that l_1 and l_2 are described by:

$$a_1x + b_1y = c_1 \quad [\text{for } l_1]$$
$$a_2x + b_2y = c_2 \quad [\text{for } l_2]$$

Show that

(a) $l_1 \parallel l_2 \longleftrightarrow a_1b_2 = b_1a_2$,
(b) $l_1 \perp l_2 \longleftrightarrow a_1a_2 + b_1b_2 = 0$,
(c) if $l_1 \nparallel l_2$ then, for $(k_1, k_2) \neq (0, 0)$, the equation:

$$(*) \qquad (a_1x + b_1y - c_1)k_1 + (a_2x + b_2y - c_2)k_2 = 0$$

describes a line through the point of intersection of l_1 and l_2.

3. Suppose that P_0 is a point of π and that the coordinates of P_0 are (x_0, y_0).

(a) Show that any line in π which contains P_0 has an equation of the form:

$$a(x - x_0) + b(y - y_0) = 0$$

and that any such equation for which $(a, b) \neq (0, 0)$ describes such a line.

(b) Suppose that l is described by the equation:

$$ax + by = c \qquad [(a, b) \neq (0, 0)]$$

Show that

(i) the line through P_0 parallel to l is described by:

$$a(x - x_0) + b(y - y_0) = 0$$

and that

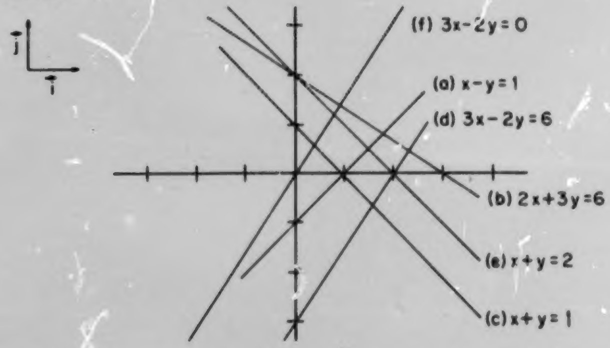
(ii) the line in π which contains P_0 and is perpendicular to l is described by:

$$b(x - x_0) - a(y - y_0) = 0$$

(c) Give a pair of direction numbers for l.

Answers for Part C

1.



2. (a) Since ℓ_1 and ℓ_2 are coplanar, $\ell_1 \parallel \ell_2$ if and only if $\ell_1 \cap \ell_2$ does not consist of a single point. So, $\ell_1 \parallel \ell_2$ if and only if the given equations do not have a unique solution — that is, if and only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

(b) $\ell_1 \perp \ell_2$ if and only if $\sigma_1 \perp \sigma_2$ [see Exercise 4 of Part B] and the latter is the case if and only if $a_1b_1 + a_2b_2 + 0 \cdot 0 = 0$ [see Exercise 5 of Part F on page 115].

(c) For $\ell_1 \nparallel \ell_2$, $\sigma_1 \nparallel \sigma_2$ and, so, for $(k_1, k_2) \neq (0, 0)$ (*) represents in $\mathcal{E}$-coordinates a plane perpendicular to π and containing the intersection of ℓ_1 and ℓ_2. So, in π-coordinates, (*) represents a line in π through the intersection of ℓ_1 and ℓ_2.

3. (a) Any line in π has an equation of the form '$ax + by = c$' and contains P_0 if and only if '$ax_0 + by_0 = c$' is satisfied.

(b) (i) All that is required is to show that the criterion of Exercise 2(a) is satisfied, and it obviously is [$ab = ab$].

(ii) All that is required is to show that the criterion of Exercise 2(b) is satisfied, and it obviously is [$ab + b \cdot -a = 0$].

(c) $(b, -a)$ [For ℓ is a normal to the line of part (b) (ii) and these are direction numbers of such a normal. [Of course, '$(-b, a)$' is an equally correct answer.]

TC 134 (1)

4. Using the results of 3(b), we have:

(a) (i) $7(x - 2) - 5(y + 7) = 0$
(ii) $-5(x - 2) - 7(y + 7) = 0$

(b) (i) $5(x - 2) + 7(y + 7) = 0$
(ii) $7(x - 2) - 5(y + 7) = 0$

(c) (i) $3(x + 1) + 6(y - 3) = 0$
(ii) $6(x + 1) - 3(y - 3) = 0$

(d) (i) $(x - 1) + 2(y - 3) = 0$
(ii) $2(x - 1) - (y - 3) = 0$

(e) (i) $5(x - 2) = 0$
(ii) $-5(y - 4) = 0$

(f) (i) $3(y - 5) = 0$
(ii) $3(x + 1) = 0$

[Think of the equation given in (e) as '$5x + 0y = 3$', so that $a = 5$ and $b = 0$.]

Of course, all of the above equations can be simplified.

4. In each part you are given the equation of a line l in π and the coordinates of a point $P \in \pi$. Find an equation for the line in π which contains P and (i) is parallel to l; (ii) is perpendicular to l.

(a) $7x - 5y = 16$; $(2, -7)$	**(b)** $5x + 7y = 3$; $(2, -7)$
(c) $3x + 6y = 5$; $(-1, 3)$	**(d)** $x + 2y = 7$; $(1, 3)$
(e) $5x = 3$; $(2, 4)$	**(f)** $3y = -2$; $(-1, 5)$

5. Assuming that l is described by

$$ax + by = c,$$

where $(a, b) \neq (0, 0)$, what can you say about l

(a) if $a = 0$? **(b)** if $b = 0$? **(c)** if $c = 0$?

6. Without finding the coordinates of the point common to the lines described by:

$$3x + 2y = 6 \quad \text{and} \quad x + 5y = 8,$$

and an equation for the line through this point which

(a) contains the origin,
(b) is perpendicular to the x-axis,
(c) is parallel to the x-axis,
(d) contains the point whose coordinates are $(-1, 2)$,
(e) contains the point whose coordinates are $(4, -3)$,
(f) is perpendicular to the line described by '$2x + 3y = 2$',
(g) is perpendicular to the line described by '$3x - 2y = 2$'.

7. Suppose that P_1 and P_2 are two points of π whose coordinates are (x_1, y_1) and (x_2, y_2), respectively.

(a) Find an equation for $\overleftrightarrow{P_1P_2}$ [*Hint*: Since $P_1 \in \overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_2}$ can be described by an equation of the form '$a(x - x_1) + b(y - y_1) = 0$'. Try to find a and b, in terms of the numbers x_1, y_1, x_2, and y_2, so that (x_2, y_2) is a solution of the resulting equation.]
(b) Express your answer for part (a) neatly by using a determinant.
(c) Show that points with coordinates (x_0, y_0), (x_1, y_1), and (x_2, y_2) are collinear if and only if

$$(y_2 - y_1)(x_1 - x_0) - (x_2 - x_1)(y_1 - y_0) = 0.$$

8. In each of the following you are given the coordinates of two points of π. Find an equation for the line containing these points.

(a) $(6, -7)$, $(-4, 3)$	**(b)** $(5, 2)$, $(-5, -2)$
(c) $(4, 3)$, $(2, 6)$	**(d)** $(17, 0)$, $(0, -12)$

9. Suppose that P_0, P_1, and P_2 are three points with coordinates (x_0, y_0), (x_1, y_1), and (x_2, y_2), respectively.

(a) Find an equation for the line in π which contains P_0 and is parallel to $\overrightarrow{P_1P_2}$.
(b) Find an equation for the line in π which contains P_0 and is perpendicular to $\overrightarrow{P_1P_2}$.

Answers for Part C [cont.]

5. (a) ℓ is perpendicular to the y-axis and parallel to the x-axis.
(b) ℓ is perpendicular to the x-axis and parallel to the y-axis.
(c) ℓ contains the origin.

6. In each part we need values of 'a' and 'b', not both 0, for:

$$(3x + 2y)a + (x + 5y)b = 6a + 8b$$

[See Exercise 2(c).]

(a) $6a + 8b = 0$; so choose $a = 4$, $b = -3$ to obtain '$9x - 7y = 0$'.

(b) $2a + 5b = 0$; so choose $a = 5$, $b = -2$ to obtain '$13x = 14$'.

(c) $3a + b = 0$; so choose $a = 1$, $b = -3$ to obtain '$-13y = -18$' [or: $13y = 18$]

(d) $a + 9b = 6a + 8b$; so choose $a = 1$, $b = 5$ to obtain '$8x + 27y = 46$'.

(e) $6a - 11b = 6a + 8b$; so choose $a = 1$, $b = 0$ to obtain '$3x + 2y = 6$'.

(f) $(3a + b)2 + (2a + 5b)3 = 0$; so choose $a = 17$, $b = -12$ to obtain '$39x - 26y = 6$'.

(g) $(3a + b)3 + (2a + 5b)\cdot -2 = 0$; so choose $a = 1$, $b = 0$ to obtain '$26x + 39y = 82$'.

[Note that the lines in parts (f) and (g) are perpendicular.]

7. (a) Following the hint, we wish to find values of 'a' and 'b' such that $a(x_2 - x_1) + b(y_2 - y_1) = 0$. The simplest choice is $a = y_2 - y_1$, $b = -(x_2 - x_1)$. So, the desired equation is '$(y_2 - y_1)(x - x_1) - (x_2 - x_1)(y - y_1) = 0$'

(b)

$$\begin{vmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix} = 0$$

[Note why this equation is satisfied (a) if $(x, y) = (x_1, y_1)$ and (b) if $(x, y) = (x_2, y_2)$.]

(c) By part (a) the given condition holds if and only if $P_0 \in \overleftrightarrow{P_1P_2}$ [where P_0 is the point with coordinates (x_0, y_0).]

8. (a) $10(x - 6) + 10(y + 7) = 0$ [or: $x + y + 1 = 0$]
(b) $-4(x - 5) + 10(y - 2) = 0$ [or: $2x - 5y = 0$]
(c) $3(x - 4) + 2(y - 3) = 0$ [or: $3x + 2y = 18$]
(d) $-12(x - 17) + 17(y - 0) = 0$ [or: $12x - 17y = 204$]

9. (a) $(y_2 - y_1)(x - x_0) - (x_2 - x_1)(y - y_0) = 0$
(b) $(x_2 - x_1)(x - x_0) + (y_2 - y_1)(y - y_0) = 0$

*

We can now illustrate the use of coordinates for a plane to obtain interesting geometric results. One of the results which we came upon in Chapter 11 is that the lines which contain the altitudes of a triangle are concurrent.

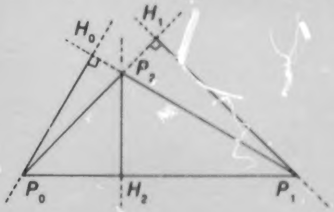


Fig. 13–5

[Recall that the altitude of $\triangle P_0P_1P_2$ from, say, P_0 is the interval $\overline{P_0H_0}$, where H_0 is the point of intersection of $\overleftrightarrow{P_1P_2}$ and the plane through P_0 perpendicular to $\overleftrightarrow{P_1P_2}$.] The argument arrived at in Part C on pages 32 and 33 made use of another result involving distance and we shall review that argument in the next chapter. Here we shall give a quite different proof of the concurrency of the lines containing the altitudes of a triangle.

Consider $\triangle P_0P_1P_2$ and the plane π which contains it. Choose an orthonormal coordinate system for π and suppose that the coordinates of P_0, P_1, and P_2 are (x_0, y_0), (x_1, y_1), and (x_2, y_2), respectively. As you showed in Exercise 9(b), the line which contains the altitude of $\triangle P_0P_1P_2$ from P_0 is described by:

$$(x_2 - x_1)(x - x_0) + (y_2 - y_1)(y - y_0) = 0$$

It follows merely by inspection of this equation that the lines containing the altitudes from P_1 and P_2 are described by the equations:

$$(x_0 - x_2)(x - x_1) + (y_0 - y_2)(y - y_1) = 0$$
$$(x_1 - x_0)(x - x_2) + (y_1 - y_0)(y - y_2) = 0$$

Our problem is to show that these three equations have a unique solution. We can do this by showing, first, that any solution of two of these equations is a solution of the third, and second, that some two of the equations have a unique solution. To establish the first of these results it is sufficient to note that on adding the left sides of the three equations and simplifying we obtain '0'. This happens because of two rather trivial real number theorems:

On adding the three equations on page 135 we obtain, after some rearrangement:

$$\begin{aligned}&[(x_2 - x_1) + (x_0 - x_2) + (x_1 - x_0)]x\\ &- [(x_2 - x_1)x_0 + (x_0 - x_2)x_1 + (x_1 - x_0)x_2]\\ &+ [(y_2 - y_1) + (y_0 - y_2) + (y_1 - y_0)]y\\ &- [(y_2 - y_1)y_0 + (y_0 - y_2)y_1 + (y_1 - y_0)y_2] = 0\end{aligned}$$

The real number theorems show that each of the bracketed expressions reduces to '0' and, so, that the equation is satisfied for all values of 'x' and 'y' no matter what the coordinates of P_0, P_1, and P_2 may be.

Values of 'x' and 'y' which satisfy the first two equations will satisfy the equation obtained by addition if and only if they satisfy the third. [This follows from the real number theorem '$(a = 0 \text{ and } b = 0) \Longrightarrow [a + b + c = 0 \Longleftrightarrow c = 0]$'.] Since any values of 'x' and 'y' satisfy the equation obtained by addition, values which satisfy the first two equations must satisfy the third.

By Theorem A on page 471 of volume 1 the first two equations have a unique solution if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_0 - x_2 & y_0 - y_2 \end{vmatrix} \neq 0.$$

This last is the case if and only if $\{P_0, P_1, P_2\}$ is not collinear. [For '$(x_2 - x_1)(y - y_2) - (y_2 - y_1)(x - x_2) = 0$' is an equation for $\overleftrightarrow{P_1P_2}$.] And, $\{P_0, P_1, P_2\}$ is noncollinear.

For a discussion of the "vector proof" see answer for Exercise 1 of Part D on page 136.

$$(*)\qquad \begin{cases}(c-b)+(a-c)+(b-a)=0\\(c-b)a+(a-c)b+(b-a)c=0\end{cases}$$

[Explain.] Explain why this shows that any solution of, say, the first two equations is also a solution of the third. That two of the equations —say, the first two—have a unique solution follows from the fact that

$$(x_2-x_1)(y_0-y_2)-(y_2-y_1)(x_0-x_2)\neq 0.$$

[Explain.] This is the case because $P_0 \notin \overleftrightarrow{P_1P_2}$. [Explain.] So, the theorem is proved.

As a matter of fact the "same" proof can be given more easily without introducing coordinates. To do so, let $\overrightarrow{p_0}$, $\overrightarrow{p_1}$, and $\overrightarrow{p_2}$ be the position vectors of P_0, P_1, and P_2 with respect to some origin. The plane σ_0 which contains P_0 and is perpendicular to $\overleftrightarrow{P_1P_2}$ is described by the vector equation:

$$(\overrightarrow{p_2}-\overrightarrow{p_1})\cdot(\overrightarrow{x}-\overrightarrow{p_0})=0$$

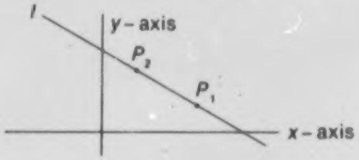
[Explain. Also, compare this equation with the coordinate equation previously given for the line containing the altitude from P_0.] If we write down similar equations for the planes σ_1 and σ_2, where $\sigma_1 = P_1[\overleftrightarrow{P_0 - P_2}]^\perp$ and $\sigma_2 = P_2[\overleftrightarrow{P_1 - P_0}]^\perp$ and use vector equations analogous to (*) it is easy to show that the intersection of σ_0, σ_1, and σ_2 is a line l. Since $l \perp \pi$, l and π intersect at a point P. Since the lines containing the altitudes of $\triangle P_0P_1P_2$ are $\sigma_0 \cap \pi$, $\sigma_1 \cap \pi$, and $\sigma_2 \cap \pi$, these lines are concurrent at P.

Part D

1. Fill in the details of the preceding vector proof that the altitudes of a triangle are concurrent.
2. Use the result just proved to show that the perpendicular bisectors of the sides of a triangle intersect in a line which is perpendicular to the plane of the triangle. [*Hint*: This is (3) on page 31. It was used on page 33 in deriving the result of Exercise 1. Try to reverse this procedure.]

Part E

Let l be a line in π which is not perpendicular to the x-axis. Sup-



Answers for Part D

1. Let $\vec{p}_0$, $\vec{p}_1$, and $\vec{p}_2$ be position vectors, with respect to some origin, of the vertices P_0, P_1, and P_2, respectively, of $\triangle P_0P_1P_2$. Then $[\overleftrightarrow{P_1P_2}] = [\vec{p}_2 - \vec{p}_1]$ and, as in (11) on page 109, $\vec{x}$ is the position vector of a point of the plane through P_0 and perpendicular to $\overleftrightarrow{P_1P_2}$ if and only if $(\vec{p}_2 - \vec{p}_1)\cdot(\vec{x} - \vec{p}_0) = 0$. . The plane in question is the plane $\sigma_0 \perp \pi$ which contains the altitude of $\triangle P_0P_1P_2$ from P_0. Similarly, the planes σ_1 and σ_2 which are perpendicular to π and contain the altitudes through P_1 and P_2, respectively, have equations
$$\text{`}(\vec{p}_0 - \vec{p}_2)\cdot(\vec{x} - \vec{p}_1) = 0\text{'} \text{ and } \text{`}(\vec{p}_1 - \vec{p}_0)\cdot(\vec{x} - \vec{p}_2) = 0\text{'}$$
respectively. Now,
$$(\vec{p}_2 - \vec{p}_1)\cdot(\vec{x} - \vec{p}_0) + (\vec{p}_0 - \vec{p}_2)\cdot(\vec{x} - \vec{p}_1) + (\vec{p}_1 - \vec{p}_0)\cdot(\vec{x} - \vec{p}_2) = 0$$
for all values of '$\vec{x}$' because
$$(\vec{p}_2 - \vec{p}_1) + (\vec{p}_0 - \vec{p}_2) + (\vec{p}_1 - \vec{p}_0) = \vec{0}$$
and $$(\vec{p}_2 - \vec{p}_1)\cdot\vec{p}_0 + (\vec{p}_0 - \vec{p}_2)\cdot\vec{p}_1 + (\vec{p}_1 - \vec{p}_0)\cdot\vec{p}_2 = 0.$$
It follows that any point common to σ_0 and σ_1 also belongs to σ_2. On the other hand, $\sigma_0 \cap \sigma_1$ is a line perpendicular to π because $\sigma_0 \perp \pi$, $\sigma_1 \perp \pi$, and $\sigma_0 \not\parallel \sigma_1$. The last is the case because, since $\{P_0, P_1, P_2\}$ is noncollinear, $(\vec{p}_2 - \vec{p}_1, \vec{p}_0 - \vec{p}_2)$ is linearly independent. By definition, the point in which $\sigma_0 \cap \sigma_1 \cap \sigma_2$ [which is the line $\sigma_0 \cap \sigma_1$] intersects π is common to the three altitudes of $\triangle P_0P_1P_2$.
2. Given $\triangle ABC$, we wish to show that the perpendicular bisectors of its sides intersect in a line which is perpendicular to $\overleftrightarrow{ABC}$. Let A' be the midpoint of $\overline{BC}$, B' be the midpoint of $\overline{AC}$, and C' be the midpoint of $\overline{AB}$. Then A', B', and C' are noncollinear, and, by the preceding theorem, the lines containing the altitudes of $\triangle A'B'C'$ are concurrent. But these lines are contained in the perpendicular bisectors of the sides of $\triangle ABC$. Therefore, the perpendicular bisectors of the sides of $\triangle ABC$ intersect in a line which is perpendicular to $\overleftrightarrow{ABC}$.

pose that the equation:

$$(*) \qquad ax + by = c$$

is an equation for l.

1. What does the assumption that l is not perpendicular to the x-axis tell you about any of the numbers a, b, and c?
2. It should follow from your answer for Exercise 1 that l has an equation of the form:

   $$(**) \qquad y = mx + d \qquad \text{[Explain.]}$$

   (a) Express 'm' and 'd' in terms of 'a', 'b', and 'c'.
   (b) The line l has many equations like (*). [Explain.] Does it have more than one equation like (**)?
3. Suppose that P_1 and P_2 are two points of l with coordinates (x_1, y_1) and (x_2, y_2).
   (a) Use (**) to express the fact that $P_1 \in l$; that $P_2 \in l$.
   (b) Can it be the case that $x_1 = x_2$? [*Hint*: Recall that P_1 and P_2 are two points.]
   (c) $P_2 - P_1$ is a non-$\vec{0}$ vector in $[l]$. What can you infer from the equations in part (a) concerning the components of $P_2 - P_1$?
   (d) Given a figure like that at the beginning of these exercises, and a ruler, how could you estimate the value of 'm' in the equation (**) for l? [Does it matter that you don't know the scale of the figure?]
4. Draw an orthonormal coordinate system for π and draw the lines with equations like (**), where $d = 0$ and m is
   (a) 1 (b) $\frac{1}{2}$ (c) 2 (d) -1 (e) -2 (f) $-\frac{1}{2}$ (g) 0
5. Repeat Exercise 4 for lines for which $m = \frac{3}{2}$ and d [illegible]
   (a) -3 (b) -1 (c) $-\frac{1}{2}$ (d) 0 (e) [illegible] (f) 2 (g) 3

*

The preceding exercises have pointed out that, with respect to any orthonormal coordinate system for π, any line l of π which is not perpendicular to the x-axis [for short: any nonvertical line] has a unique equation of the form:

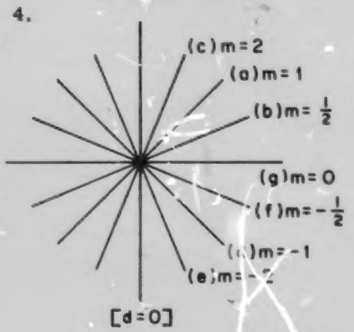
$$(4) \qquad y = mx + d$$

The number m is called *the slope of l* [*with respect to the given coordinate system*] and the number d is *the y-intercept of l*. [The word 'intercept' has occurred earlier in our discussion of planes. Exercises 3(d) and 4 suggest the reason for using the word 'slope'.] In consequence, (4) is called *the slope-intercept equation for l*, and any equation like (4) is said to be "in slope-intercept form". Note that lines perpendicular

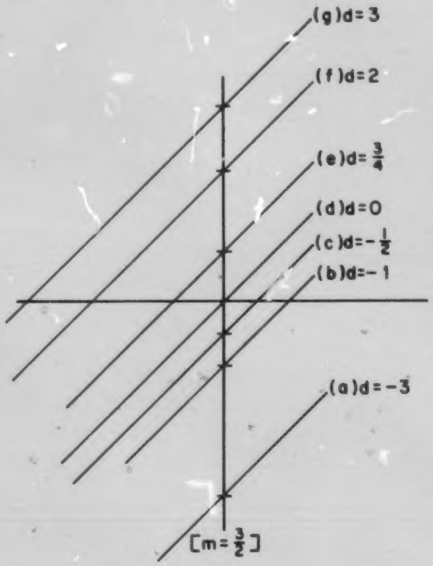
Answers for Part E

1. $b \neq 0$ [See Exercise 5 of Part C.]
2. (a) $m = -a/b$, $d = c/b$
   (b) [$(ak)x + (bk)y = ck \Longleftrightarrow ax + by = c$, for $k \neq 0$]; No.
3. (a) $y_1 = mx_1 + d$; $y_2 = mx_2 + d$
   (b) No. [If $x_1 = x_2$ then, by (a), $y_1 = y_2$ and, so $P_1 = P_2$. But, P_1 and P_2 are two points.]
   (c) The components [with respect to $(\vec{i}, \vec{j})$] of $P_2 - P_1$ are $(x_2 - x_1, y_2 - y_1)$ — that is, $(x_2 - x_1, m(x_2 - x_1))$.
   (d) From (a), $m = (y_2 - y_1)/(x_2 - x_1)$ and this ratio can be obtained by measuring $|y_2 - y_1|$ and $|x_2 - x_1|$ to any scale and paying proper regard to whether $x_2 - x_1$ and $y_2 - y_1$ are positive or negative.

4.



5.



to the x-axis do not have slopes and cannot be described by equations in slope-intercept form. [But, whether a line l in π has a slope-intercept equation depends entirely on how l is related to the coordinate system. With respect to one coordinate system l may have a slope-intercept equation, while with respect to another it may not.]

Part F

[As always in this section, the following exercises refer to some one orthonormal coordinate system for π.]

1. (a) What lines of π have slope 0 with respect to the given system?
   (b) What lines of π do not have slopes? [Note carefully the difference between the questions in parts (a) and (b). Zero is not nothing.]
2. Suppose that l_1 and l_2 are lines of π which have slopes m_1 and m_2, respectively, and suppose that $m_2 > m_1$.
   (a) Which of the two lines might you describe as being "steeper" than the other?
   (b) Reconsider the question in part (a), assuming that m_1 and m_2 are both negative. Assuming that both are positive.
   (c) How would you answer part (a) if you were told that $|m_2| > |m_1|$ [rather than that $m_2 > m_1$]?
3. Show that if l_1 and l_2 are lines of π which have slopes m_1 and m_2 then
   (a) $l_1 \parallel l_2 \longleftrightarrow m_1 = m_2$, and
   (b) $l_1 \perp l_2 \longleftrightarrow m_1 m_2 = -1$.
4. Suppose that l is the line in π which has slope m and contains the point whose coordinates are (x_0, y_0). Use this data to write an equation for l. [*Hint*: See Exercise 3(a) of Part C.]
5. Suppose that l is the line in π which contains two points with coordinates (x_0, y_0) and (x_1, y_1).
   (a) How can you tell whether l has a slope? [*Hint*: All you know are the coordinates of the two given points of l.]
   (b) If l has a slope, how can you find it?
   (c) Write an equation for l. [*Hint*: Use Exercise 4.]

*

Summarizing we have seen that any line l in π has an equation of the form:

$$ax + by = c \qquad [(a, b) \neq (0, 0)]$$

where (a, b) are direction numbers for the normals to l in π and $(b, -a)$ are direction numbers of l itself. If l contains the point with coordinates (x_0, y_0) then l is described by:

Answers for Part F

1. (a) The lines which are parallel to the x-axis.
   (b) The lines parallel to the y-axis.
2. (a) [Some students might venture that ℓ_2 would be steeper than ℓ_1, but the answer depends on more information than we're given. Hopefully 2(b) and (c) will clarify this.]
   (b) If both m_1 and m_2 are negative then ℓ_1 is steeper. If both m_1 and m_2 are positive then ℓ_2 is steeper.
   [We see therefore that simply knowing that $m_2 > m_1$ tells us nothing about which line is steeper.]
   (c) If $|m_2| > |m_1|$ then ℓ_2 is steeper than ℓ_1. [To convince oneself, one should also consider situations in which $m_2 < 0$ and $m_1 > 0$.]
3. Suppose that equations for ℓ_1 and ℓ_2 are '$y = m_1x + d_1$' and '$y = m_2x + d_2$', respectively. Then by the results of Exercise 2, page 133, it follows that
   (a) $\ell_1 \parallel \ell_2 \Longleftrightarrow m_1(-1) = (-1)m_2$. That is, $\ell_1 \parallel \ell_2 \Longleftrightarrow m_1 = m_2$.
   (b) $\ell_1 \perp \ell_2 \Longleftrightarrow m_1m_2 + (-1)(-1) = 0$. That is, $\ell_1 \perp \ell_2 \Longleftrightarrow m_1m_2 = -1$.
4. $y - y_0 = m(x - x_0)$
5. (a) ℓ has a slope if and only if $x_1 \neq x_0$.
   (b) $m = (y_1 - y_0)/(x_1 - x_0)$
   (c) $y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$ [See, for example, the answer for Exercise 7(a) of Part C.]

$$a(x - x_0) + b(y - y_0) = 0,$$

and, if (x_1, y_1) are the coordinates of another point of l then:

$$(y_1 - y_0)(x - x_0) - (x_1 - x_0)(y - y_0) = 0$$

is, also, an equation for l. [The preceding is called *the two-point form* for the equation of a line. It yields a simple criterion for collinearity of points.]

A line which is not perpendicular to the x-axis has an equation of the form:

$$y = mx + b$$

[*the slope-intercept form*]. If such a line contains the point with coordinates (x_0, y_0), it is described by the equation:

$$y - y_0 = m(x - x_0)$$

[*the point-slope form*]. Finally, if (x_0, y_0) and (x_1, y_1) are the coordinates of two points of a nonvertical line l then the slope m of l is given by:

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

and l is described by:

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

[This is a variant of the two-point form. Another such variant is:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$$

The latter suggests parametric equations for l:

$$\begin{cases} x = x_0 + (x_1 - x_0)t \\ y = y_0 + (y_1 - y_0)t \end{cases}$$

Of course, $(x_1 - x_0, y_1 - y_0)$ are direction numbers of l and can be replaced, in any of the preceding equations by the components of any non-$\overrightarrow{0}$ vector in $[l]$.]

Part G

1. In each part, you are given the coordinates of two points. In each case, write an equation in slope-intercept form for the line which contains the given points.
   - **(a)** (3, −2); (9, 3)
   - **(b)** (4, 8); (12, 8)
   - **(c)** (−3, 0); (7, 2)
   - **(d)** (−5, 0); (0, 6)
2. For each of the lines described in Exercise 1, determine the coordinates of the points of intersection of the line with the coordinate axes.
3. Find the coordinates of the point of intersection of the lines described in
   - **(a)** parts (a) and (c) of Exercise 1;
   - **(b)** parts (b) and (c) of Exercise 1;
   - **(c)** parts (a) and (d) of Exercise 1;
   - **(d)** parts (b) and (d) of Exercise 1.
4. For each of the lines described in Exercise 1, write an equation in slope-intercept form of the line which contains the point with coordinates (6, −7) and is perpendicular to the given line.

13.05 Chapter Summary

Vocabulary Summary

normal
  to a plane
  to a line in π
trace of a plane
π-coordinates of a point
slope of a line
point-slope equation for l
two-point equation for l
direction numbers
  of a line
  of l with respect to π
projecting planes for l
$\mathcal{E}$-coordinates of a point
slope-intercept equation for l
symmetric equations for l

Theorems

13-1. With respect to an orthonormal coordinate system, an equation:

$$\text{(i)}\qquad (x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0$$

or:

$$\text{(ii)}\qquad x_1m_1 + x_2m_2 + x_3m_3 = c,$$

where $(m_1, m_2, m_3) \neq (0, 0, 0)$, describes a plane whose normals have the direction numbers (m_1, m_2, m_3); and, any such plane can be described by equations like (i) and, also, by equations like (ii).

Answers for Part G

1.	(a) $y = \frac{5}{6}x - \frac{9}{2}$	(b) $y = 8$	
	(c) $y = \frac{1}{5}x + \frac{3}{5}$	(d) $y = \frac{6}{5}x + 6$	
2.	(a) (27/5, 0), (0, −9/2)	(b) (0, 8)	
	(c) (−3, 0), (0, 3/5)	(d) (−5, 0), (0, 6)	
3.	(a) (153/19, 42/19)	(b) (37, 8)	
	(c) (−315/11, −312/11)	(d) (5/3, 8)	
4.	(a) $y = -\frac{6}{5}x + \frac{1}{5}$	(b) [This has no equation in slope-intercept form.]	
	(c) $y = -5x + 23$	(d) $y = -\frac{5}{6}x - 2$	

13-2. If, with respect to an orthonormal basis for $\mathscr{T}$, the components of the linearly independent vectors $\vec{p}$ and $\vec{q}$ are (p_1, p_2, p_3) and (q_1, q_2, q_3), respectively, then $[\vec{p}, \vec{q}]^{\perp} = [\vec{m}]$, where $\vec{m}$ is the vector whose components are

$$\left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}\right).$$

13-3. If, with respect to an orthonormal basis for $\mathscr{T}$, the components of the non-$\vec{0}$ vector $\vec{m}$ are (m_1, m_2, m_3) then $[\vec{m}]^{\perp} = [\vec{p}_1, \vec{p}_2, \vec{p}_3]$, where the components of $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ are $(0, m_3, -m_2)$, $(-m_3, 0, m_1)$, and $(m_2, -m_1, 0)$, respectively.

13-4. If $(\vec{p}, \vec{q}, \vec{r})$ is any basis for $\mathscr{T}$ and $\vec{i} = \vec{p}/\|\vec{p}\|$, $\vec{j} = \vec{q}_1/\|\vec{q}_1\|$, $\vec{k} = \vec{r}_1/\|\vec{r}_1\|$, where $\vec{q}_1 = \vec{q} - \vec{i}(\vec{q} \cdot \vec{i})$ and $\vec{r}_1 = \vec{r} - \vec{i}(\vec{r} \cdot \vec{i}) - \vec{j}(\vec{r} \cdot \vec{j})$, then $(\vec{i}, \vec{j}, \vec{k})$ is an orthonormal basis for $\mathscr{T}$ such that $[\vec{i}] = [\vec{p}]$ and $[\vec{i}, \vec{j}] = [\vec{p}, \vec{q}]$. Moreover, $\vec{p} \cdot \vec{i}$, $\vec{q} \cdot \vec{j}$, and $\vec{r} \cdot \vec{k}$ are positive.

Corollary. $\|\vec{q}_1\|^2 = \begin{vmatrix} \vec{p} \cdot \vec{p} & \vec{p} \cdot \vec{q} \\ \vec{q} \cdot \vec{p} & \vec{q} \cdot \vec{q} \end{vmatrix} / (\vec{p} \cdot \vec{p})$, and

$$\|\vec{r}_1\|^2 = \begin{vmatrix} \vec{p} \cdot \vec{p} & \vec{p} \cdot \vec{q} & \vec{p} \cdot \vec{r} \\ \vec{q} \cdot \vec{p} & \vec{q} \cdot \vec{q} & \vec{q} \cdot \vec{r} \\ \vec{r} \cdot \vec{p} & \vec{r} \cdot \vec{q} & \vec{r} \cdot \vec{r} \end{vmatrix} \Bigg/ \begin{vmatrix} \vec{p} \cdot \vec{p} & \vec{p} \cdot \vec{q} \\ \vec{q} \cdot \vec{p} & \vec{q} \cdot \vec{q} \end{vmatrix}$$

Chapter Test

Assume that all coordinates of points and components of vectors are given with respect to an orthonormal coordinate system.

1. (a) Let π be the plane which contains the point whose coordinates are $(5, 9, -8)$ and whose normals have direction numbers $(-2, 3, 2)$. Write an equation which describes π.
   (b) Does the plane π, described in (a), contain the origin? Explain your answer.
   (c) Let B be the point whose coordinates are $(-3, -4, -5)$. Compute the distance from B to π.
2. Suppose that σ_1 and σ_2 are planes which are described by these equations:

$$\begin{array}{rl} 3x_1 - 4x_2 - 8x_3 = 7 & [\text{for } \sigma_1] \\ -4x_1 + 5x_2 - 4x_3 = 5 & [\text{for } \sigma_2] \end{array}$$

   (a) Show that $\sigma_1 \perp \sigma_2$.
   (b) Compute direction numbers for the line $\sigma_1 \cap \sigma_2$.
   (c) Compute the coordinates of the point of intersection of the line $\sigma_1 \cap \sigma_2$ and the plane which is described by the equation '$x_3 = -1$'.

Answers for Chapter Test

1. (a) $(x_1 - 5) \cdot -2 + (x_2 - 9) \cdot 3 + (x_3 + 8) \cdot 2 = 0$
   [or: $2x_1 - 3x_2 - 2x_3 = -1$]
   (b) No, for $2 \cdot 0 - 3 \cdot 0 - 2 \cdot 0 = 0 \neq -1$.
   (c) P is a point of the normal to π which contains B if and only if P has coordinates $(-3 - 2r, -4 + 3r, -5 + 2r)$, for some r. So, the foot of the perpendicular from B to π is such that $2(-3 - 2r) - 3(-4 + 3r) - 2(-5 + 2r) = -1$ for some r. Since the latter is the case if and only if $r = 1$, the foot of the perpendicular to π from B has coordinates $(-5, -1, -3)$. So, the distance from B to π is $\sqrt{17}$. $[\sqrt{(-3+5)^2 + (-4+1)^2 + (-5+3)^2}]$ [Alternatively, an equation of the plane parallel to π which cont[illegible] $(-3, -4, -5)$ is '$2x_1 - 3x_2 - 2x_3 = 16$'. Since $16 > 0$ a[illegible] 0 the origin is between this plane and π. Sin[illegible] distances between these planes and the origin are $16/\sqrt{17}$ and $1/\sqrt{17}$, the distance between the planes is $16/\sqrt{17} + 1/\sqrt{17}$ — that is, $\sqrt{17}$. (See page 1[illegible]4.)]
2. (a) Since $3 \cdot -4 + -4 \cdot 5 + -8 \cdot -4 = 0$, $\sigma_1 \perp \sigma_2$. [See Exercise 5 of Part F on page 115.]
   (b) Since $\left(\begin{vmatrix} -4 & -8 \\ 5 & -4 \end{vmatrix}, \begin{vmatrix} -8 & 3 \\ -4 & -4 \end{vmatrix}, \begin{vmatrix} 3 & -4 \\ -4 & 5 \end{vmatrix}\right) = (56, 44, -1)$, $\sigma_1 \cap \sigma_2$ has direction numbers $(56, 44, -1)$. [By Theorem 10-16.]
   (c) $(1, 1, -1)$ [To find x_1 and x_2 solve '$3x_1 - 4x_2 + 8 = 7$ and $-4x_1 + 5x_2 + 4 = 5$'.]

3. If $\vec{p}$ has components (p_1, p_2, p_3) then $\vec{p} \in [\vec{m}]^{\perp}$ if and only if $6p_1 + 2p_2 - 5p_3 = 0$. So, in particular the vectors $\vec{p}_1$ and $\vec{p}_2$ with components $(1, -3, 0)$ and $(5, 0, 6)$ belong to $[\vec{m}]^{\perp}$. Since $(\vec{p}_1, \vec{p}_2)$ is obviously linearly independent and since $[\vec{m}]^{\perp}$ is a bidirection, $[\vec{m}]^{\perp} = [\vec{p}_1, \vec{p}_2]$. [Students may invoke Theorem 14-3.]
4. (a) The distance between π_1 and the origin is $10/\sqrt{9 + 16 + 25} = \sqrt{2}$; that between π_2 and the origin is $26/\sqrt{169 + 25 + 144} = \sqrt{2}$. So, π_1 and π_2 are equidistant from the origin.
   (b) $3 \cdot 13 + 4 \cdot 5 - 5 \cdot 12 = -1 \neq 0$. So, $\pi_1 \not\perp \pi_2$.
   (c) $x_1 = 5 + 73r$, $x_2 = -2 - 101r$, $x_3 = -3 - 37r$ [Find direction numbers of $\pi_1 \cap \pi_2$ as in Exercise 2(b).]

3. Suppose that $\vec{m}$ has components (6, 2, −5). Give the components of two vectors—say, $\vec{p}_1$ and $\vec{p}_2$—such that $[\vec{m}]^\perp = [\vec{p}_1, \vec{p}_2]$. You must, of course, show that $(\vec{p}_1, \vec{p}_2)$ is linearly independent and that $\vec{m}$ is orthogonal to both $\vec{p}_1$ and $\vec{p}_2$.
4. Suppose that π_1 and π_2 are the two planes described by these equations:

$$\begin{array}{rl} 3x_1 + 4x_2 - 5x_3 = 10 & [\text{for } \pi_1] \\ 13x_1 + 5x_2 + 12x_3 = 26 & [\text{for } \pi_2] \end{array}$$

   (a) Show that π_1 and π_2 are equidistant from the origin.
   (b) Determine whether or not $\pi_1 \perp \pi_2$.
   (c) Write parametric equations for the line which is parallel to $\pi_1 \cap \pi_2$ and contains the point whose coordinates are (5, −2, −3).
5. Suppose that $\vec{p}$ and $\vec{q}$ have components (5, −3, 2) and (−10, 6, 5), respectively, and that A has coordinates (1, 2, −3).
   (a) Show that $(\vec{p}, \vec{q})$ is linearly independent.
   (b) Write an equation for the plane through A with bidirection $[\vec{p}, \vec{q}]$.
   (c) Write parametric equations for the line through A which is perpendicular to the plane described in (b).
6. Suppose that (−5, 14, −13) are direction numbers for a line l and that (2, 7, −2) and (3, 0, 3) are the components of the terms of a basis for the bidirection of a plane π. Show that $l \parallel \pi$.

Background Topic

In Volume 1 [Section 5.06 on pages 206 through 207] we discussed a 2-dimensional vector space whose vectors—called *measure vectors*—are the ordered pairs of real numbers. In this space $\mathcal{R} \times \mathcal{R}$ of measure vectors addition and multiplication by a real number were defined by:

$$(1) \qquad (a, b) + (c, d) = (a + c, b + d)$$

and

$$(2) \qquad (a, b)c = (ac, bc)$$

We defined $\vec{0}$ [for this vector space] and oppositing of measure vectors by:

$$(3) \qquad \vec{0} = (0, 0) \quad \text{and:} \quad -(a, b) = (-a, -b)$$

With these four definitions it was easy to see that Postulates 4_0(a)–(d) and 4_1–4_8 are satisfied. For example, as to Postulate 4_8,

Answers for Chapter Test [cont.]

5. (a) Since $\left(\begin{vmatrix} -3 & 2 \\ 6 & 5 \end{vmatrix}, \begin{vmatrix} 2 & 5 \\ 5 & -10 \end{vmatrix}, \begin{vmatrix} 5 & -3 \\ -10 & 6 \end{vmatrix}\right) = (-27, -45, 0)$

   $\neq (0, 0, 0)$, $(\vec{p}, \vec{q})$ is linearly independent.

   (b) $(x_1 - 1)\cdot 3 + (x_2 - 2)\cdot 5 + (x_3 + 3)\cdot 0 = 0$ [or: $3x_1 + 5x_2 = 13$] [By part (a) any normal to the plane in question has direction numbers (−27, −45, 0). So, it also has direction numbers (3, 5, 0).]

   (c) $x_1 = 1 + 3r$, $x_2 = 2 + 5r$, $x_3 = -3$ [In this, (3, 5) may be replaced by any proportional pair of numbers — say, (−27, −45).]

6. We need to show that the three vectors with the given triples as components are linearly dependent. This is the case because

$$\begin{vmatrix} -5 & 14 & -13 \\ 2 & 7 & -2 \\ 3 & 0 & 3 \end{vmatrix} = 0.$$

BACKGROUND TOPIC

This background work constitutes a short introduction to complex numbers through the use of the quadratic formula to solve quadratic equations with real number coefficients and real or complex roots. This is not needed in the sequel but is included for teachers whose students need to learn about the subject at this time. Although not needed in the sequel, the subject is closely related to the study of 2-dimensional vector spaces and plane geometry.

Several interesting structures can be imposed on the set $\mathcal{R} \times \mathcal{R}$ of all ordered pairs of real numbers. For example, we have seen at the end of Chapter 5 that $\mathcal{R} \times \mathcal{R}$ becomes a vector space when, for ordered pairs, addition, multiplication by real numbers, $\vec{0}$, and oppositing are defined as in (1) - (3). In this context we have called the ordered pairs of real numbers measure vectors since they can be used as measures of velocities, forces, plane displacements, etc. much as real numbers are used to measure, for example, masses. The resulting vector space is easily seen to be 2-dimensional. For example, the measure vectors (1, 0) and (0, 1) form a basis since they obviously span $\mathcal{R} \times \mathcal{R}$ and are linearly independent. [$(1, 0)a + (0, 1)b = (a, b)$ and $(a, b) = \vec{0}$ if and only if $a = 0 = b$.]

The space $\mathcal{R} \times \mathcal{R}$ of measure vectors becomes an inner product space when we define dot multiplication by (4). Alternatively, the space $\mathcal{R} \times \mathcal{R}$ of measure vectors becomes the algebra of complex numbers when we define [cross] multiplication by (5) on page 143. [An algebra is a vector space together with a definition of multiplication (... × ...) which is distributive "both ways" with respect to addition and which behaves with respect to multiplication by real numbers [(2)] in such a way that $(\vec{a} \times \vec{b})c = (\vec{a}c) \times \vec{b} = \vec{a} \times (\vec{b}c)$. In an algebra, multiplication of vectors need not be either commutative or associative.] When we choose (5) — rather than (4) — as our definition of multiplication we speak of the members of $\mathcal{R} \times \mathcal{R}$ as complex numbers — rather than as measure vectors. [In Chapter 19 we shall impose still a third structure on $\mathcal{R} \times \mathcal{R}$ under which it will be a Euclidean plane. Then we shall call the members of $\mathcal{R} \times \mathcal{R}$ "points".]

$$[(a, b)c]d = (ac, bd)d = ((ac)d, (bc)d) = (a(cd), b(cd)) = (a, b)(cd).$$

If we decide, as we shall, to take $(\overrightarrow{u}, \overrightarrow{v})$ where $\overrightarrow{u} = (1, 0)$ and $\overrightarrow{v} = (0, 1)$ as an orthonormal basis for the space of 2-dimensional measure vectors, we can define dot multiplication by:

$$(4) \qquad (a, b) \cdot (c, d) = ac + bd$$

[See Theorem 11-12.]

Exercises

Part A

1. Compute.
   (a) $(2, -3) + (-4, 5)$ (b) $(6, 8) + (0, 0)$ (c) $(-4, 6) + (4, -6)$
   (d) $(7, \frac{9}{2})2$ (e) $(6, -5)0$ (f) $(4, 2) \cdot (-3, 5)$
2. Show that, under our definitions, operations on measure vectors satisfy Postulates 4_4 and 4_6. [*Hint:* To save letters and make it easier to follow your computations, let $\overrightarrow{a} = (a_1, a_2)$ and $\overrightarrow{b} = (b_1, b_2)$.]
3. Show that the defined operations on measure vectors satisfy Postulates 4_0(e) and 4_{11}–4_{14}. [Use notation like that suggested for Exercise 2.]
4. (a) Show that $(a, b) \perp (-b, a)$.
   (b) Show that $\overrightarrow{u}$ and $\overrightarrow{v}$ are, as claimed above, orthogonal unit vectors.
5. Show that $(\overrightarrow{u}, \overrightarrow{v})$ does form a basis for the measure vector space $\mathscr{R} \times \mathscr{R}$ by showing that it is linearly independent and that, for any a and b, $(a, b) = \overrightarrow{u}a + \overrightarrow{v}b$.

*

It is possible—and very useful—to define another kind of multiplication in the space $\mathscr{R} \times \mathscr{R}$ of measure vectors. For the time being we shall use '×' to refer to this operation and define it by:

$$(5) \qquad (a, b) \times (c, d) = (ac - bd, ad + bc)$$

Notice that this is a way to multiply two measure vectors so as to obtain a product which is also a measure vector, rather than a real number. When this kind of multiplication is used, the earlier dot multiplication turns out to be not very important and one considers only the operations defined by (1), (2), (3), and (5). In this case it is customary to speak of $\mathscr{R} \times \mathscr{R}$ with these operations as *the complex number system* and to denote $\overrightarrow{u}$ and $\overrightarrow{v}$ by '*1*' and '*i*'. Using this notation it follows as in Exercise 5 of Part A that $(a, b) = 1a + ib$. Note that $1a = (a, 0)$ and ib is $(0, b)$. It is also customary to denote '(0, 0)' by '*0*' rather than by '$\overrightarrow{0}$'.

Answers for Part A

1. (a) $(-2, 2)$ (b) $(6, 8)$ (c) $(0, 0)$
   (d) $(14, 9)$ (e) $(0, 0)$ (f) -2

2. 4_4: $(a_1, a_2)+(b_1, b_2) = (a_1+b_1, a_2+b_2) = (b_1+a_1, b_2+a_2) = (b_1, b_2)+(a_1, a_2)$
   4_6: $(a_1, a_2)(b + c) = (a_1(b + c), a_2(b + c)) = (a_1b + a_1c, a_2b + a_2c)$
   $= (a_1b, a_2b) + (a_1c, a_2c) = (a_1, a_2)b + (a_1, a_2)c$

3. 4_0(e): $(a, b) \cdot (c, d) = ac + bd \in \mathcal{R}$ by 4_0(d) and 4_0(a).
   4_{11}: $(a, b) \cdot (a, b) = a^2 + b^2 \neq 0$ for $a \neq 0$ or $b \neq 0$ [by Exercise 1 on page 7].
   4_{12}: $[(a_1, a_2) + (b_1, b_2)] \cdot (c_1, c_2) = (a_1 + b_1, a_2 + b_2) \cdot (c_1, c_2)$
   $= (a_1 + b_1)c_1 + (a_2 + b_2)c_2$
   $= (a_1c_1 + a_2c_2) + (b_1c_1 + b_2c_2)$
   $= (a_1, a_2) \cdot (c_1, c_2) + (b_1, b_2) \cdot (c_1, c_2)$
   4_{13}: $[(a_1, a_2)a] \cdot (b_1, b_2) = (a_1a, a_2a) \cdot (b_1, b_2) = (a_1a)b_1 + (a_2a)b_2$
   $= (a_1b_1 + a_2b_2)a = [(a_1, a_2) \cdot (b_1, b_2)]a$
   4_{14}: $(a_1, a_2) \cdot (b_1, b_2) = a_1b_1 + a_2b_2 = b_1a_1 + b_2a_2 = (b_1, b_2) \cdot (a_1, a_2)$

4. (a) Since $(a, b) \cdot (-b, a) = a \cdot -b + ba = -(ab) + (ab) = 0$, $(a, b) \perp (-b, a)$.
   (b) $\vec{u} \cdot \vec{v} = (1, 0) \cdot (0, 1) = 1 \cdot 0 + 0 \cdot 1 = 0$;
   $\vec{u} \cdot \vec{u} = (1, 0) \cdot (1, 0) = 1 \cdot 1 + 0 \cdot 0 = 1$;
   $\vec{v} \cdot \vec{v} = (0, 1) \cdot (0, 1) = 0 \cdot 0 + 1 \cdot 1 = 1$.

5. $\vec{u}a + \vec{v}b = (1, 0)a + (0, 1)b = (a, 0) + (0, b) = (a, b)$, and $(a, b) = \vec{0}$ if and only if $a = 0 = b$. [This proves, at the same time, that $(\vec{u}, \vec{v})$ spans $\mathcal{R} \times \mathcal{R}$ and that $(\vec{u}, \vec{v})$ is linearly independent.]

The definition (5) can be motivated by showing that the only definitions of multiplication under which multiplication is commutative, associative, and distributive with respect to addition, and have two other desirable properties [see below] are of the form:

$$(5_0) \qquad (a, b) \times (c, d) = (ac + (bd)k, ad + bc + (bd)\ell)$$

where ℓ and k are given real numbers such that $\ell^2 + 4k > 0$. The desirable properties are that $(a, b) \times (c, 0) = (a, b)c$ and that there is a definition of reciprocating such that, for $(a, b) \neq \underline{0}$, $(a, b) \times /(a, b) = \underline{1}$. [It is for this last property that we need the restriction '$\ell^2 + 4k > 0$'.] From this result one can attain to (5) in many ways. The briefest is by noting that the choice of $k = -1$ and $\ell = 0$ yields the simplest suitable definition.

Although the preceding justification is too long for the text, or for general class use, we shall sketch it for the possible use of interested and bright students. To begin with, one uses the second of the "other desirable properties" together with the assumed associativity of multiplication to show that $[(a, b)e] \times [(c, d)f] = [(a, b) \times (c, d)](ef)$ and that $(a, b) \times \underline{1} = (a, b) = \underline{1} \times (a, b)$. Next, using these results, noting that $(a, b) = \underline{1}a + \underline{i}b$ and $(c, d) = \underline{1}c + \underline{i}d$, and using the distributivity [both ways] of multiplication over addition and the usual properties of addition, one shows that

$$(a, b) \times (c, d) = \underline{1}(ac) + \underline{i}(ad + bc) + (\underline{i} \times \underline{i})(bd).$$

Assuming that $\underline{i} \times \underline{i} = (k, \ell)$ leads at once to (5_0).

To define reciprocating one must be able to solve the equation '$(a, b) \times (c, d) = \underline{1}$' for '$c$' and '$d$' unless $a = 0 = b$. This equation is equivalent to the system:

$$\begin{aligned} ac + (bk)d &= 1 \\ bc + (a + b\ell)d &= 0 \end{aligned}$$

which, we know, is solvable if and only if the determinant $a(a + b\ell) - b(bk) \neq 0$. This is the case if and only if $a^2 + \ell(ab) - kb^2 \neq 0$. And, with some knowledge of quadratic equations, it is easy to see that this last is the case for $a \neq 0$ or $b \neq 0$ if and only if the discriminant $\ell^2 + 4k > 0$.

In discussing complex numbers it is customary to indicate multiplication either by juxtaposition of terms [as in: $(a, b)(c, d)$] or, since (4) is not used, by a '$\cdot$', as well as by a '×'. We shall refer to the operation defined by (5) as *multiplication of complex numbers*.

Part B

1. Compute.
   (a) $(2, 3) \times (-2, 1)$ (b) $(-2, 1) \times (2, 3)$ (c) $(7, -5) \times (1, 2)$
   (d) $(a, b) \times 1$ (e) $(a, b) \times 0$ (f) $(a, b) \times (a, -b)$
   (g) $(a, 0) \times (c, 0)$ (h) $(a, b) \times i$ (i) $i \times i$
2. (a) Compare your answer for Exercise 1(h) with the result you obtained in Exercise 4(a) of Part A.
   (b) Can you simplify your answer for Exercise 1(i) still further?
3. Show that multiplication of complex numbers is
   (a) commutative.
   *(b) distributive with respect to addition.
   *(c) associative.
   [For parts (b) and (c) use subscript notation as suggested for Exercise 2 of Part A.]

*

The results of Exercise 2, some of those of Exercise 1, and properties of the operations defined by (1) and (2), make it very easy to carry out multiplication of complex numbers without remembering the definition (5):

$$\begin{aligned}(a, b) \times (c, d) &= (1a + ib)(1c + id) && \text{[Why?]}\\ &= (1a)(1c + id) + (ib)(1c + id) && \text{[Why?]}\\ &= (1a)(1c) + (ib)(id) + (1a)(id) + (ib)(1c) && \text{[Why?]}\\ &= 1(ac) + -1(bd) + i(ad) + i(bc) && \text{[Why?]}\\ &= 1(ac - bd) + i(ad + bc) && \text{[Why?]}\\ &= (ac - bd, ad + bc). && \text{[Why?]}\end{aligned}$$

Evidently, what one needs principally to know is that multiplication is commutative and is distributive with respect to addition, that $1 \times 1 = 1$, that $i \times 1 = i$, and that $i \times i = -1$.

The procedure just illustrated can be made still simpler if we note that

$$(a, 0) + (b, 0) = (a + b, 0),\ (a, 0)(b, 0) = (ab, 0) = (a, 0)b,$$
$$\text{and } -(a, 0) = (-a, 0).$$

In words, the complex numbers with second component zero behave with respect to addition, multiplication, and oppositing just like their first components and multiplication of such a complex member by a real number is equivalent to multiplying by the "corresponding" complex number. Because of this, complex numbers with second com-

Answers for Part B

1. (a) $(-7, -4)$ (b) $(-7, -4)$ (c) $(17, 9)$ (d) (a, b) (e) $(0, 0)$
   (f) $(a^2 + b^2, 0)$ (g) $(ac, 0)$ (h) $(-b, a)$ (i) $(-1, 0)$
2. (a) Multiplying a given measure vector by $\underline{i}$ yields a measure vector orthogonal to the given one.
   (b) $-\underline{1}$
3. commutative: $(a_1, a_2) \times (b_1, b_2) = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1)$
   $= (b_1a_1 - b_2a_2, b_1a_2 + b_2a_1) = (b_1, b_2) \times (a_1, a_2)$
   distributive: $[(a_1, a_2) + (b_1, b_2)] \times (c_1, c_2)$
   $= (a_1 + b_1, a_2 + b_2) \times (c_1, c_2)$
   $= ([a_1 + b_1]c_1 - [a_2 + b_2]c_2, [a_1 + b_1]c_2 + [a_2 + b_2]c_1)$
   $= ([a_1c_1 - a_2c_2] + [b_1c_1 - b_2c_2], [a_1c_2 + a_2c_1] + [b_1c_2 + b_2c_1])$
   $= (a_1c_1 - a_2c_2, a_1c_2 + a_2c_1) + (b_1c_1 - b_2c_2, b_1c_2 + b_2c_1)$
   $= (a_1, a_2) \times (c_1, c_2) + (b_1, b_2) \times (c_1, c_2)$
   associative: $[(a_1, a_2) \times (b_1, b_2)] \times (c_1, c_2)$
   $= ([a_1b_1 - a_2b_2]c_1 - (a_1b_2 + a_2b_1)c_2, [a_1b_1 - a_2b_2]c_2 + [a_1b_2 + a_2b_1]c_1)$
   $= (a_1[b_1c_1 - b_2c_2] - a_2[b_2c_1 + b_1c_2], a_1[b_1c_2 + b_2c_1] + a_2[b_1c_1 - b_2c_2])$
   $= (a_1, a_2) \times [(b_1, b_2) \times (c_1, c_2)]$
   [Several (perhaps 4) steps have been omitted from the proof of associativity.]

The answers for the 'Why?'s are: Exercise 5 of Part A; distributivity; left distributivity as well as associativity of addition; properties of addition and of multiplication by real numbers as well as the fact that $\underline{i}^2 = -\underline{1}$ [Exercise 1(i) of Part B]; associativity of addition and properties of multiplication by real numbers; Exercise 5 of Part A.

Complex numbers are often written in the form '$a + b\underline{i}$' rather than '$a + \underline{i}b$'. You may do as you choose. We shall, in the text, maintain the convention that vectors are multiplied by real numbers on the right.

ponent zero are called *real-complex numbers.* Since the arithmetic of these real-complex numbers is the same as the arithmetic of the corresponding real numbers which are their first components it is customary to pretend that the real-complex numbers are the same as the real numbers. This pretense is, of course, a fiction, but it is a harmless and useful one. It leads us to agree that

$$(a, b) = a + ib$$

and gives us the rules:

$$\begin{array}{ll} (1') & (a + ib) + (c + id) = (a + c) + i(b + d) \\ (2') & (a + ib)c = ac + i(bc) \\ (3') & -(a + ib) = -a - ib \\ (5') & (a + ib) \times (c + id) = (ac - bd) + i(ad + bc) \end{array}$$

The last rule is easy to recall by thinking of distributivity, commutativity, etc., and the fact that $i^2 = -1$.

Part C

1. Compute.

(a) $(3 + i7) + (-9 + i6)$ **(b)** $(9 - i2) + (7 - i3)$
(c) $(8 + i) + (5 - i)$ **(d)** $(6 - i5) - (7 - i2)$
(e) $(3 + i6) - (5 + i4)$ **(f)** $(7 - i) - (7 - i)$
(g) $(2 + i5) \times (6 + i7)$ **(h)** $(-2 + i7)(5 - i4)$
(i) $(3 + i2)(-8 - i3)$ **(j)** $(5 - i7)(6 + i5)$
(k) $(3 + i4)(3 + i4)$ **(l)** $(7 + i5)^2$
(m) $(5 + i0)(3 + i8)$ **(n)** $9(6 + i11)$
(o) $2i(3 + i5)$ **(p)** $(3 + i2)(5 + i4)(2 - i3)$
(q) $(8 + i3)[(2 + i5) - (7 - i3)]$ **(r)** $(3 + i4)(3 - i4)$
(s) $(1 + i)(1 - i)$ **(t)** $(2 - i3)(2 + i3)$

2. For each of the given complex numbers find another so that the product will be a nonzero real [complex] number.

(a) $1 - i$ **(b)** $9 + i4$ **(c)** $3 + i7$ **(d)** $6 - i5$
(e) $7 + i$ **(f)** $6 + i5$ **(g)** $2 + i3$ **(h)** $a + ib$

[*Hint*: The last three parts of Exercise 1 may help. To see what is going on, consider the problem of finding, for $3 + i4$, a nonzero complex number $a + ib$ such that $(3 + i4)(a + ib) = c + i0$ where $c \neq 0$. This problem reduces to finding real numbers a and b, not both 0, such that $3b + 4a = 0$. [Explain.] One choice, then, is given by '$a = 3$, $b = -4$'. (What other choices are there?)]

3. Pairs of complex numbers like $3 + i4$ and $3 - i4$ are called pairs of *conjugate* complex numbers. By definition, the conjugate of $a + ib$ is $a - ib$; for short, $\overline{a + ib} = a - ib$. [Read '$\overline{}$' as you would 'the conjugate of'.] Compute.

(a) $\overline{7 + i2}$ **(b)** $\overline{7 - i2}$ **(c)** $\overline{9}$ **(d)** $\overline{i2}$

4. Show that $(a + ib) \times \overline{a + ib}$ is a real [complex] number.

Answers for Part C

1. (a) $-6 + \underline{i}13$ (b) $16 - \underline{i}5$ (c) 13 [or: $13 + \underline{i}0$]
(d) $-1 - \underline{i}3$ (e) $-2 + \underline{i}2$ (f) $\underline{0}$ [or: $0 + \underline{i}0$]
(g) $-23 + \underline{i}44$ (h) $18 + \underline{i}43$ (i) $-18 - \underline{i}25$
(j) $65 - \underline{i}17$ (k) $-7 + \underline{i}24$ (ℓ) $24 + \underline{i}70$
(m) $15 + \underline{i}40$ (n) $54 + \underline{i}99$ (o) $-10 + \underline{i}6$
(p) $80 + \underline{i}23$ (q) $-64 + \underline{i}49$ (r) 25
(s) 2 (t) 13

2. (a) $1 + \underline{i}$ (b) $9 - \underline{i}4$ (c) $3 - \underline{i}7$ (d) $6 + \underline{i}5$
(e) $7 - \underline{i}$ (f) $6 - \underline{i}5$ (g) $2 - \underline{i}3$ (h) $a - \underline{i}b$

[The result of multiplying any of the answers given for Exercise 2 by a nonzero real number will also be a correct answer. The given answers are, however, the "desired" ones. Following out the hint, note that $(3 + \underline{i}4)(a + \underline{i}b) = (3a - 4b) + \underline{i}(3b + 4a)$ and, so, is a nonzero real complex number if and only if $3a - 4b \neq 0$ and $3b + 4a = 0$. The solutions of the latter equation are given by $(a, b) = (3k, -4k)$ for any real number k. The pair $(3, -4)$ obtained for $k = 1$ also satisfies the inequation. $(3 \cdot 3 - 4 \cdot -4 \neq 0)$]

3. (a) $7 - \underline{i}2$ (b) $7 + \underline{i}2$ (c) 9 [or: $9 + \underline{i}0$] (d) $-\underline{i}2$

4. $(a + \underline{i}b) \times \overline{a + \underline{i}b} = (a + \underline{i}b) \times (a - \underline{i}b) = a^2 + b^2$. [The product is, actually, the real complex number $(a^2 + b^2, 0)$, but we are pretending that such a number is the same as the real number which is its first component.]

5. Show that
   (a) $\overline{(a + ib) + (c + id)} = \overline{a + ib} + \overline{c + id}$, and
   (b) $\overline{(a + ib) \times (c + id)} = \overline{a + ib} \times \overline{c + id}$.
6. Find a and b such that $(3 + i4)(a + ib) = 1$ [*Hint*: Your work with conjugates may suggest how.]

*

At the beginning of these exercises we noticed that $\mathscr{R} \times \mathscr{R}$ is an inner product space under the definitions (1) – (4) on pages 142 and 143. In particular, each ordered pair of real numbers has, as a vector, a norm such that

$$(*) \qquad \|(a, b)\|^2 = a^2 + b^2. \qquad \text{[Explain.]}$$

In your work with conjugates you have found that, also,

$$(**) \qquad (a + ib) \times \overline{a - ib} = a^2 + b^2.$$

This suggests that we define the norm—or, more usually, *the absolute value* of a complex number by:

$$(6) \qquad |a + ib| = \sqrt{a^2 + b^2}$$

and record the theorem:

$$|a + ib|^2 = (a + ib) \times \overline{a + ib}$$

It follows from (**) that, if we define reciprocating for complex numbers by:

$$(7) \qquad /(a + ib) = \overline{a + ib}/(a^2 + b^2) = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$$

then, for $a + ib \neq 0$,

$$(a + ib) \times /(a + ib) = 1.$$

This, in addition to other properties of complex numbers which we have previously established shows that, with our definitions of addition and multiplication, oppositing, and reciprocating, $0[(0, 0)]$ and $1[(1, 0)]$, the complex numbers *form a field*. In particular all the properties of real numbers and the operations on them which follow from Postulates 5_0(a) – (f) and 5_1 – 5_7 hold equally well for the complex numbers and the corresponding operations on them.

Answers for Part C [cont.]

5. (a) $\overline{(a + \underline{i}b) + (c + \underline{i}d)} = \overline{(a + c) + \underline{i}(b + d)} = (a + c) - \underline{i}(b + d)$
   $= (a - \underline{i}b) + (c - \underline{i}d) = \overline{a + \underline{i}b} + \overline{c + \underline{i}d}$.
   (b) $\overline{(a + \underline{i}b) \times (c + \underline{i}d)} = \overline{(ac - bd) + \underline{i}(ad + bc)} = (ac - bd) - \underline{i}(ad + bc)$;
   $\overline{a + \underline{i}b} \times \overline{c + \underline{i}d} = (a - \underline{i}b) \times (c - \underline{i}d) = (ac - bd) - \underline{i}(ad + bc)$
6. $a = 3/25$, $b = -4/25$ [From our work with conjugates we know that $(3 + \underline{i}4)(3 - \underline{i}4) = 25$.]

*

Note that the right side of (*) is '$a^2 + b^2$' by definition but that the right side of (**) is '$a^2 + b^2$' by definition and by our agreement to identify each real complex number with its first component.

Part D

1. Compute [that is, put in the form '$a + ib$' (or '$a - ib$')].
   - (a) $/(3 + i4)$ (b) $/(4 + i2)$
   - (c) $/i$ (d) $/(i6)$
   - (e) $(5 + i7) \div (7 - i3)$ (f) $(3 + i4) \div (-2 - i5)$
   - (g) $\dfrac{9 + i3}{6 + i5}$ (h) $\dfrac{2 - i5}{i3}$
2. Show that, for any complex numbers z_1 and z_2,
   - (a) $|z_1z_2| = |z_1| \cdot |z_2|$ [*Hint*: Exercise 5(b) of Part C will be helpful.]
   - (b) $|z_1 + z_2| \leq |z_1| + |z_2|$ [*Hint*: Recall that the absolute value of a member of $\mathscr{R} \times \mathscr{R}$, considered as a complex number, is the same as its norm when it is considered as a measure vector. Look ahead to Theorem 14-1 on page 152.]
3. Is it possible to find an order relation for the set of complex numbers with respect to which the complex number system is an ordered field—that is, so that Postulates $5_8 - 5_{12}$ are satisfied? [*Hint*: Recall that $i^2 = -1$.]
4. Show that for each real number $a < 0$, $(i\sqrt{-a})^2 = a$. How many complex numbers have -4 as their square? How many have 4 as their square?
5. Exercise 4 suggests that, for real [complex] numbers $x < 0$, we define:

$$\sqrt{x} = i\sqrt{-x}$$

where, since $-x > 0$, $\sqrt{-x}$ is the "ordinary" principal square root of x. You know that, for $a \geq 0$ and $b \geq 0$, $\sqrt{a}\sqrt{b} = \sqrt{ab}$. Find a similar result [$\sqrt{a}\sqrt{b} = ?$] in case $a < 0$ and $b < 0$.

6. Compute.
   - (a) $\sqrt{-9}$ (b) $\sqrt{-25}$ (c) $\sqrt{-2}$ (d) $\sqrt{-5}$

*

In the Background Topic at the end of Chapter 12 we studied quadratic functions. A function f is a quadratic function if there are real numbers a, b, and c, $a \neq 0$, such that, for each x,

$$(*) \qquad f(x) = ax^2 + bx + c.$$

We found a process which we called 'completing the square' which enabled us to represent any quadratic function f in a form:

$$(**) \qquad f(x) = a(x - p)^2 + q$$

where $p = -b/2a$ and $q = (4ac - b^2)/(4a)$.

Completing the square can be used in many kinds of problems. One, which we have not previously discussed, is that of finding the argu-

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Answers for Part D

1. (a) $\frac{3}{25} - \underline{i}\frac{4}{25}$ (b) $\frac{2}{\sqrt{5}} - \underline{i}\frac{1}{\sqrt{5}}$ (c) $0 - \underline{i}$ [or: $-\underline{i}$] (d) $-\underline{i}/6$
   (e) $\frac{7}{29} + \underline{i}\frac{32}{29}$ (f) $-\frac{26}{29} + \underline{i}\frac{7}{29}$ (g) $\frac{69}{61} - \underline{i}\frac{27}{61}$ (h) $-\frac{5}{3} - \underline{i}\frac{2}{3}$

   [Division problems like the above can be solved by computing the reciprocal of the divisor and multiplying the dividend by it. Alternatively, one may multiply both dividend and divisor by the conjugate of the divisor and simplify. This alternative procedure is somewhat more efficient than the first, and you may wish to give it to your students.]

2. (a) By the theorem following (6) we know that $|z_1z_2|^2 = (z_1z_2) \times (\overline{z_1z_2})$. By Exercise 5(b) of Part C, $\overline{z_1z_2} = \bar{z}_1 \times \bar{z}_2$. Substituting and using associativity and commutativity, we find that $|z_1z_2|^2 = (z_1\bar{z}_1) \times (z_2\bar{z}_2)$ $= |z_1|^2|z_2|^2 = (|z_1| \cdot |z_2|)^2$. Since absolute values are nonnegative it follows that $|z_1z_2| = |z_1| \cdot |z_2|$.

   (b) [The hint forces students to look ahead at a theorem they have not yet proved. This is good experience — mathematics is such a large field that one often has to accept theorems on faith.] $|z_1 + z_2| = \|z_1 + z_2\| \leq \|z_1\| + \|z_2\| = |z_1| + |z_2|$

3. No. In an ordered field squares are nonnegative and, in the complex number field both $\underline{1}$ and $-\underline{1}$ are squares. But, nonzero opposites cannot both be nonnegative [if $a > 0$ then $-a < 0$].

4. For $a < 0$, $-a > 0$ and, so, $(\underline{i}\sqrt{-a})^2 = \underline{i}^2(\sqrt{-a})^2 = -1 \cdot -a = a$.; There are two numbers, $\underline{i}2$ and $-\underline{i}2$, whose square is -4.; There are two numbers, 2 and -2, whose square is 4. [In any field a nonzero number which has a square root has two square roots and these are opposites of one another. For, in any field, $b^2 = a^2$ if and only if $b = a$ or $b = -a$.]

5. For $a < 0$ and $b < 0$, $\sqrt{a}\sqrt{b} = -\sqrt{ab}$. [For $(\underline{i}\sqrt{-a})(\underline{i}\sqrt{-b})$ $= \underline{i}^2\sqrt{-a \cdot -b} = -\sqrt{ab}$.]

6. (a) $\underline{i}3$ (b) $\underline{i}5$ (c) $\underline{i}\sqrt{2}$ (d) $\underline{i}\sqrt{5}$

*

You may wish to review briefly the procedure of completing the square before taking up the material on quadratic functions and quadratic equations.

ments [if there are any] for which the functions f of (*) [or of (**)] has the value zero. These arguments are called *zeros* of the function f; alternatively, they are called *roots*, or *solutions*, of the quadratic equation:

$$(\dagger) \qquad ax^2 + bx + c = 0$$

These zeros or roots are easily found if we complete the square and consider the equivalent equation:

$$(\dagger\dagger) \qquad a(x - p)^2 + q = 0$$

Transforming the latter we obtain:

$$(x - p)^2 = -q/a$$

Now, if $-q/a > 0$ we can carry the transformation a step further:

$$x - p = \sqrt{-q/a} \text{ or } x - p = -\sqrt{-q/a}$$

Substituting the expressions for 'p' and 'q' in terms of 'a', 'b', and 'c' we obtain:

$$x = -\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}} \text{ or } x = -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

and, finally:

$$(8) \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{[Explain.]}$$

Formula (8) gives us readily the zeros of a quadratic function f [as in (*)] or of a quadratic equation [as in (†)] as long as a, b, and c are real numbers such that $b^2 - 4ac > 0$. [It also works in case $b^2 - 4ac = 0$. Explain.]

The preceding could as well have been said in the Background Topic where we first studied quadratic functions. What is new here is that if we allow 'x' to have complex numbers as values we can still interpret (8) as giving zeros of the function f of (*), and roots of the equation (†), even in case $b^2 - 4ac < 0$. All we need do in this case is to transform '$\sqrt{b^2 - 4ac}$' into '$i\sqrt{4ac - b^2}$' in accord with the definition of Exercise 5 of Part D.

The sentence (8) which [for $a \neq 0$] is equivalent to the equation (†) is often simplified to:

$$(8') \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{[Read '}\pm\text{' as 'plus or minus'.]}$$

The explanation asked for in connection with (8) involves the fact that

$$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2|a|}.$$

Since, for $a > 0$, $|a| = a$, the alternation sentence preceding (8) reduces to (8) in case $a > 0$. In case $a < 0$, when $|a| = -a$, the sentence preceding (8) reduces to:

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

and this is equivalent to (8).

Be sure your students understand that, for example, the equation '$(x - 1)(x - 2) = 0$' is equivalent to the sentence '$x = 1$ or $x = 2$' and that this sentence has the numbers 1 and 2 as roots. [There is occasional confusion between 'or' and 'and' here. For example, some students will argue that if 1 and 2 are roots then '$x = 1$ and $x = 2$' is equivalent to the given equation — and, so that the equation implies that $1 = 2$.]

The quadratic formula works equally well if a, b, and c are arbitrary complex numbers, with $a \neq 0$. [But, unless a, b, and c are real (complex) numbers, the discriminant may not discriminate between real and complex (non-real) roots.] To apply the quadratic formula effectively in this general case one needs to know how to compute square roots of complex numbers. Teaching this is beyond our present aims.

and called *the quadratic formula*. Inspection of this formula shows that, for any real numbers a, b, and c, with $a \neq 0$, equation (†) has two roots if and only if $b^2 - 4ac \neq 0$ and has a single root [which has some pretensions of being "counted twice"] if $b^2 - 4ac = 0$. [The preceding holds, in fact, even if a, b, and c are complex numbers.] Moreover, the equation has two real roots if and only if $b^2 - 4ac > 0$ and has two complex roots if and only if $b^2 - 4ac < 0$. In the latter case, the roots are a pair of conjugate complex numbers. Because the quantity $b^2 - 4ac$ can be used, as above, to discriminate among the possible types of roots (†) may have, $b^2 - 4ac$ is called *the discriminant* of (†).

Part E

1. For each of the following quadratic equations, use its discriminant to determine the nature of its roots. Then, use (8) or (8′) to find these roots.
   - **(a)** $x^2 - 4x + 29 = 0$
   - **(b)** $x^2 - 2x - 15 = 0$
   - **(c)** $x^2 - 6x + 25 = 0$
   - **(d)** $x^2 + 16 = 0$
   - **(e)** $x^2 - 7x + 9 = 0$
   - **(f)** $x^2 + x + 1 = 0$
   - **(g)** $x^2 + 9x + 9 = 0$
   - **(h)** $x^2 - 4x + 4 = 0$
   - **(i)** $5x^2 + 6x - 9 = 0$
   - **(j)** $6x^2 - 7x + 1 = 0$
2. Show, for equation '$ax^2 + bx + c = 0$', that
   - **(a)** the sum of the roots is $-b/a$, and that
   - **(b)** the product of the roots is c/a.
3. **(a)** Solve:

$$s = v_0 t - \tfrac{1}{2}gt^2$$

for 't'. [When g is approximately 32, s feet is the height reached by a body t seconds after it has been thrown upward with a velocity of v_0 feet per second.]

(b) Simplify your solution in part (a) under the assumption that when $s = 0$, $t = 0$.

(c) How high does the body rise before it starts to fall back?

Answers for Part E

1. (a) $2 + \underline{i}5,\ 2 - \underline{i}5$

   (b) 5, −3 [Students should see that this equation could be solved by factoring.]

   (c) $3 + \underline{i}4,\ 3 - \underline{i}4$

   (d) $\underline{i}4,\ -\underline{i}4$

   (e) $(7 + \sqrt{13})/2,\ (7 - \sqrt{13})/2$

   (f) $(-1 + \underline{i}\sqrt{3})/2,\ (-1 - \underline{i}\sqrt{3})/2$

   (g) $(-9 + 3\sqrt{5})/2,\ (-9 - 3\sqrt{5})/2$

   (h) 2

   (i) $(-3 + 3\sqrt{6})/5,\ (-3 - 3\sqrt{6})/5$

   (j) 1, 1/6

2. (a) $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} = \dfrac{-2b}{2a} = -b/a$

   (b) $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} = \dfrac{b^2 - (b^2 - 4ac)}{4a^2} = \dfrac{4ac}{4a^2} = c/a$

3. (a) $t = \dfrac{v_0 + \sqrt{v_0^2 - 2gs}}{g}$ or $t = \dfrac{v_0 - \sqrt{v_0^2 - 2gs}}{g}$

   (b) $t = \dfrac{v_0 - \sqrt{v_0^2 - 2gs}}{g}$

   (c) $v_0^2/(2g)$

Chapter Fourteen

Distance

14.01 Two Useful Inequalities

It will turn out that Theorem 11-8—the Schwarz Inequality—is one of the more useful theorems concerning dot multiplication. According to this theorem, for any vectors $\vec{a}$ and $\vec{b}$,

(1) $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\|\,\|\vec{b}\|$, and

(2) $|\vec{a} \cdot \vec{b}| = \|\vec{a}\|\,\|\vec{b}\| \longleftrightarrow (\vec{a}, \vec{b})$ is linearly dependent.

This result can be stated in another way:

(3) $-(\|\vec{a}\|\,\|\vec{b}\|) \leq \vec{a} \cdot \vec{b} \leq \|\vec{a}\|\,\|\vec{b}\|$

(4) $(\vec{a}, \vec{b})$ is linearly dependent $\longleftrightarrow \|\vec{a}\|^2\|\vec{b}\|^2 = (\vec{a} \cdot \vec{b})^2$.

Notice that (4) can be stated conveniently by introducing a determinant and, in fact, so can the whole of Theorem 11-8. [Do this. Check your result with (2) on page 127.]

In the following exercises you will find a proof of the Schwarz Inequality which is different from that given for Theorem 11-8 and which yields additional information. You will also prove another important inequality theorem.

Exercises

Part A

In discussing the Schwarz Inequality it is sometimes convenient to consider two cases—the case in which $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, and the case in which $\vec{a} \neq \vec{0} \neq \vec{b}$.

1. (a) Show that (1), above, holds in the first case.

Comments on text: Recalling that $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$, and $(\vec{a} \cdot \vec{b})^2 = (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{a})$, we see that (4) can be restated as: $(\vec{a}, \vec{b})$ is linearly dependent $\Longleftrightarrow \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix} = 0$. Schwarz's inequality can be restated as: $\begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix} \geq 0$. Note that this agrees with (2) on page 127.

Here are suggestions for the exercises in section 14.01:

(i) Part A is best treated in class discussion.
(ii) Parts B and C can be used for homework except for Exercise 4 of Part C. It is best to do this exercise under teacher direction.
(iii) Exercise 2 of Part D should also be teacher directed.
(iv) The remainder of Part D, and Part E may be used either in class or as homework. These exercises should be easy.

Answers for Part A

1. (a) If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, then $|\vec{a} \cdot \vec{b}| = 0$ and $\|\vec{a}\|\,\|\vec{b}\| = 0$, so that (1) holds. [We have equality].

(b) Show that (2), above, holds in the first case. [*Hint*: Recalling some of the rules of logic from Volume 1, show that any sentence of the form '(*p* and *q*) ⟶ [*p* ⟷ *q*]' is a valid sentence.]

2. As an example of the second case, suppose that $\vec{a}$ and $\vec{b}$ are unit vectors – say, $\vec{u}$ and $\vec{v}$. State the corresponding instances of (3) and (4), above.
3. Assuming that $\vec{u}$ and $\vec{v}$ are unit vectors, prove:
   (a) $\vec{u} \cdot \vec{v} \leq 1$ and $[\vec{u} \cdot \vec{v} = 1 \longleftrightarrow \vec{u} = \vec{v}]$ [*Hint*: Compute $\|\vec{u} - \vec{v}\|^2$.]
   (b) $\vec{u} \cdot \vec{v} \geq -1$ and $[\vec{u} \cdot \vec{v} = -1 \longleftrightarrow \vec{u} = -\vec{v}]$ [*Hint*: Use the result of part (a).]
4. Use the results obtained in Exercise 3 to prove (1) in case $\vec{a} \neq \vec{0} \neq \vec{b}$.
5. Show that, for unit vectors $\vec{u}$ and $\vec{v}$, $\vec{u}$ and $\vec{v}$ have the same sense if and only if $\vec{u} = \vec{v}$. [*Hint*: Recall that $\vec{a}$ and $\vec{b}$ have the same sense if and only if $\vec{a}$ is the product of $\vec{b}$ by some positive number. Also, recall that $\|\vec{b}a\| = \|\vec{b}\|\,|a|$.]
6. Use the results obtained in Exercises 3 and 5 to prove:
   (a) $\vec{a} \cdot \vec{b} = \|\vec{a}\|\,\|\vec{b}\| \longleftrightarrow \vec{a}$ and $\vec{b}$ have the same sense $[\vec{a} \neq \vec{0} \neq \vec{b}]$
   (b) $\vec{a} \cdot \vec{b} = -(\|\vec{a}\|\,\|\vec{b}\|) \longleftrightarrow \vec{a}$ and $\vec{b}$ have opposite senses $[\vec{a} \neq \vec{0} \neq \vec{b}]$
7. Use the results of Exercise 6 to prove (2) in case $\vec{a} \neq \vec{0} \neq \vec{b}$.

*

From the preceding exercises it should be clear that Theorem 11–8 is easily derivable from the following:

> **Lemma** $\vec{a} \cdot \vec{b} \leq \|\vec{a}\|\,\|\vec{b}\|$, and
> $\vec{a} \cdot \vec{b} = \|\vec{a}\|\,\|\vec{b}\| \longleftrightarrow (\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$
> or $\vec{a}$ and $\vec{b}$ have the same sense)

[Explain.] The principal result used in establishing this lemma is that proved in Exercise 3(a). This, in turn, resulted from the fact that, for unit vectors $\vec{u}$ and $\vec{v}$,

$$\begin{aligned}\|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2(\vec{u} \cdot \vec{v}) \\ &= 2 - 2(\vec{u} \cdot \vec{v}) = 2(1 - \vec{u} \cdot \vec{v}),\end{aligned}$$

together with the fact that $\vec{0}$ is the only vector whose norm is 0.

This suggests that it may be worthwhile to investigate $\|\vec{a} - \vec{b}\|^2$ [and $\|\vec{a} + \vec{b}\|^2$] without the assumption that $\vec{a}$ and $\vec{b}$ are unit vectors. Doing so will, indeed, lead to another useful theorem.

Answers for Part A [cont.]

1. (b) Let p be the statement '$|\vec{a} \cdot \vec{b}| = \|\vec{a}\|\,\|\vec{b}\|$' and q be the statement '$(\vec{a}, \vec{b})$ is linearly dependent'. Each of p and q follows from '$\vec{a} = \vec{0}$' and from '$\vec{b} = \vec{0}$'. So, the conjunction of p and q follows from '$\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$'. The argument for this case of (2) is completed by noting the validity of the scheme:

$$\dfrac{\dfrac{\dfrac{p \text{ and } q}{p}}{q \Longrightarrow p} \qquad \dfrac{\dfrac{p \text{ and } q}{q}}{p \Longrightarrow q}}{p \Longleftrightarrow q}$$

2. $-1 \leq \vec{u} \cdot \vec{v} \leq 1$ and: $(\vec{u}, \vec{v})$ is linearly dependent $\Longleftrightarrow (\vec{u} \cdot \vec{v})^2 = 1$.

3. (a) $\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2(\vec{u} \cdot \vec{v})$. Since $\vec{u} \cdot \vec{u} = 1 = \vec{v} \cdot \vec{v}$ it follows that $\|\vec{u} - \vec{v}\|^2 = 2(1 - \vec{u} \cdot \vec{v})$. Since $\|\vec{u} - \vec{v}\|^2 \geq 0$ it follows that $\vec{u} \cdot \vec{v} \leq 1$. Also, $\|\vec{u} - \vec{v}\|^2 = 0$ if and only if $\vec{u} - \vec{v} = \vec{0}$ — that is, if and only if $\vec{u} = \vec{v}$. Hence, $\vec{u} = \vec{v}$ if and only if $\vec{u} \cdot \vec{v} = 1$.

   (b) Assuming that $\vec{u}$ and $\vec{v}$ are unit vectors, so are $\vec{u}$ and $-\vec{v}$. So, by part (a), $\vec{u} \cdot -\vec{v} \leq 1$ and $\vec{u} \cdot -\vec{v} = 1$ if and only if $\vec{u} = -\vec{v}$. Since $\vec{u} \cdot -\vec{v} = -(\vec{u} \cdot \vec{v})$ it follows that $\vec{u} \cdot \vec{v} \geq -1$ and $\vec{u} \cdot \vec{v} = -1$ if and only if $\vec{u} = -\vec{v}$.

4. Suppose that $\vec{a} \neq \vec{0} \neq \vec{b}$ and let $\vec{u} = \vec{a}/\|\vec{a}\|$ and $\vec{v} = \vec{b}/\|\vec{b}\|$. It follows that $\vec{u}$ and $\vec{v}$ are unit vectors. By Exercise 3, $|\vec{u} \cdot \vec{v}| \leq 1$ [see Exercise 10 of Part B on page 8] and, so, $|(\vec{a}/\|\vec{a}\|) \cdot (\vec{b}/\|\vec{b}\|)| \leq 1$. Hence $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\|\,\|\vec{b}\|$.

5. Let $\vec{u}$ and $\vec{v}$ be unit vectors. $\vec{u}$ and $\vec{v}$ have the same sense if and only if there is a positive number — say, c — such that $\vec{u} = \vec{v}c$. Since $\vec{v}1 = \vec{v}$, the latter is the case if $\vec{u} = \vec{v}$. On the other hand, if $\vec{u} = \vec{v}c$ then $\|\vec{u}\| = \|\vec{v}\|\,|c|$ and, since $\|\vec{u}\| = \|\vec{v}\| \neq 0$ and $c > 0$, $c = 1$, and $\vec{u} = \vec{v}$.

Answers for Part A [cont.]

6. (a) Suppose that $\vec{a} \neq \vec{0} \neq \vec{b}$ and let $\vec{u}$ and $\vec{v}$ be as in Exercise 4. As in this exercise, $\vec{a} \cdot \vec{b} = \|\vec{a}\|\,\|\vec{b}\|$ if and only if $\vec{u} \cdot \vec{v} = 1$. By Exercise 3(a), $\vec{u} \cdot \vec{v} = 1$ if and only if $\vec{u} = \vec{v}$ and, by Exercise 6(a), $\vec{u} = \vec{v}$ if and only if $\vec{u}$ and $\vec{v}$ have the same sense. Since $/\|\vec{a}\| > 0$ it follows that $\vec{u}$ and $\vec{a}$ have the same sense and, similarly, $\vec{v}$ and $\vec{b}$ have the same sense. Hence, for $\vec{a} \neq \vec{0} \neq \vec{b}$, $\vec{a} \cdot \vec{b} = \|\vec{a}\|\,\|\vec{b}\|$ if and only if $\vec{a}$ and $\vec{b}$ have the same sense.

   (b) In part (a), substitute '$-\vec{b}$' for '$\vec{b}$' and note that $\vec{a} \cdot -\vec{b} = -(\vec{a} \cdot \vec{b})$ and that, for $\vec{a} \neq \vec{0} \neq \vec{b}$, $\vec{a}$ and $-\vec{b}$ have the same sense if and only if $\vec{a}$ and $\vec{b}$ have opposite senses.

7. By Exercise 6, it follows, for $\vec{a} \neq \vec{0} \neq \vec{b}$, that $|\vec{a} \cdot \vec{b}| = \|\vec{a}\|\,\|\vec{b}\|$ if and only if $\vec{a}$ and $\vec{b}$ have the same sense or have opposite senses. If this is the case then $(\vec{a}, \vec{b})$ is certainly linearly dependent. On the other hand, since $\vec{a} \neq \vec{0} \neq \vec{b}$, if $(\vec{a}, \vec{b})$ is linearly dependent then $\vec{a}$ and $\vec{b}$ have the same or opposite senses.

*

Substituting, in the lemma, '$-\vec{b}$' for '$\vec{b}$' and "simplifying" yields:

$$\vec{a} \cdot \vec{b} \geq -\|\vec{a}\|\,\|\vec{b}\|, \text{ and}$$

$$\vec{a} \cdot \vec{b} = -\|\vec{a}\|\,\|\vec{b}\| \iff (\vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \text{ and } \vec{b} \text{ have opposite senses}).$$

From this and the lemma it follows at once that $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\|\,\|\vec{b}\|$ and that $|\vec{a} \cdot \vec{b}| = \|\vec{a}\|\,\|\vec{b}\|$ if and only if ($\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have the same sense or opposite senses). The latter is the case if and only if $(\vec{a}, \vec{b})$ is linearly dependent.

Part B

1. (a) Express $\|\overrightarrow{a} + \overrightarrow{b}\|^2$ in terms of '$\|\overrightarrow{a}\|$', '$\|\overrightarrow{b}\|$', and '$\overrightarrow{a} \cdot \overrightarrow{b}$'.
   (b) Prove that $\|\overrightarrow{a} + \overrightarrow{b}\| \leq \|\overrightarrow{a}\| + \|\overrightarrow{b}\|$, with equality if and only if $\overrightarrow{a} = \overrightarrow{0}$ or $\overrightarrow{b} = \overrightarrow{0}$, or $\overrightarrow{a}$ and $\overrightarrow{b}$ have the same sense. [*Hint*: Apply the lemma to the result obtained in part (a) and use the fact that, for nonnegative numbers a and b, if $a^2 \leq b^2$ then $a \leq b$.]
2. Obtain a result analogous to that in Exercise 1 by computing $\|\overrightarrow{a} - \overrightarrow{b}\|^2$. [*Hint*: When applying the lemma, recall that if $a \leq b$ then $-a \geq -b$.]
3. Obtain a result concerning $\|\overrightarrow{a} + \overrightarrow{b}\|$ as an instance of the result of Exercise 2.

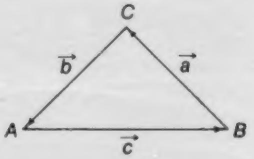
*

The results obtained in Exercises 1 and 3 can be combined into:

> **Theorem 14–1** [The Triangle Inequality]
>
> $$\bigl|\|\overrightarrow{a}\| - \|\overrightarrow{b}\|\bigr| \leq \|\overrightarrow{a} + \overrightarrow{b}\| \leq \|\overrightarrow{a}\| + \|\overrightarrow{b}\|,$$
>
> with equality on the left if and only if $\overrightarrow{a} = \overrightarrow{0}$ or $\overrightarrow{b} = \overrightarrow{0}$ or $\overrightarrow{a}$ and $\overrightarrow{b}$ have opposite senses, and on the right if and only if $\overrightarrow{a} = \overrightarrow{0}$ or $\overrightarrow{b} = \overrightarrow{0}$ or $\overrightarrow{a}$ and $\overrightarrow{b}$ have the same sense.

Part C

Consider $\triangle ABC$, where $\overrightarrow{a} = C - B$, $\overrightarrow{b} = A - C$, and $\overrightarrow{c} = B - A$.



1. Show that $\|\overrightarrow{c}\| = \|\overrightarrow{a} + \overrightarrow{b}\|$.
2. What does Theorem 14–1 tell you about $\|\overrightarrow{a}\|$, $\|\overrightarrow{b}\|$, and $\|\overrightarrow{c}\|$? [*Hint*: Make use of the fact that ABC is a triangle.]
3. Re-phrase Theorem 14–1 to give information concerning $\|\overrightarrow{a} - \overrightarrow{b}\|$ rather than $\|\overrightarrow{a} + \overrightarrow{b}\|$.

*4. Derive the left-hand inequality in Theorem 14–1 from the right-hand inequality. [*Hint*: In the right-hand inequality, replace '$\overrightarrow{b}$' by '$-\overrightarrow{b}$' and then replace '$\overrightarrow{a}$' by '$\overrightarrow{a} + \overrightarrow{b}$'.]

Part D

1. Simplify:
   (a) $|3 + 5|$, $|3| + |5|$, $|3| - |5|$, $\bigl||3| - |5|\bigr|$
   (b) $|-6 + 4|$, $|-6| + |4|$, $|-6| - |4|$, $\bigl||-6| - |4|\bigr|$
   (c) $|-2 + -3|$, $|-2| + |-3|$, $|-2| - |-3|$, $\bigl||-2| - |-3|\bigr|$

Answers for Part B

1. (a) $\|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a}\cdot\vec{b}$, since $\|\vec{a} + \vec{b}\|^2 = (\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b}) = \vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$, and $\vec{x}\cdot\vec{x} = \|\vec{x}\|^2$, for each $\vec{x}$.
   (b) By the lemma, $\vec{a}\cdot\vec{b} \leq \|\vec{a}\|\,\|\vec{b}\|$. Applying this to the results of part (a), we have $\|\vec{a} + \vec{b}\|^2 \leq \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\|\vec{a}\|\,\|\vec{b}\| = (\|\vec{a}\| + \|\vec{b}\|)^2$. Since $\|\vec{a} + \vec{b}\|$ and $(\|\vec{a}\| + \|\vec{b}\|)$ are both nonnegative, it follows that $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$.
   By the lemma, we will have equality if and only if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have the same sense.
2. $\|\vec{a} - \vec{b}\| \geq \bigl|\|\vec{a}\| - \|\vec{b}\|\bigr|$, with equality if and only if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have the same sense. For, $\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a}\cdot\vec{b} \geq \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\,\|\vec{b}\| = (\|\vec{a}\| - \|\vec{b}\|)^2$. Since $\|\vec{a}\| - \|\vec{b}\|$ may be negative, we must use its absolute value when taking its square root.
3. $\|\vec{a} + \vec{b}\| \geq \bigl|\|\vec{a}\| - \|\vec{b}\|\bigr|$, with equality if and only if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have opposite senses. [Substitute '$-\vec{b}$' for '$\vec{b}$' in Exercise 2 and note that $\|-\vec{b}\| = \|\vec{b}\|$.]

Answers for Part C

1. Since $B - A = (C - A) + (B - C)$, $\vec{c} = -\vec{b} + -\vec{a} = -(\vec{a} + \vec{b})$. Hence, $\|\vec{c}\| = \|-(\vec{a} + \vec{b})\| = \|\vec{a} + \vec{b}\|$.
2. Since ABC is a triangle, $(\vec{a}, \vec{b})$ is linearly independent. Hence, by Theorem 14-1 and Exercise 1, $\bigl|\|\vec{a}\| - \|\vec{b}\|\bigr| < \|\vec{c}\| < \|\vec{a}\| + \|\vec{b}\|$. [In view of the definition of distance to be introduced in the next section, this result may be stated as: The measure of one side of a triangle is greater than the difference of the measures of the other two sides and is less than the sum of these measures.]
3. $\bigl|\|\vec{a}\| - \|\vec{b}\|\bigr| \leq \|\vec{a} - \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$, with equality on the left if and only if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have the same sense, and on the right if and only if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have opposite senses. [Replace '$\vec{b}$' in Theorem 14-1 by '$-\vec{b}$' and recall that $\|-\vec{b}\| = \|\vec{b}\|$.]
4. By the right-hand inequality in Theorem 14-1, $\|\vec{a} - \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$. So, $\|(\vec{a} + \vec{b}) - \vec{b}\| \leq \|\vec{a} + \vec{b}\| + \|\vec{b}\|$ — that is, $\|\vec{a}\| - \|\vec{b}\| \leq \|\vec{a} + \vec{b}\|$. From this it follows that $\|\vec{b}\| - \|\vec{a}\| \leq \|\vec{b} + \vec{a}\| = \|\vec{a} + \vec{b}\|$. Since $\bigl|\|\vec{a}\| - \|\vec{b}\|\bigr|$ is either $\|\vec{a}\| - \|\vec{b}\|$ or $\|\vec{b}\| - \|\vec{a}\|$ it follows that $\bigl|\|\vec{a}\| - \|\vec{b}\|\bigr| \leq \|\vec{a} + \vec{b}\|$.

Answers for Part D

1. (a) 8, 8, −2, 2 (b) 2, 10, 2, 2 (c) 5, 5, −1, 1

2. Prove:
   (a) $\left||a| - |b|\right| \le |a + b| \le |a| + |b|$, with equality on the left if and only if $ab \le 0$ and on the right if and only if $ab \ge 0$.
   (b) $\left||a| - |b|\right| \le |a - b| \le |a| + |b|$, with equality on the left if and only if $ab \ge 0$ and on the right if and only if $ab \le 0$.
   [*Hint*: Note that Theorem 14–1 holds for *any* vector space for which there is a multiplication operation which has the properties we have postulated for dot multiplication.]
3. Give the range of values for the measure of the third side of a triangle given that the measures of two sides of the triangle are
   (a) 8 and 6 (b) 5 and 9 (c) 19 and 4 (d) 19 and 19
4. In each of the following, tell whether the given measures can be the measures of the sides of a triangle.
   (a) 6, 3, 4 (b) 6, 4, 2 (c) 3, 4, 5 (d) 3, 7, 3

Part E

By definition, for any real numbers *a* and *b*,

$$(*) \qquad \max(a, b) = \frac{(a + b) + |a - b|}{2}$$

1. Compute $\max(a, b)$ for each of the following pairs (a, b).
   (a) (3, 2) (b) (2, 3) (c) (−1, 3) (d) (−4, 3) (e) (−6, 6)
2. Prove each of the following.
   (a) $\max(a, b) = \max(b, a)$ (b) $\max(a, b) = a \quad [a \ge b]$
   (c) $\max(a, b) = b \quad [a \le b]$ (d) $|a| = \max(a, -a)$
3. As you may have guessed, 'max' is short for 'maximum' and the max of two numbers is the greater of them.
   (a) Formulate a definition similar to (*) for '$\min(a, b)$' so that the min of two numbers is the lesser of them.
   (b) Check your definition by using it to compute $\min(a, b)$ for each of the pairs in Exercise 1.
   (c) Use (*) and your definition to show that $\min(a, b) + \max(a, b) = a + b$.

14.02 Some Fundamental Properties of Distance

In Chapter 11 we noted some intuitive notions about the distance, $d(P, Q)$, from a point P to a point Q. These notions suggested a way of defining distance, and we are now ready to profit from introducing this definition formally:

‖ Definition 14–1 $d(P, Q) = \|Q - P\|$

Answers for Part D [cont.]

2. The real numbers form an inner product space with $a \cdot b$ defined to be ab and, so, $\|a\| = \sqrt{a^2} = |a|$. Moreover, $ab \ge 0$ if and only if $a = 0$ or $b = 0$ or a and b have the same sense, and $ab \le 0$ if and only if $a = 0$ or $b = 0$ or a and b have opposite senses. So, applying Theorem 14-1 to this inner product space, we obtain the result in part (a). Applying Exercise 3 of Part C to this inner product space yields the result of part (b).

[Note that a formal treatment of Exercises 3 and 4 would require definitions given in the next section. Students should, however, be able to operate informally in solving these exercises.]

3. Let s be the measure of the third side. Then
   (a) $2 < s < 14$ (b) $4 < s < 14$
   (c) $15 < s < 23$ (d) $0 < s < 38$
4. A triangle is possible in cases (a) and (c) but not in cases (b) and (d).

Answers for Part E

1. (a) 3 (b) 3 (c) 3 (d) 3 (e) 6
2. (a) This follows from (*) and the fact that $a + b = b + a$ and $|a - b| = |b - a|$.
   (b) [For $a \ge b$, $|a - b| = a - b$.]
   (c) [For $a \le b$, $|a - b| = b - a$.]
   (d) $\max(a, -a) = [(a + -a) + |a - -a|]/2 = |2a|/2 = |a|$
3. (a) $\min(a, b) = \dfrac{(a + b) - |a - b|}{2}$ [One motivation suggesting this formula is that it should be the case that $\min(a, b) = (a + b) - \max(a, b)$.
   (b), (c) Obvious.

Using theorems about the norm function, including Theorem 14-1, it is easy to prove:

> **Theorem 14-2** (a) $d(P, Q) \geq 0$
> (b) $d(P, Q) = 0 \longleftrightarrow Q = P$
> (c) $d(Q, P) = d(P, Q)$
> (d) $d(P, R) \leq d(P, Q) + d(Q, R)$

In addition to Theorem 14-2(d), Theorem 14-1 yields:

> **Theorem 14-3** $d(P, R) = d(P, Q) + d(Q, R)$
> $$\Updownarrow$$
> $$Q \in \overline{PR}$$

Notions of "distance" occur in many parts of mathematics other than geometry and the four parts of Theorem 14-2 state the fundamental properties of any "distance" function. In certain cases, Theorem 14-3 can be used to give a definition of 'segment'.

Exercises

Part A

1. Prove Theorem 14-2.
2. Prove Theorem 14-3.

*3. Consider the sentence.

(e) $d(P, R) \leq d(P, Q) + d(R, Q)$

(i) Show that (e) is a consequence of parts (c) and (d) of Theorem 14-2.
(ii) Show that (e) implies part (a) of Theorem 14-2.
(iii) Show that (e) and '$d(A, A) = 0$' imply part (c) of Theorem 14-2.
(iv) Conclude that a real-valued function d from $\mathscr{E} \times \mathscr{E}$ to $\mathscr{R}$ satisfies Theorem 14-2(a)-(d) if and only if it satisfies (b) and (e).

Part B

Suppose that M is the midpoint of $\overleftrightarrow{AB}$.

1. Show that $P - M = [(P - A) + (P - B)]/2$.
2. Show that $P - M \in [B - A]^{\perp}$ if and only if $d(A, P) = d(B, P)$. [*Hint*: $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) =$ [illegible] if and only if $\|\vec{a}\| = \|\vec{b}\|$. Can you prove this?]
3. (a) What kind of s[illegible]s $\{X : X - M \in [B - A]^{\perp}\}$ in case $A \neq B$?
   (b) What is $\{X : X$ [illegible] $M \in [B - A]^{\perp}\}$ in case $A = B$? [*Hint*: What is $[\vec{0}]^{\perp}$?]

*4. Suppose that $P - M \in [B - A]^{\perp}$ and that $P \in \overleftrightarrow{AB}$. What else can you say about P?

*

The conditions collected into Theorem 14-2 are precisely those which mathematicians choose to characterize distance functions or *metrics*. More explicitly any set $\mathscr{E}$ together with a function d defined on $\mathscr{E} \times \mathscr{E}$ and satisfying Theorem 14-2 is called *a metric space* and, for any P and Q in $\mathscr{E}$, d(P, Q) is called *the distance* [in this metric space] *between* P *and* Q. For example, the set of all continuous real-valued functions defined on the segment $\overline{0, 1}$ together with the function d defined by:

$$d(f, g) = \max(|f(x) - g(x)|)$$

is a metric space whose "points" are functions. There is a very extensive and fruitful theory of what might be called 'the geometry of metric spaces'.

It is amusing to note that the four parts of Theorem 14-2 can be derived from part (b) and:

(d′) $d(P, R) \leq d(Q, P) + d(Q, R)$

[Hint. Substitute 'P' for 'R' and 'Q' in (d′) and show that $d(P, P) \geq 0$. Use this and the result of substituting 'P' for 'R' in (d′) to show that $d(Q, P) \geq 0$. Use the result of substituting 'Q' for 'R' in (d′) to show that $d(P, Q) \leq d(Q, P)$ and, from this, infer (c). From (c) and (d′) infer (d).]

* * *

Suggestions for use of the exercises of section 14.02 are as follows:

(i) Part A and B, and the discussion on pages 155-157 are best treated under teacher direction so that students become properly acquainted with distance and its properties.
(ii) Part C can be used for homework. Students may be permitted to work in teams on this.
(iii) Part D can be used in class to illustrate some applications of distance properties to triangles.
(iv) Part E may be used for homework.
(v) Parts F and G can be used as supervised class exercises.
(vi) Part H has important applications, but may be used for homework.
(vii) Part I should be treated under teacher direction.
(viii) Part J may be used for homework.

Answers for Part A

1. (a) $d(P, Q) = ||Q - P|| \geq 0$

   (b) By definition, $d(P, Q) = 0$ if and only if $||Q - P|| = 0$. The latter is the case if and only if $Q - P = \vec{0}$ — that is, if and only if $Q = P$.

   (c) $d(Q, P) = ||P - Q|| = ||-(P - Q)|| = ||Q - P|| = d(P, Q)$

   (d) $d(P, R) = ||R - P|| = ||(Q - P) + (R - Q)|| \leq ||Q - P||$

   $+ ||R - Q|| = d(P, Q) + d(Q, R)$

2. $d(P, R) = d(P, Q) + d(Q, R)$ if and only if $||R - P|| = ||Q - P|| + ||R - Q||$. Since $R - P = (Q - P) + (R - Q)$ it follows from Theorem 14-1 that $d(P, R) = d(P, Q) + d(Q, R)$ if and only if $Q = P$ or $Q = R$ or $Q - P$ and $R - Q$ have the same sense. In case $Q \neq P$, $Q \neq R$, and $O - P$ and $R - Q$ have the same sense, $Q \in \overline{PR}$. [For, in this case, $Q - P = (R - Q)c$ for some $c > 0$ and, so, $Q = P + (R - P)[c/(1 + c)]$, where, since $c > 0$, $0 < c/(1 + c) < 1$.] On the other hand, if $Q \in \overline{PR}$ then $Q = P$ or $Q = R$ or $Q = P + (R - P)r$ for some r between 0 and 1. In the last case $Q = P + (R - P)r$ and, so, $Q = P + (R - Q)[r/(1 - r)]$ where, since $0 < r < 1$, $r/(1 - r) > 0$. Combining this with the preceding results we see that $Q = P$ or $Q = R$ or $Q - P$ and $R - P$ have the same sense, if and only if $Q \in \overline{PR}$. Consequently, Theorem 14-3.

3. (i) By (c), $d(Q, R) = d(R, Q)$ and, so, by (d), $d(P, R) \leq d(P, Q) + d(R, Q)$.

   (ii) With $R = Q$ it follows from (e) [assuming that $d(P, Q) \in \mathcal{R}$] that $d(Q, Q) \geq 0$. With $R = P$ it follows from (e) that $2d(P, Q) \geq d(P, P) \geq 0$. So, $d(P, Q) \geq 0$.

   (iii) With $Q = P$ it follows from (e) that $d(P, R) \leq d(P, P) + d(R, P)$ and, so, since $d(P, P) = 0$, that $d(P, R) \leq d(R, P)$. By symmetry, $d(R, P) \leq d(P, R)$. Hence, $d(R, P) = d(P, R)$ and, so, $d(Q, P) = d(P, Q)$.

   (iv) Since, by (i), (c) and (d) imply (e) it follows that (b) - (d) imply (b) and (e). On the other hand, assuming that $d(P, Q) \in \mathcal{R}$ it follows by (ii) that (e) implies (a) and, by (iii), that (e) and (b) imply (c). Since, as in (i), (e) and (c) imply (d) it follows that (e) and (b) imply (d). Hence, (e) and (b) [and '$d(P, Q) \in \mathcal{R}$'] imply (a) - (d).

Answers for Part B

1. Since M is the midpoint of $\overleftrightarrow{AB}$ it follows by Definition 7-15 that $M - A = B - M$. So, $(P - A) + (M - P) = (P - M) + (B - P)$ and it follows that $(P - M)2 = (P - A) + (P - B)$. Hence, the result.

2. By Exercise 1, $P - M \in [B - A]^{\perp}$ if and only if $[(P - A) + (P - B)] \cdot (B - A) = 0$. Since $B - A = (P - A) - (P - B)$, the latter is the case if and only if $(P - A) \cdot (P - A) = (P - B) \cdot (P - B)$. $[(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{b}$;

   $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0 \Longleftrightarrow \vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b}$; $\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} \Longleftrightarrow ||\vec{a}|| = ||\vec{b}||$ (since $||\vec{a}||$ and $||\vec{b}||$ are nonnegative)]. So, $P - M \in [B - A]^{\perp}$ if and only if $||P - A|| = ||P - B||$ — that is, if and only if $d(A, P) = d(B, P)$.

3. (a) In case $A \neq B$ the set is a plane. [For, in this case, $[B - A]^{\perp}$ is a proper bidirection.]

   (b) In case $A = B$ the set in question is $\mathcal{E}$. [From the hint, $[\vec{0}]^{\perp} = \mathcal{T}$.]

4. $P = M$. [For if $P \in \overleftrightarrow{AB}$ then, since $M \in \overleftrightarrow{AB}$, $P - M \in [B - A]$. And $[B - A] \cap [B - A]^{\perp} = \{\vec{0}\}$.] [Note that by Exercises 2 and 4, M is the only point of $\overleftrightarrow{AB}$ which is equidistant from A and B. We shall garner this result, formally, as a corollary to Theorem 14-6.]

A set which contains the midpoint of a segment but contains no other point of the segment is said to *bisect* the segment. [Intuitively, such a set "separates" the segment into "equal" parts.] In order to avoid introducing restrictions into our definitions it will be convenient to agree that any set which contains the point A bisects the "degenerate segment" $\overline{AA}$. With this agreement it is reasonable to describe the set discussed in Exercise 3 of Part B as *the perpendicular bisector of* $\overline{AB}$. We formalize this notion in:

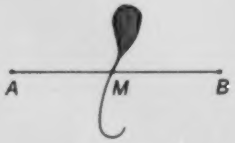


Fig. 14–1

Definition 14–2 The perpendicular bisector of $\overline{AB}$ is $\{X : X - M \in [B - A]^{\perp}$, where M is the midpoint of $\overline{AB}\}$.

Since, for $A \neq B$, $[B - A]^{\perp}$ is the bidirection of planes which are perpendicular to $\overleftrightarrow{AB}$, we have immediately:

Theorem 14–4 For $A \neq B$, the perpendicular bisector of $\overline{AB}$ is the plane which contains the midpoint of $\overline{AB}$ and is perpendicular to $\overleftrightarrow{AB}$.

[As you discovered in Exercise 3(b), for $A = B$, the perpendicular bisector of $\overline{AB}$ is $\mathscr{E}$. Making things turn out this way is not very important – all it does is simplify our formalism slightly.]

Exercise 2 of Part B gives us:

Theorem 14–5 The perpendicular bisector of $\overline{AB}$ is $\{X : d(A, X) = d(B, X)\}$.

[This theorem tells us that the perpendicular bisector of a segment is the set of all points which are equidistant from the endpoints of the segment.] And, from Theorems 14–4 and 14–5, we obtain:

Theorem 14–6 A point P is equidistant from two points A and B if and only if it belongs to the plane which is perpendicular to $\overleftrightarrow{AB}$ at the midpoint of $\overline{AB}$.

Since $\overline{AA} = \overline{BB}$ for any A and B, the convention as to the perpendicular bisector of $\overline{AB}$ would lead to trouble if it were not the case that, for any A and B, the perpendicular bisector of $\overline{AA}$ is the perpendicular bisector of $\overline{BB}$. This, as remarked on page 155, is the case.

Note that if $P \in \ell$ [or π] then the altitude from P to ℓ [or π] is $\overline{PP}$ and the foot of this altitude is P.

Answers to questions:

A triangle has three altitudes. An isosceles triangle [Definition 14-6] has a median which is an altitude. A right triangle [Definition 14-7] has an altitude which is a side. No triangle has an altitude which is both a median and a side, because no triangle has a side as a median. An isosceles right triangle has an altitude which is a median and an [other] altitude which is a side.

Corollary There is one and only one point of $\overleftrightarrow{AB}$ which is equidistant from A and B, and this point is the midpoint of $\overset{\bullet\!-\!\bullet}{AB}$.

We shall frequently wish to speak of the perpendicular bisector of $\overline{AB}$ [rather than of $\overset{\bullet\!-\!\bullet}{AB}$]. Without introducing a formal definition we shall agree that the perpendicular bisector of $\overline{AB}$ is the perpendicular bisector of $\overset{\bullet\!-\!\bullet}{AB}$. With this convention, the following theorem makes sense and follows easily using Theorem 14-6 and Theorem 12-6.

Theorem 14-7 The intersection of the three perpendicular bisectors of the sides of a triangle is a line which is perpendicular to the plane of the triangle.

In studying "plane geometry", where one considers only points belonging to a given plane, the perpendicular bisector of an interval is a line, rather than a plane. Instead of Theorem 14-7 one has the statement:

> The perpendicular bisectors of the sides of a triangle are concurrent.

In our 3-dimensional geometry an analogous statement is a corollary of Theorem 14-7:

Corollary 1 The perpendicular bisectors of the sides of a triangle intersect the plane of the triangle in three concurrent lines.

Theorem 14-7 and its Corollary 1 were arrived at on intuitive grounds in Chapter 11. There our purpose was to establish a theorem concerning the altitudes of a triangle. Since, as the argument given in Chapter 11 shows, this result depends mostly on Corollary 1 of Theorem 14-7, it is convenient to list it as a corollary of the same theorem. First, we need a definition of 'altitude'. For later use it is best to introduce a slightly more general notion. As we know by a corollary to Theorem 12-15, given a line l and a point $P \notin l$, there is exactly one point $Q \in l$ such that $\overleftrightarrow{PQ} \perp l$. We shall be interested in the interval $\overline{PQ}$, and shall call it *the perpendicular from P to l*. To avoid introducing restrictions we adopt:

Definition 14-3
(a) $\overline{PQ}$ is the perpendicular from P to l
$\Longleftrightarrow (Q \in l \text{ and } P - Q \in [l]^{\perp})$
(b) $\overline{PQ}$ is the perpendicular from P to π
$\Longleftrightarrow (Q \in \pi \text{ and } P - Q \in [\pi]^{\perp})$

[The point Q referred to in either part of the definition is called the *foot* of the corresponding perpendicular.]

Definition 14-4 The altitude of a triangle from one of its vertices [or, to the opposite side of the triangle] is the perpendicular from that vertex to the line containing the opposite side.

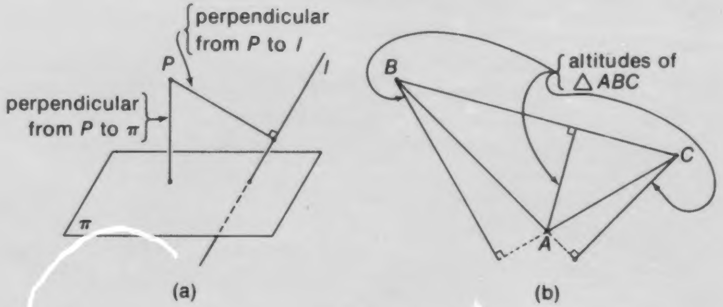


Fig. 14-2

How many altitudes does a triangle have? Can a triangle have a median which is also an altitude? Can a triangle have an altitude which is also a side of the triangle? Can a triangle have an altitude which is both a median and a side of the triangle? Can a triangle have an altitude which is a median and an altitude which is a side of the triangle? Draw pictures to illustrate each of your answers. We can now state the second corollary of Theorem 14-7:

Corollary 2 The lines containing the altitudes of a triangle are concurrent.

[You may recall that we have given a proof of Corollary 2 in Chapter 13 which is different from the one given in Chapter 11.]

Theorem 14-7 and its corollaries depend heavily on Theorem 14-6. As we shall see, Theorem 14-6 is the basis for many interesting geometric results. It is perhaps fair to say that Theorems 14-3 and 14-6 are two of the most useful theorems we shall have concerning distance.

Part C

1. Prove the corollary of Theorem 14-6. [*Hint*: Consider separately the case in which $A \neq B$ and that in which $A = B$.]
2. (a) Prove Theorem 14-7.
   (b) Prove Corollary 1 of Theorem 14-7.
3. Show that the lines containing the altitudes of a triangle are the intersections, with the plane of the triangle, of the planes through the vertices which are perpendicular to the opposite sides.
4. (a) Use Corollary 2 of Theorem 14-7 to derive Corollary 1. [*Hint*: Show that, given $\triangle ABC$, there are points M, N, and P in $\overleftrightarrow{ABC}$ such that MNP is a triangle whose altitudes are contained in the perpendicular bisectors of the sides of $\triangle ABC$.]
   (b) Use Corollary 1 to prove Corollary 2.

*5. Use Corollary 1 to derive Theorem 14-7.

*

Given two points P and Q, the distance $d(P, Q)$ is, also, the lengthmeasure of the segment $\overline{PQ}$. Since it is customary to refer to this lengthmeasure as 'PQ' we shall adopt:

Definition 14-5 $PQ = d(P, Q)$

For example, using this notation, Theorem 14-3 can be rewritten as:

$$PR = PQ + QR \longleftrightarrow Q \in \overline{PR}$$

You may recall that in Chapter 8 [see Part D on page 363 of Volume 1] we have used 'PQ' to refer to the *sensed distance* from P to Q. The introduction of sensed distances on a line l depends on choosing a non-$\vec{0}$ vector $\vec{a} \in [l]$. With respect to $\vec{a}$, the sensed distance from P to Q, for $\{P, Q\} \subseteq l$, is $(Q - P) : \vec{a}$. In case $\vec{a}$ happens to be the unit vector in $[Q - P]^+$, $(Q - P) : \vec{a} = \|Q - P\| = d(P, Q)$. If $\vec{a} = \vec{u}a$, where $\vec{u}$ is the unit vector in $[Q - P]^+$, then $(Q - P) : \vec{a} = d(P, Q)/a$. In the present volume we shall use 'PQ' as defined in Definition 14-5 rather than as in Chapter 8.

The following definitions are the basis of a number of useful theorems concerning triangles:

Definition 14-6
(a) $\triangle ABC$ is an isosceles triangle with base $\overline{AB}$
$\longleftrightarrow BC = CA$
(b) $\triangle ABC$ is an equilateral triangle
$\longleftrightarrow BC = CA = AB$

Answers for Part C

1. Suppose that $A \neq B$. Let π be the perpendicular bisector of $\overleftrightarrow{AB}$, and let M be the midpoint of $\overleftrightarrow{AB}$. Suppose that P is an element of $\overleftrightarrow{AB}$ which is equidistant from A and B. Then $P - M \in [\pi] \cap [\overleftrightarrow{AB}] = \{\vec{0}\}$. Therefore $P = M$. Therefore, there is one and only one point of $\overleftrightarrow{AB}$ which is equidistant from A and B, and this point is M. If $A = B$, $\overleftrightarrow{AB} = \{A\}$, and the corollary holds trivially.
2. (a) Let ABC be a triangle, and let π_1 and π_2 be the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$, respectively. By Theorem 12-6, $\pi_1 \cap \pi_2$ is a line ℓ which is perpendicular to the plane of $\triangle ABC$. Let $P \in \ell$. Then since $P \in \pi_1$, $PA = PB$ and since $P \in \pi_2$, $PB = PC$. Therefore, $PA = PC$. So, $P \in \pi_3$, where π_3 is the perpendicular bisector of $\overline{AC}$. It follows that $\ell \subseteq \pi_3$, since P was an arbitrary point of ℓ. Since $\pi_1 \cap \pi_2 = \ell \subseteq \pi_3$ it follows that $\pi_1 \cap \pi_2 \cap \pi_3 = \ell$.
   (b) Since each perpendicular bisector is perpendicular to the plane of the triangle, each intersects this plane in a line. Since a line perpendicular to a plane intersects it at a single point it follows by Theorem 14-7 that the perpendicular bisectors and the plane of the triangle intersect in a single point. So, the lines in which the perpendicular bisectors intersect the plane of the triangle intersect in a single point, $[(\pi_1 \cap \pi_2 \cap \pi_3) \cap \sigma = \{P\} \Longrightarrow (\pi_1 \cap \sigma) \cap (\pi_2 \cap \sigma) \cap (\pi_3 \cap \sigma) = \{P\}]$
3. Let ABC be a triangle, and let π be the plane containing A which is perpendicular to $\overleftrightarrow{BC}$. Then $\pi \cap \overleftrightarrow{ABC}$ is a line containing A. Since $\pi \perp \overleftrightarrow{BC}$, $\pi \cap \overleftrightarrow{BC}$ is a point — say Q. Since $Q \in \pi$ and $A \in \pi$, $\overline{AQ} \perp \overline{BC}$. Since $\overline{AQ} \subseteq \overleftrightarrow{ABC}$, it follows that $\overline{AQ}$ is the altitude from A (to $\overline{BC}$). Therefore, $\pi \cap \overleftrightarrow{ABC}$ is the line $\overleftrightarrow{AQ}$ which contains the altitude from A.
4. (a) [See answer for Exercise 9 of Part C on page 33.]
   (b) [See answer for Exercise 2 of Part D on page 136.]
5. Assuming Corollary 1, let P be the point of intersection of perpendicular bisectors and the plane of the triangle. The line ℓ through P and perpendicular to the plane of the triangle belongs to each plane through P perpendicular to this latter plane. So, ℓ is contained in each of the perpendicular bisectors and is their intersection.

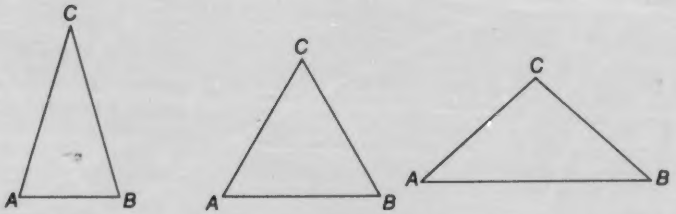


Fig. 14-3

Definition 14-7

$\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$

$$\Longleftrightarrow$$

$$\overleftrightarrow{CA} \perp \overleftrightarrow{BC}$$

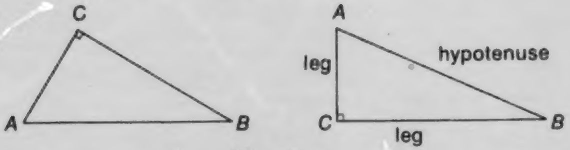
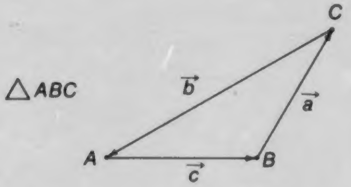


Fig. 14-4

[The sides of a right triangle other than its hypotenuse are sometimes called the *legs* of the triangle.]

When proving theorems concerning a triangle it is often helpful to use the notation illustrated in the following figure:



$\|\vec{a}\| = BC = a$

$\|\vec{b}\| = CA = b$

$\|\vec{c}\| = AB = c$

Fig. 14-5

Note that, since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$,

(*) $\|\vec{c}\| = \|\vec{a} + \vec{b}\|$. [Explain.]

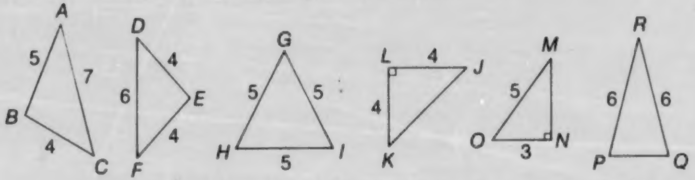
Recall, also, that if D is the point which divides the interval from A to B in $r : (1 - r)$ then

(**) $D - C = \vec{b} + \vec{c}r = \vec{b}(1 - r) - \vec{a}r$ [Explain.],

and $D \in \overline{AB}$ if and only if $0 < r < 1$.

Part D

1. Consider the triangles pictured here:



   (a) Which are isosceles triangles? In each case, give the base.
   (b) Which are equilateral triangles?
   (c) Which are right triangles? In each case, give the hypotenuse.

2. Consider $\triangle ABC$ pictured in Exercise 1. Suppose that $\overline{AS}$ is the median from A and that $\overline{BT}$ is the median from B.
   (a) Show that $AS < 7$. [*Hint*: Use the triangle inequality.]
   (b) Show that $AS > 5$.
   (c) Show that $\frac{1}{2} < BT < \frac{15}{2}$.
   (d) Suppose that $\overline{CU}$ is the median from C. Make a conjecture as to which of AS, BT, and CU is the greater number.

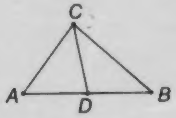
3. Consider $\triangle DEF$ pictured in Exercise 1. Suppose that $\overline{DS}$ and $\overline{FT}$ are the medians from D and F, respectively.
   (a) Show that $2 < DS < 6$ and $2 < FT < 6$.
   (b) Make a conjecture about DS and FT.

4. Consider $\triangle PQR$ pictured in Exercise 1. Give the range of values which are possible measures of $\overline{PQ}$. Is there any value for 'PQ' such that $\triangle PQR$ is a right triangle?

Part E

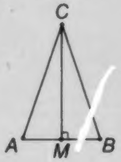
Each part of the following theorem uses the notation introduced in Definition 14–5 and can be proved by using the triangle inequality [Theorem 14–1]:

Theorem 14–8 In $\triangle ABC$,
(a) $|BC - CA| < AB < BC + CA$ and
(b) $D \in \overline{AB} \Longrightarrow CD < \max(BC, CA)$.



1. Prove part (a).
2. Prove part (b). [*Hint*: Use (**), above, and recall that each of $\|\vec{a}\|$ and $\|\vec{b}\|$ is at most $\max(\|\vec{a}\|, \|\vec{b}\|)$.]
3. The following theorem is almost a restatement of Theorem 14–6:

Theorem 14–9 $\triangle ABC$ is an isosceles triangle with base $\overline{AB}$ if and only if its median from C is its altitude from C.



Prove this theorem.

Answers for Part D

1. (a) From left to right: No; Yes - base $\overline{DF}$; Yes - each of the three sides could be considered a base; Yes - base $\overline{JK}$; No; Yes - base $\overline{PQ}$.
   (b) The second one is. Depending upon PQ, the last one might be.
   (c) The fourth and fifth ones are with hypotenuses $\overline{JK}$ and $\overline{MO}$, respectively.

2. (a) By the triangle inequality [in particular, Exercise 2 of Part C on page 152] $SB + BA > AS$ — that is, $2 + 5 > AS$. So, $AS < 7$.
   (b) By the triangle inequality, $AS + SC > AC$. That is, $AS + 2 > 7$. So, $AS > 5$.
   (c) Since $BT + 7/2 > 4$, $BT > 1/2$. Since $BT < 7/2 + 4$, $BT < 15/2$.
   (d) AS

3. (a) By the triangle inequality, $4 - 2 < DS < 4 + 2$ and $4 - 2 < FT < 4 + 2$.
   (b) $DS = FT$

4. $0 < PQ < 12$. It should be intuitively clear that, among the possible triangles fitting the data given for $\triangle PQR$, there should be just one which is a right triangle. After proving Theorem 14-10 students can be certain that $\triangle PQR$ is a right triangle if and only if $PQ = 6\sqrt{2}$.

TC 160 (2)

Sample Quiz

1. Suppose that ABCD is a quadrilateral whose sides have measures as follows:

   $AB = 5, \quad BC = 6, \quad CD = 3, \quad DA = 7$

   Complete the following sentences.

   (a) ___ < AC < ___ (b) ___ < BD < ___

2. Given that an isosceles triangle has sides whose measures are 2 and 5, tell what the measure of the third side is. Justify your answer.

3. Suppose that O is the origin, that A has coordinates $(3, 4, 12)$, and that B has coordinates $(8, 3, b)$, for some b, with respect to an orthonormal coordinate system.

   (a) Determine b such that $\triangle AOB$ is a right triangle with hypotenuse $\overline{AB}$.

   (b) Let T be the point on $\overleftrightarrow{OB}$ such that right $\triangle OAT$ is isosceles with base $\overline{AT}$. Determine the number t such that $T - O = (B - O)t$. What are the coordinates of T?

Key to Sample Quiz

1. (a) $\underline{\ 4\ } < AC < \underline{\ 10\ }$ (b) $\underline{\ 3\ } < BD < \underline{\ 9\ }$

2. The measure of the third side is 5. Either the third side has measure 2 or it has measure 5. But, it cannot have measure 2 as $2 + 2 \not> 5$.

3. (a) b is such that $3 \cdot 8 + 4 \cdot 3 + 12 \cdot b = 0$, for $(A - O) \cdot (B - O) = 0$. Thus, $b = -3$.

   (b) $OT = OA = 13$, so that $t = OT/OB = 13/\sqrt{82}$. T has coordinates $(104/\sqrt{82}, 39/\sqrt{82}, -39/\sqrt{82})$.

TC 160 (3)

Answers for Part E

1. Let $\vec{a} = C - B$ and $\vec{b} = A - C$. Then $\|\vec{a}\| = BC$, $\|\vec{b}\| = CA$, and $\vec{a} + \vec{b} = A - B$, so that $\|\vec{a} + \vec{b}\| = AB$. Making these substitutions into the first part of Theorem 14-1, yields (a), since $(\vec{a}, \vec{b})$ is linearly independent.

2. Let $\vec{a} = C - B$, and $\vec{b} = A - C$. Then by (**), $D - C = \vec{b}(1 - r) - \vec{a}r$ for some r, $0 < r < 1$. Therefore, $CD = \|D - C\| = \|\vec{b}(1 - r) - \vec{a}r\| < \|\vec{b}\|(1 - r) + \|\vec{a}\|r$. [Since $(\vec{a}, \vec{b})$ is linearly independent and $0 < r < 1$, $(\vec{b}(1 - r), \vec{a}r)$ is linearly independent.] If $\max(\|\vec{a}\|, \|\vec{b}\|) = \|\vec{a}\|$, then $\|\vec{b}\|(1 - r) + \|\vec{a}\|r \le \|\vec{a}\|(1 - r) + \|\vec{a}\|r = \|\vec{a}\|$. If $\max(\|\vec{a}\|, \|\vec{b}\|) = \|\vec{b}\|$, then $\|\vec{b}\|(1 - r) + \|\vec{a}\|r \le \|\vec{b}\|(1 - r) + \|\vec{b}\|r = \|\vec{b}\|$. Therefore, in either case, $CD < \max(\|\vec{a}\|, \|\vec{b}\|) = \max(BC, CA)$.

3. Let M be the midpoint of $\overleftrightarrow{AB}$. By Theorem 14-6, C is equidistant from A and B if and only if $\overleftrightarrow{CM} \perp \overleftrightarrow{AB}$. So, $\triangle ABC$ is isosceles with base $\overline{AB}$ if and only if its median from C is its altitude from C.

TC 161 (1)

4. Suppose that, in $\triangle ABC$, $\vec{c} = B - A$, $\vec{b} = A - C$, and $\vec{a} = C - B$. It follows that $\|\vec{c}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a} \cdot \vec{b}$. So, $\vec{a} \cdot \vec{b} = 0$ if and only if $\|\vec{c}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$. Hence, $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$ if and only if $AB^2 = BC^2 + CA^2$.

5. (a) $\sqrt{2}$ (b) 13 (c) 26 [Compare with (b).] (e) $\sqrt{10}$ (f) 10

   [The numbers given in (d) are not the measures of the sides of any triangle, since $\sqrt{5} + \sqrt{5} < 10$.]

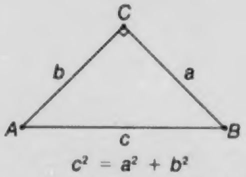
6. (a) 15 (b) 25 (c) $6\sqrt{2}$ (d) 25 (e) 50 (f) $24\sqrt{2}$

7. (a) Let ABC be an isosceles right triangle with base $\overline{AB}$. Let $a = AC = BC$, and let $c = AB$. Then by Theorem 14-10, $c^2 = a^2 + a^2 = 2a^2$, and so, $c/a = \sqrt{2}$.

   (b) Let ABC be an equilateral triangle, whose sides each have measure a. Suppose that the altitude from — say, A — has measure h. By Theorem 14-9, the altitude $\overline{AH}$ from A is also the median from A. Therefore $HB = HC = a/2$. Applying Theorem 14-10 to $\triangle AHB$, we have $h^2 + (a/2)^2 = a^2$. Therefore, $h^2 = 3a^2/4$, so $h/a = \sqrt{3}/2$.

4. Prove:

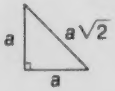
Theorem 14–10 [The Pythagorean Theorem]
$\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$ if and only if $AB^2 = BC^2 + CA^2$.



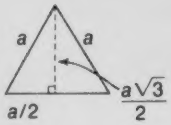
5. Notice that 3, 4, and 5 are the measures of the sides of a right triangle, for $3^2 + 4^2 = 5^2$. Determine which of the following are measures of sides of a right triangle. For those which are, give the measure of the hypotenuse.
   (a) $1, 1, \sqrt{2}$ (b) $12, 13, 5$ (c) $24, 10, 26$
   (d) $\sqrt{5}, \sqrt{5}, 10$ (e) $\sqrt{5}, \sqrt{5}, \sqrt{10}$ (f) $10, 6, 8$

6. Compute the length of the hypotenuse of a right triangle whose legs have lengths
   (a) 9 and 12 (b) 24 and 7 (c) 6 and 6
   (d) 15 and 20 (e) 48 and 14 (f) 24 and 24

7. Prove the following corollaries of Theorem 14–10:
   (a) The ratio of the measure of the hypotenuse of an isosceles right triangle to the measure of either of its legs is $\sqrt{2}$.

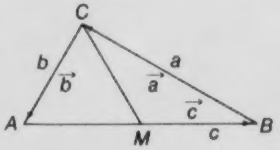


   (b) The ratio of the measure of an altitude of an equilateral triangle to the measure of any of its sides is $\sqrt{3}/2$.



Part F

Consider $\triangle ABC$ and its median, $\overline{CM}$, from C. Use the standard notation shown in the figure. [In particular, $\vec{c} = B - A = -(\vec{a} + \vec{b})$.]



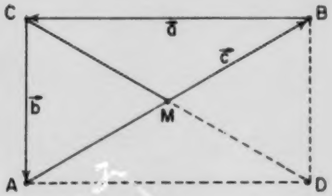
1. Show that $M - C = (\vec{b} - \vec{a})/2$.
2. Copy the figure and locate the point D such that $D - C = \vec{b} - \vec{a}$. What kind of quadrilateral is $ADBC$?
3. Find an expression for $\|M - C\|^2$
   (a) in terms of 'a', 'b', and '$\vec{a} \cdot \vec{b}$'. [*Hint*: $\vec{a} \cdot \vec{a} = a^2$]
   (b) in terms of 'c' and '$\vec{a} \cdot \vec{b}$'. [*Hint*: $\|\vec{a} - \vec{b}\|^2 = \|\vec{a} + \vec{b}\|^2 - 4(\vec{a} \cdot \vec{b})$ [Why?].]
4. Prove:

Theorem 14–11 A triangle is a right triangle with a given side as hypotenuse if and only if the measure of the given side is twice the measure of the median to that side.

Answers for Part F

1. $M - C = (A - C) + (M - A) = \vec{b} + \vec{c}/2 = \vec{b} - (\vec{a} + \vec{b})/2 = (\vec{b} - \vec{a})/2.$

2.



ADBC is a parallelogram.
[For, $D - A = (D - C) - (A - C)$
$= (\vec{b} - \vec{a}) - \vec{b} = -\vec{a} = B - C$.]

3. (a) Since $M - C = (\vec{b} - \vec{a})/2$, $\|M - C\|^2 = (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a})/4$
$= [a^2 + b^2 - 2(\vec{a} \cdot \vec{b})]/4$
   (b) By simple algebra, $\|\vec{a} + \vec{b}\|^2 - \|\vec{a} - \vec{b}\|^2 = 4(\vec{a} \cdot \vec{b})$. Since $\|\vec{a} + \vec{b}\|^2 = \|\vec{c}\|^2 = c^2$ and $\|\vec{a} - \vec{b}\|^2 = \|M - C\|^2 \cdot 4$ it follows that $\|M - C\|^2 = c^2/4 - (\vec{a} \cdot \vec{b})$.

4. Since $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$ if and only if $\vec{a} \cdot \vec{b} = 0$ it follows from Exercise 3(b) that $\triangle ABC$ is a right triangle with $\overline{AB}$ as hypotenuse if and only if $\|M - C\| = c/2$. Hence, the theorem.

TC 162 (1)

5. By definition, $\{A, B, C\}$ is noncollinear if and only if $(B - A, C - A)$ is linearly independent — that is, if and only if $(\vec{c}, -\vec{b})$ is linearly independent. The latter is the case if and only if $(\vec{b}, \vec{c})$ is linearly independent. By the cyclic symmetry of our notation it follows that, since $\{B, C, A\}$ is noncollinear, $(\vec{c}, \vec{a})$ is linearly independent; and, since $\{C, A, B\}$ is noncollinear, $(\vec{a}, \vec{b})$ is linearly independent. [To gain confidence in such arguments from symmetry of notation, write the first two sentences of the preceding argument on the board and beneath them write the sentence which results from substituting 'B' for 'A', 'C' for 'B', 'A' for 'C', '$\vec{a}$' for '$\vec{c}$' and '$\vec{c}$' for '$\vec{b}$'.]

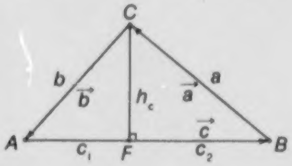
[Since $(\vec{c}, \vec{a})$ is linearly independent it follows from the Schwarz inequality that $|\vec{c} \cdot \vec{a}| < \|\vec{c}\|\,\|\vec{a}\|$ and, so, that $(\vec{c} \cdot \vec{a})^2 < \|\vec{c}\|^2\|\vec{a}\|^2$. So, since $\|\vec{c}\|^2 = c^2$ and $\|\vec{a}\|^2 = a^2$ it follows that $c^2a^2 - (\vec{c} \cdot \vec{a})^2 > 0$. By symmetry of notation, $a^2b^2 - (\vec{a} \cdot \vec{b})^2 > 0$ and $b^2c^2 - (\vec{b} \cdot \vec{c})^2 > 0$.]

Since $\vec{c} = -(\vec{a} + \vec{b})$, $c^2 = \|\vec{c}\|^2 = a^2 + b^2 + 2\vec{a} \cdot \vec{b}$ and $(\vec{c} \cdot \vec{a})^2$
$= (a^2 + \vec{a} \cdot \vec{b})^2 = a^4 + (\vec{a} \cdot \vec{b})^2 + 2a^2(\vec{a} \cdot \vec{b})$. Hence, $c^2a^2 - (\vec{c} \cdot \vec{a})^2$
$= a^4 + a^2b^2 + 2a^2(\vec{a} \cdot \vec{b}) - a^4 - (\vec{a} \cdot \vec{b})^2 - 2a^2(\vec{a} \cdot \vec{b}) = a^2b^2 - (\vec{a} \cdot \vec{b})^2$.
By symmetry of notation it follows that $a^2b^2 - (\vec{a} \cdot \vec{b})^2 = b^2c^2 - (\vec{b} \cdot \vec{c})^2$.
[The common value of the expressions in (*) is four times the square of the area-measure of the region bounded by $\triangle ABC$. ($K = \frac{1}{2}bc \sin \angle A$) We make this identification in Chapter 16.]

5. Since $\{A, B, C\}$ is noncollinear, $(\vec{c}, \vec{a})$, $(\vec{a}, \vec{b})$, and $(\vec{b}, \vec{c})$ are all linearly independent. [Explain.] It follows that $c^2a^2 - (\vec{c} \cdot \vec{a})^2$, $a^2b^2 - (\vec{a} \cdot \vec{b})^2$, and $b^2c^2 - (\vec{b} \cdot \vec{c})^2$ are all positive. [Why?] Use the fact that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ to show that

$$(*) \qquad b^2c^2 - (\vec{b} \cdot \vec{c})^2 = c^2a^2 - (\vec{c} \cdot \vec{a})^2 = a^2b^2 - (\vec{a} \cdot \vec{b})^2.$$

Part G



Consider $\triangle ABC$ and its altitude, $\overline{CF}$, from C. Note that we let $h_c = \|C - F\|$, $c_1 = \|F - A\|$, and $c_2 = \|B - F\|$. [$c_1 + c_2 = c$ if and only if $F \in \overleftrightarrow{AB}$!]

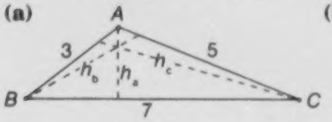
1. (a) Show that $F - A = -\vec{c}\left(\dfrac{\vec{b} \cdot \vec{c}}{c^2}\right) = \vec{c}\left(\dfrac{\vec{b} \cdot \vec{a} + b^2}{c^2}\right)$
   (b) Establish a similar result for $B - F$.
   (c) Show that $(F - A) : (B - F) = b^2/a^2$ if $\vec{a} \cdot \vec{b} = 0$ or $a = b$.
2. Suppose that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$. Show that
   (a) $F \in \overrightarrow{AB}$
   (b) $c = c_1 + c_2$
   (c) $c_1/c_2 = b^2/a^2$
   (d) $c_2c = a^2$ [*Hint*: By (b) and (c), $c/c_2 = (b^2/a^2) + 1$.]
   (e) $c_1c = b^2$
   (f) $h_c^2 = c_1c_2$
   (g) $ch_c = ab$
   (h) $h_c/b = a/c = c_2/a$
3. Show that, for any $\triangle ABC$,

$$h_c^2 = \frac{a^2b^2 - (\vec{a} \cdot \vec{b})^2}{c^2}$$

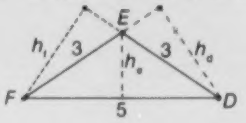
[*Hint*: $h_c^2 = b^2 - c_1^2$. Eliminate 'c_1^2' by using the *first* formula in Exercise 1(a). Then use a result from Part F.]

4. Make use of the result in Exercise 3 to compute the measures of the three altitudes in each of the triangles pictured here. [*Hint*: As in the proof of the Pythagorean Theorem, use the fact that, in $\triangle ABC$, $\|\vec{c}\| = \|\vec{a} + \vec{b}\|$ to compute '$\vec{a} \cdot \vec{b}$' in terms of 'a', 'b', and 'c'.]

(a)



(b)



5. By Exercise 2(a) the foot of one altitude of a right triangle belongs to the side of the triangle. The foot of the other two altitudes is the vertex opposite the hypotenuse.
   (a) Show that each triangle has at least one altitude whose foot belongs to a side of the triangle. [*Hint*: If the foot of some alti-

Answers for Part G

1. (a) $F - A = \text{proj}_{[\vec{c}]}(-\vec{b}) = \vec{c}\ \text{comp}_{\vec{c}}(-\vec{b}) = \vec{c}\,[(-\vec{b} \cdot \vec{c})/(\vec{c} \cdot \vec{c})] = -\vec{c}(\vec{b} \cdot \vec{c})/c^2$. Since $\vec{c} = -(\vec{a} + \vec{b})$, $\vec{b} \cdot \vec{c} = -(\vec{b} \cdot \vec{a} + b^2)$ and it follows that $F - A = \vec{c}(\vec{b} \cdot \vec{a} + b^2)/c^2$.
   (b) $B - F = \text{proj}_{[\vec{c}]}(-\vec{a}) = -\vec{c}(\vec{a} \cdot \vec{c})/c^2 = \vec{c}(\vec{a} \cdot \vec{b} + a^2)/c^2$ [Merely interchanging '$\vec{a}$' and '$\vec{b}$' in Exercise 1.]
   (c) $(F - A):(B - F) = (\vec{b} \cdot \vec{a} + b^2)/(\vec{a} \cdot \vec{b} + a^2)$ and $(\vec{b} \cdot \vec{a} + b^2)/(\vec{a} \cdot \vec{b} + a^2) = b^2/a^2$ if and only if $(\vec{b} \cdot \vec{a} + b^2)a^2 = (\vec{a} \cdot \vec{b} + a^2)b^2$ which is the case if and only if $(a^2 - b^2)(\vec{a} \cdot \vec{b}) = 0$ — that is, if and only if $\vec{a} \cdot \vec{b} = 0$ or $a = b$. [Recall that a and b are both positive.]
2. (a) By Exercise 1(c), $(F - A):(B - F) > 0$. Hence, by Theorem 8-5, $F \in \overline{AB}$.
   (b) By part (a) and Theorem 14-3, $d(A, B) = d(A, F) + d(F, B)$ — that is, $c = c_1 + c_2$.
   (c) $c_1/c_2 = \|F - A\|/\|B - F\| = \overline{AF}:\overline{FB}$, by Exercise 1 of Part B on page 55. By definition, $\overline{AF}:\overline{FB} = |(F - A):(B - F)| = b^2/a^2$ by Exercise 1(c) [and the fact that $b^2/a^2 \geq 0$. Hence, $c_1/c_2 = b^2/a^2$.
   (d) Beginning as suggested in the hint, $c/c_2 = (c_1 + c_2)/c_2 = c_1/c_2 + 1 = b^2/a^2 + 1 = (b^2 + a^2)/a^2 = c^2/a^2$. Since $c \neq 0$ it follows that $/c_2 = c/a^2$ — that is, that $c_2c = a^2$.
   (e) What is true of c_2 and a is equally true of c_1 and b. So, by part (d), $c_1c = b^2$.
   (f) By Theorem 14-10, part (e), and part (a), $h_c^2 = b^2 - c_1^2 = c_1c - c_1^2 = c_1(c - c_1) = c_1c_2$.
   (g) By parts (f), (d), and (e), $c^2h_c^2 = c^2(c_1c_2) = (c_2c)(c_1c) = a^2b^2 = (ab)^2$. Hence [everything being positive], $ch_c = ab$.
   (h) By parts (g) and (d), $h_c/b = a/c$ and $a/c = c_2/a$.
3. Since, by Exercise 1(a), $c_1^2 = \|F - A\|^2 = (\vec{b} \cdot \vec{c})^2/c^2$ it follows that $h_c^2 = b^2 - c_1^2 = b^2 - (\vec{b} \cdot \vec{c})^2/c^2 = [b^2c^2 - (\vec{b} \cdot \vec{c})]^2/c^2 = [a^2b^2 - (\vec{a} \cdot \vec{b})^2]/c^2$ by Exercise 5 of Part F. [This exercise shows to you more directly the relation between the common value of the expressions in Exercise 5 of Part F and the area-measure of the region bounded by $\triangle ABC$. This relation is remarked on in the commentary for Exercise 5.]

tude of $\triangle ABC$ is a vertex then $\triangle ABC$ is a right triangle and, by Exercise 2(a), has an altitude of the kind we are looking for. By Exercises 1(a) and (b) it follows that $F \notin \overleftrightarrow{AB}$ if and only if $(\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{c}) < 0$. What we need to show is that there is no triangle which satisfies this and two similar conditions.]

*(b) Show that each triangle has either exactly one or all three of its altitudes as described in (a).

*

When a, b, and c are positive numbers such that $ab = c^2$ then c is said to be *the mean proportional between a and b*, or *the geometric mean of a and b*. [The words 'mean proportional' refer to the fact that '$ab = c^2$' is equivalent to the "proportion" '$a : c = c : b$'. Here, the word 'mean' refers to the middle positions of 'c' in the proportion. In 'geometric mean', 'mean' has the sense of 'average'—the number $\sqrt{ab}$ is one kind of average of a and b. Another is the arithmetic mean, $(a + b)/2$, of a and b.]

The results obtained in Exercise 2(d), (e), and (f) are often stated in terms of mean proportionals:

Theorem 14-12

(a) The altitude to the hypotenuse of a right triangle is the mean proportional between the measures of the intervals into which its foot divides the hypotenuse.

(b) Either leg of a right triangle is the mean proportional between the hypotenuse and the measure of that one of the two intervals, into which the foot of the altitude divides the hypotenuse, which is adjacent to the given leg.

Notice that in the preceding theorem we have used 'altitude', 'hypotenuse' and 'leg' sometimes to refer to the intervals we have defined them to be and sometimes to refer to the measures of those intervals. This practice is a common one and we shall adopt it. Theorem 14-12 might be shortened a bit more by deleting the phrase 'the measure of' in the two places where it still occurs. However, the most easily understood statement of the theorem is merely:

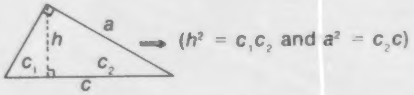


Fig. 14-6

Answers for Part G [cont.]

4. (a) Using the hint we find that $\vec{a} \cdot \vec{b} = (c^2 - b^2 - a^2)/2$. [You may recognize this as a version of the Cosine Law which will be introduced in Chapter 17.] So [in part (a)], $\vec{a} \cdot \vec{b} = [3^2 - 7^2 - 5^2]/2 = -65/2$, $\vec{b} \cdot \vec{c} = [7^2 - 5^2 - 3^2]/2 = 15/2$, and $\vec{c} \cdot \vec{a} = [5^2 - 3^2 - 7^2]/2 = -33/2$. Hence, by Exercise 3, $h_c^2 = [49 \cdot 25 - (65)^2/4]/9 = 75/4$. So, $h_c = 5\sqrt{3}/2$. Similarly, $h_a = 15\sqrt{3}/14$ and $h_b = 3\sqrt{3}/2$.

(b) $h_e = \sqrt{11}/2$; $h_d = 5\sqrt{11}/6 = h_f$

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5. (a) [For F not to belong to $\overleftrightarrow{AB}$ it is necessary and sufficient that F - A and B - F have opposite senses. From Exercises 1(a) and 1(b), this is the case if and only if $(\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{c}) < 0$.] If, in $\triangle ABC$, we had $(\vec{b} \cdot \vec{c})(\vec{c} \cdot \vec{a})$, $(\vec{c} \cdot \vec{a})(\vec{a} \cdot \vec{b})$, and $(\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{c})$ all negative then the product of these three numbers would be negative. But, the product is the product of the squares of three numbers and, so, must be positive or zero.

(b) By the reasoning given in answer to part (a), either none of the products $(\vec{b} \cdot \vec{c})(\vec{c} \cdot \vec{a})$, $(\vec{c} \cdot \vec{a})(\vec{a} \cdot \vec{b})$, and $(\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{c})$ is negative or just two are negative. So, either all altitudes of a triangle have feet which belong to the segments determined by the opposite sides or just one does. If one foot is a vertex then the triangle is a right triangle and has just one altitude whose foot belongs to the opposite side. If no foot is a vertex then the feet belonging to the segments determined by the opposite sides actually belong to the sides themselves and, so, there are three or just one such altitude.

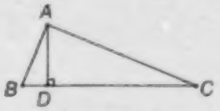
From Exercise 3 and (*) of Exercise 5, Part F, we have;

> **Theorem 14–13** In any triangle, the product of any side by the altitude to it is the same as the product of any other side by the altitude to it.

As a corollary to this [or, directly from Exercise 2(g)] we have:

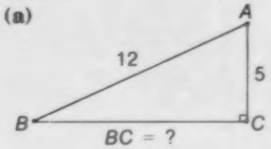
> **Corollary** In a right triangle, the product of the hypotenuse by the altitude to it is the same as the product of the legs.

Part H

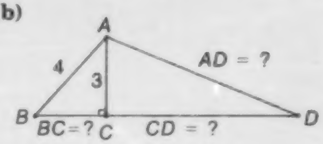


1. Here is a picture of right triangle, $\triangle ABC$, with hypotenuse $\overline{BC}$ and altitude $\overline{AD}$.
   - (a) Given that $BD = 6$ and $DC = 3$, compute AD, AC, and AB.
   - (b) Given that $BD = 6$ and $AB = 9$, compute BC, AD, and AC.
   - (c) Given that $AC = 6$ and $AB = 2$, compute BD, DC, and AD.
   - (d) Given that $AD = 12$ and $DC = 3$, compute BD, AB, and AC.
2. In each of the following, you are given a picture of a geometric figure and some information about it. Compute the indicated distances.

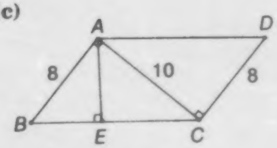
(a)



(b)

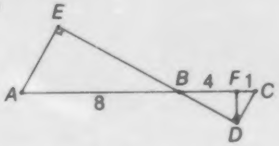


(c)



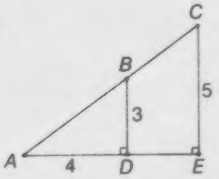
Compute AE, BE, AD.

(d)



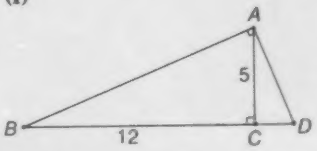
Compute DF, DC, DB, BE, EA.

(e)



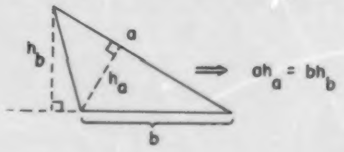
Compute AB, BC, DE.

(f)

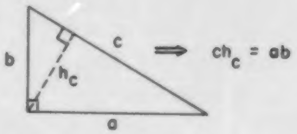


Compute AB, CD, AD.

Theorem 14-13 in the form of Figure 14-6 is:



The corollary is:



Answers for Part H

1. (a) $AD = 3\sqrt{2}$, $AC = 3\sqrt{3}$, $AB = 3\sqrt{6}$

(b) $BC = 81/6$, $AD = 3\sqrt{5}$, $AC = 9\sqrt{5}/2$

(c) $BD = 2/\sqrt{10}$, $DC = 18/\sqrt{10}$, $AD = 6/\sqrt{10}$

(d) $BD = 48$, $AB = 12\sqrt{17}$, $AC = 3\sqrt{17}$

2. (a) $BC = \sqrt{119}$

(b) $BC = \sqrt{7}$, $CD = 9/\sqrt{7}$, $AD = 12/\sqrt{7}$

(c) $AE = 40/\sqrt{41}$, $BE = 32/\sqrt{41}$, $AD = 2\sqrt{41}$

(d) $DF = 2$, $DC = \sqrt{5}$, $DB = 2\sqrt{5}$, $BE = 16/\sqrt{5}$, $EA = 8/\sqrt{5}$

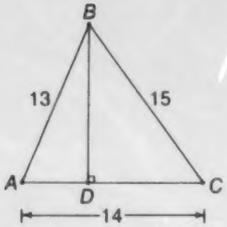
(e) $AB = 5$, $BC = 10/3$, $DE = 8/3$

(f) $AB = 13$, $CD = 25/12$, $AD = 65/12$

(g) $BD = 12$, $AD = 5$, $DC = 9$

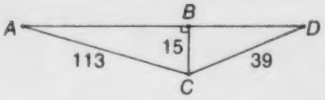
(h) $AB = 112$, $BD = 36$

(g)



Compute *BD*, *AD*, *DC*.
[*Hint*: *DC* is 14 − *AD*. Use
the Pythagorean Theorem.]

(h)



Compute *AB*, *BD*.

***Part I**

In Chapter 11 we were able to show that certain of our intuitive notions concerning perpendicularity and distance in $\mathscr{E}$ imply the existence of an operation on $\mathscr{T}$, which we called dot multiplication, with properties expressed in Postulates 4_0(e) and 4_{11}–4_{14}. And we showed that, conversely, given such an operation, we could justify our intuitive notions on the basis of such definitions as:

$$(1) \qquad \overleftrightarrow{A[\vec{a}]} \perp \overleftrightarrow{B[\vec{b}]} \longleftrightarrow \vec{a} \cdot \vec{b} = 0$$

and:

$$(2) \qquad d(P, Q) = \|Q - P\| = \sqrt{(Q - P) \cdot (Q - P)}.$$

Now, in physical space we may measure distances in centimeters, miles, or light-years and, for each such choice of a unit of length, there is a corresponding distance function. The intuitive notions of distance which we used in arriving at dot multiplication—and which we get back from (2)—presuppose the choice of some unit of length. A different choice would lead to a different multiplication operation in $\mathscr{T}$ which would have the same properties which we have postulated for dot multiplication. If the length of this new unit were *c* times that of the one leading to dot multiplication then the corresponding distance function—say, d^*—would be such that

$$(3) \qquad d^*(P, Q) = d(P, Q)/c.$$

Nevertheless, this new multiplication—let's call it 'star multiplication'—would have all the properties we have postulated of dot multiplication and, since our notions of perpendicularity are the same no matter what unit of length we choose, it would be the case that

$$(4) \qquad \vec{a} * \vec{b} = 0 \longleftrightarrow \vec{a} \cdot \vec{b} = 0.$$

As mentioned on page 42, an operation which satisfies Postulates 4_0(e) and 4_{11} - 4_{14} is called an inner product and a vector space on which one has defined an inner product is called an inner product space. There are many ways of defining an inner product on $\mathcal{T}$ [or on any other finite-dimensional vector space]. For example, one may choose quite arbitrarily a basis $(\vec{i}, \vec{j}, \vec{k})$ for $\mathcal{T}$ and use part (a) of Theorem 11-12 as a definition of '$\vec{a} \cdot \vec{b}$'. With this definition, the chosen basis will turn out to be "orthonormal" — however skew it may "look" to you. One can then go on to define a kind of "perpendicularity" in $\mathcal{E}$ and all our theorems will continue to hold — but lines will be called perpendicular which appear not to be, and intervals in different directions which are said to have the same length will appear to have different lengths.

Our procedure for obtaining an inner product has, of course, been the reverse of that described above. We have "looked at" distance and perpendicularity in $\mathcal{E}$ and chosen the inner product which our intuitions suggested. In particular, we have chosen an inner product for which orthogonality corresponds with our intuitive notions of perpendicularity in $\mathcal{E}$. The purpose of the discussion preceding the exercises of Part I is to point out that we still had a wide choice among inner products. The reason for this is that, after satisfying our notions as to perpendicularity, we have a wide choice of units of distance. The exercises themselves show that this is the only choice which is left. Any inner product in $\mathcal{T}$ with respect to which vectors are orthogonal if and only if they are in the directions of perpendicular lines in $\mathcal{E}$ is related to dot-multiplication just as *-multiplication is in (5). Somewhat more abstractly, two inner products for $\mathcal{T}$ which yield the same relation of orthogonality also yield proportional norms.

As a matter of fact, there are many multiplication operations in $\mathscr{T}$ which have the properties expressed in Postulates 4_0(e) and $4_{11}-4_{14}$, and it is by no means the case that all of them satisfy (4). However, as you will see in the following exercises, for each one which *does* satisfy (4), there is a number $c > 0$ such that

$$\vec{a} * \vec{b} = (\vec{a} \cdot \vec{b})/c^2. \tag{5}$$

In particular, if star multiplication has the properties postulated of dot multiplication, and is such that

$$\vec{a} * \vec{b} = 0 \longleftrightarrow \overleftrightarrow{A[\vec{a}]} \perp \overleftrightarrow{B[\vec{b}]},$$

then the use of star multiplication in (2) in place of dot multiplication will yield the distance function d^* of (3). In short, any two multiplications in $\mathscr{T}$ which have the properties postulated for dot multiplication, and which yield the same notion of orthogonality for members of $\mathscr{T}$, assign proportional norms to members of $\mathscr{T}$.

To establish this result, suppose that star multiplication has the properties postulated of dot multiplication and, in addition, satisfies (4).

1. Suppose that $\vec{u}$ and $\vec{v}$ are unit vectors—that is, that $\vec{u} \cdot \vec{u} = 1 = \vec{v} \cdot \vec{v}$. Show that
   (a) $(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v}) = 0$,
   (b) $(\vec{u} - \vec{v}) * (\vec{u} + \vec{v}) = \vec{u} * \vec{u} - \vec{v} * \vec{v}$,
   (c) $\vec{u} * \vec{u} = \vec{v} * \vec{v}$, and
   (d) $\vec{v} * \vec{v} > 0$.
2. Suppose that $\vec{v} \cdot \vec{v} = 1$. Let $c = /\sqrt{\vec{v} * \vec{v}}$. Show that
   (a) $c > 0$,
   (b) If $\vec{u} \cdot \vec{u} = 1$ then $\vec{u} * \vec{u} = /c^2$,
   (c) $(\vec{a}/\sqrt{\vec{a} \cdot \vec{a}}) * (\vec{a}/\sqrt{\vec{a} \cdot \vec{a}}) = /c^2 \quad [\vec{a} \neq \vec{0}]$, and
   (d) $\vec{a} * \vec{a} = (\vec{a} \cdot \vec{a})/c^2$. [*Hint*: Consider two cases, either $\vec{a} \neq \vec{0}$ or $\vec{a} = \vec{0}$.]
3. (a) Show that $\vec{a} \cdot \vec{b} = [(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) - (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})]/4$.
   (b) Show that, for any $\vec{a}$ and $\vec{b}$, $\vec{a} * \vec{b} = (\vec{a} \cdot \vec{b})/c^2$. [*Hint*: By our assumption, the result of part (a) holds for star multiplication. Use this and Exercise 2(d).]
4. You have shown that if star multiplication has the properties postulated of dot multiplication and satisfies (4) then there is a number $c > 0$ such that (5) is satisfied. Now, assume that star multiplication is defined by (5). Show that it has the properties postulated of dot multiplication and satisfies (4).

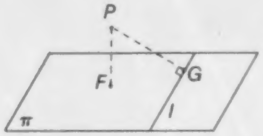
Answers for Part I

1. (a) $(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} = \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{v} = 1 - 1 = 0$
   (b) $(\vec{u} - \vec{v}) * (\vec{u} + \vec{v}) = \vec{u} * \vec{u} + \vec{u} * \vec{v} - \vec{v} * \vec{u} - \vec{v} * \vec{v} = \vec{u} * \vec{u} - \vec{v} * \vec{v}$
   (c) By (4), since $(\vec{u} - \vec{v}) \cdot (\vec{u} + \vec{v}) = 0$, $(\vec{u} - \vec{v}) * (\vec{u} + \vec{v}) = 0$ and, by (b), $\vec{u} * \vec{u} = \vec{v} * \vec{v}$.
   (d) By (4), since $\vec{v} \cdot \vec{v} \neq 0$, $\vec{v} * \vec{v} \neq 0$. Since $\vec{v} * \vec{v} \geq 0$ it follows that $\vec{v} * \vec{v} > 0$. [$\vec{v} * \vec{v} \geq 0$ because of Exercise 1 of Part A on page 47 and the fact that $*$-multiplication has the postulated properties of dot-multiplication.]
2. (a) We know from Exercise 1(d) that $\vec{v} * \vec{v} > 0$. Since square roots and reciprocals of positive numbers are positive it follows that $/\sqrt{\vec{v} * \vec{v}} > 0$. So, $c > 0$.
   (b) By Exercise 1(c), $\vec{u} * \vec{u} = \vec{v} * \vec{v}$. So, $c = /\sqrt{\vec{u} * \vec{u}}$ and, hence, $\vec{u} * \vec{u} = /c^2$.
   (c) This follows from (b) since $(\vec{a}/\sqrt{\vec{a} \cdot \vec{a}}) \cdot (\vec{a}/\sqrt{\vec{a} \cdot \vec{a}}) = 1$.
   (d) For $\vec{a} \neq \vec{0}$ this follows from part (c) in view of the fact that $*$-multiplication has the postulated properties of dot-multiplication. For $\vec{a} = \vec{0}$, $\vec{a} \cdot \vec{a} = 0$ by Theorem 11-1(c). So, for $\vec{a} = \vec{0}$, $\vec{a} * \vec{a} = 0$.
3. (a) [Expand right side and simplify.]
   (b) Since $*$-multiplication has the properties of dot-multiplication needed in part (a) it follows that

$$\begin{aligned}\vec{a} * \vec{b} &= [(\vec{a} + \vec{b}) * (\vec{a} + \vec{b}) - (\vec{a} - \vec{b}) * (\vec{a} - \vec{b})]/4\\ &= [(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) - (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})]/(4c^2)\\ &= (\vec{a} \cdot \vec{b})/c^2\end{aligned}$$

4. Suppose that, by definition, $\vec{a} * \vec{b} = (\vec{a} \cdot \vec{b})/c^2$, where $c > 0$.
   4_0(e): Since $\vec{a} \cdot \vec{b} \in \mathcal{R}$ and $c \in \mathcal{R}$, $\vec{a} * \vec{b} \in \mathcal{R}$ [by various parts of 5_0].
   4_{11}: Since, for $\vec{a} \neq \vec{0}$, $\vec{a} \cdot \vec{a} > 0$ and since $c^2 > 0$ it follows that, for $\vec{a} \neq \vec{0}$, $\vec{a} * \vec{a} > 0$.
   4_{12}: $[(\vec{a} + \vec{b}) \cdot \vec{c}]/c^2 = (\vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c})/c^2 = (\vec{a} \cdot \vec{c})/c^2 + (\vec{b} \cdot \vec{c})/c^2$
   4_{13}: $[(\vec{a}a) \cdot \vec{b}]/c^2 = [(\vec{a} \cdot \vec{b})a]/c^2 = [(\vec{a} \cdot \vec{b})/c^2]a$
   4_{14}: $(\vec{a} \cdot \vec{b})/c^2 = (\vec{b} \cdot \vec{a})/c^2$
   (4): $(\vec{a} \cdot \vec{b})/c^2 = 0 \Longleftrightarrow \vec{a} \cdot \vec{b} = 0 \quad [c \neq 0]$

Part J



Consider a plane π, a line $l \subseteq \pi$, and a point $P \notin \pi$. Let F be the foot of the perpendicular from P to π, and let G be the foot of the perpendicular from P to l.

1. (a) Show that the distance PG is less than the distance between P and any other point of l. [*Hint*: Use a theorem about right triangles.]
   (b) Show that the distance PF is less than the distance between P and any other point of π.
   (c) In what circumstances will it be the case that $PG = PF$?
2. (a) Suppose that Q and R are points of l. What can you say about PQ and PR if $QG = RG$? If $QG > RG$?
   (b) Suppose that Q and R are points of π. What can you say about QF and RF if $PQ = PR$? If $PQ > PR$?
3. (a) Suppose that $l = \overleftrightarrow{A[\vec{c}]}$ and that $\vec{p} = P - A$. Find a formula for the distance between P and G in terms of '$\vec{p}$' and '$\vec{c}$'.
   (b) Compare your answer with Exercise 3 of Part G. [*Hint*: Recall Exercise 5 of Part F.]
4. Suppose that $l = \overleftrightarrow{A[\vec{u}]}$ and $\pi = \overline{A[\vec{u}\ \vec{v}]}$, where $(\vec{u}, \vec{v})$ is orthonormal. Let $\vec{p} = P - A$.
   (a) Proceeding as you may have done in Exercise 3(a), it is easy to see that $P - G = \vec{p} - \vec{u}(\vec{p} \cdot \vec{u})$. Starting from this, find a similar formula for $P - F$.
   (b) Show that $FP = \sqrt{\vec{p} \cdot \vec{p} - (\vec{p} \cdot \vec{u})^2 - (\vec{p} \cdot \vec{v})^2}$.
5. Given that A, P, l, π, and $(\vec{u}, \vec{v})$ are as described in Exercise 4, and that $(\vec{u}, \vec{v}, \vec{k})$ is an orthonormal basis, compute GP and FP, where G and F are the feet of the perpendiculars from P to l and to π, respectively, when A and P have coordinates as follows:
   (a) $A(1, -2, 0)$; $P(5, 4, 0)$ (b) $A(6, -3, 0)$; $P(5, -2, 1)$
   (c) $A(0, 0, 0)$; $P(5, -5, -1)$ (d) $A(3, -3, 0)$; $P(4, -4, -1)$
   (e) $A(5, -12, 0)$; $P(2, -8, 8)$ (f) $A(24, -25, 0)$; $P(19, -13, 12)$

14.03 Distances Between Points and Lines or Planes

In the preceding exercises you have seen that, given P and π, there is a point of π which is closer to P than is any other point of π and that, in fact, this point is the foot of the perpendicular from P to π. [In the exercises you assumed that $P \notin \pi$. Is the preceding statement correct without this assumption?] You also obtained similar results for a point and a line.

Answers for Part J

1. (a) Let H be any other point of l. Then $\triangle PHG$ is a right triangle with hypotenuse $\overline{PH}$. By the Pythagorean Theorem, $PH^2 = PG^2 + GH^2$. Since $GH \neq 0$, $PH^2 > PG^2$ and, so, $PH > PG$.
   (b) Let R be any other point of π. Then $\triangle PFR$ is a right triangle with hypotenuse $\overline{PR}$. By the Pythagorean Theorem, $PR^2 = PF^2 + FR^2$. So $PR^2 > PF^2$ and, so, $PR > PF$.
   (c) By the result of part (b), $PG \neq PF$ unless $G = F$, in which case $PG = PF$.
2. (a) Using the Pythagorean Theorem for $\triangle PGQ$ and $\triangle PGR$, we see that if $QG = RG$ then $PQ = PR$ and that if $QG > RG$ then $PQ > PR$.
   (b) Using the Pythagorean Theorem for $\triangle PFQ$ and $\triangle PFR$, we see that if $PQ = PR$ then $QF = RF$ and that if $PQ > PR$ then $QF > RF$.
3. (a) $(PG)^2 = ||\vec{p}||^2 - ||G - A||^2$, where $G - A = \text{proj}_{[\vec{c}]}(\vec{p}) - \vec{c}[(\vec{p} \cdot \vec{c})/(\vec{c} \cdot \vec{c})]$. So, $(PG)^2 = ||\vec{p}||^2 - (\vec{p} \cdot \vec{c})^2/||\vec{c}||^2 = [||\vec{p}||^2||\vec{c}||^2 - (\vec{p} \cdot \vec{c})^2]/||\vec{c}||^2$. Hence, $PG = \sqrt{||\vec{p}||^2||\vec{c}||^2 - (\vec{p} \cdot \vec{c})^2}/||\vec{c}||$.
   (b) In Exercise 3 of Part G, we have that, for any $\triangle ABC$, $h_c{}^2 = \dfrac{a^2b^2 - (\vec{a} \cdot \vec{b})^2}{c^2}$. And, by Exercise 5 of Part F, the latter is equivalent to: (*) $h_c{}^2 = [a^2c^2 - (\vec{a} \cdot \vec{c})^2]/c^2$

   This compares favorably with the formula derived in part (a), for there we are dealing with the triangle, $\triangle PA(A + \vec{c})$, and the result we obtained is the instance of (*) replacing 'A' by '$A + \vec{c}$', 'B' by 'A', 'C' by 'P', 'h_c' by 'PG', 'a' by '$||\vec{p}||$', and 'c' by '$||\vec{c}||$'.
4. (a) [The suggested solution for Exercise 3(a) is: $P - G = \vec{p} - (G - A) = \vec{p} - \vec{c}[(\vec{p} \cdot \vec{c})/(\vec{c} \cdot \vec{c})]$ and, so, $(PG)^2 = (P - G) \cdot (P - G) = ||\vec{p}||^2 - (\vec{p} \cdot \vec{c})^2/||\vec{c}||^2$, etc. Since, in the present exercise the unit vector $\vec{u}$ plays the role of $\vec{c}$, $P - G = \vec{p} - \vec{u}(\vec{p} \cdot \vec{u})$.]
   $P - F = (P - G) - (F - G) = \vec{p} - \vec{u}(\vec{p} \cdot \vec{u}) - \vec{v}([\vec{p} - \vec{u}(\vec{p} \cdot \vec{u})] \cdot \vec{v})$
   $= \vec{p} - \vec{u}(\vec{p} \cdot \vec{u}) - \vec{v}(\vec{p} \cdot \vec{v})$ [since $\vec{u} \cdot \vec{v} = 0$].
   (b) Since $(\vec{u}, \vec{v})$ is orthonormal, $||\vec{p} - \vec{u}(\vec{p} \cdot \vec{u}) - \vec{v}(\vec{p} \cdot \vec{v})||^2$
   $= \vec{p} \cdot \vec{p} - 2(\vec{p} \cdot \vec{u})^2 - 2(\vec{p} \cdot \vec{v})^2 + (\vec{p} \cdot \vec{u})^2 + (\vec{p} \cdot \vec{v})^2 + 2(\vec{u} \cdot \vec{v})(\vec{p} \cdot \vec{u})(\vec{p} \cdot \vec{v})$
   $= \vec{p} \cdot \vec{p} - (\vec{p} \cdot \vec{u})^2 - (\vec{p} \cdot \vec{v})^2$. Hence, $FP = \sqrt{\vec{p} \cdot \vec{p} - (\vec{p} \cdot \vec{u})^2 - (\vec{p} \cdot \vec{v})^2}$.
5. (a) 6, 0 (b) $\sqrt{2}$, 1 (c) $\sqrt{26}$, 1
   (d) $\sqrt{2}$, 1 (e) $4\sqrt{5}$, 8 (f) $12\sqrt{2}$, 12

   [Here are the details of the solution for part (a): $P - A = \vec{p}$, and has components (4, 6, 0). So, $\vec{p} \cdot \vec{p} = 4 \cdot 4 + 6 \cdot 6 + 0 \cdot 0 = 52$. Also, $\vec{p} \cdot \vec{u} = 4$ and $\vec{p} \cdot \vec{v} = 6$ so that $PG = \sqrt{\vec{p} \cdot \vec{p} - (\vec{p} \cdot \vec{u})^2} = \sqrt{52 - 16} = 6$ and $PF = \sqrt{\vec{p} \cdot \vec{p} - (\vec{p} \cdot \vec{u})^2 - (\vec{p} \cdot \vec{v})^2} = \sqrt{52 - 16 - 36} = 0$.]

So we have:

> **Theorem 14–14** The perpendicular from a given point to a given line or plane is the shortest of the intervals whose endpoints are the given point and a point of the given line or plane.

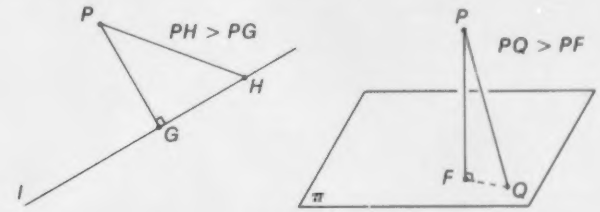


Fig. 14–7

In establishing this result you probably used the Pythagorean Theorem, arguing that if F is the foot of the perpendicular from P to π and $Q \in \pi$ then $\triangle PQF$ is a right triangle with hypotenuse $\overline{PQ}$ and, so,

$$(1) \qquad PQ = \sqrt{QF^2 + FP^2}.$$

It follows, since P and F are given, that the distance PQ is least when $QF = 0$ – that is, when Q is the foot of the perpendicular from P. [Explain.] It also follows, that, P and F being given, PQ depends only on QF and is greater the further Q is from F. More explicitly, if, for each $x \geq 0$,

$$(2) \qquad f(x) = \sqrt{x^2 + FP^2}$$

then, for any $Q \in \pi$, $PQ = f(QF)$; moreover, for $x > y \geq 0$, $f(x) > f(y)$. [Explain.] A function whose arguments and values are real numbers and which has larger values for larger arguments is said to be *an increasing function*. So, we can formulate the result of Exercise 2, page 102 as:

> **Theorem 14–15** The distance between a given point and a point of a given line or plane is an increasing function of the distance between the second point and the foot of the perpendicular to the given line or plane.

[Modify the preceding discussion of a given point P and plane π to obtain the result stated in this theorem for a given point P and line l.]

Explanation called for: Suppose that $x > y \geq 0$. It follows by Exercise 8(a) on page 8 that $x^2 > y^2$ and, so, by Postulate 5_{12} that $x^2 + FP^2 > y^2 + FP^2$. From this, by a theorem like Exercise 8(b) on page 8, it follows that $\sqrt{x^2 + FP^2} > \sqrt{y^2 + FP^2}$ — that is, that $f(x) > f(y)$. [In applying Exercise 8(b), take $a = \sqrt{x^2 + FP^2}$ and $b = \sqrt{y^2 + FP^2}$. Then, by (3) of Part B on page 8, $a^2 = x^2 + FP^2$ and $b^2 = y^2 + FP^2$. Since, $a^2 > b^2$ it follows by Exercise 8(b) that, since $a \geq 0$, $a \geq b$. Now, in case $a = b$, $a^2 = b^2$. So, since $a^2 \neq b^2$ it follows that $a \neq b$ and, hence, that $a > b$. (Note that we have proved a "refinement" of the theorem of Exercise 8(b): $a^2 > b^2 \Longrightarrow a > b\ [a \geq 0]$)].

In Theorem 14-15, we use the phrase 'is an increasing function of' to express the fact that there exists an increasing function — say, f — such that $d(P, \ell) = f(d(P, G))$ [or such that $d(P, \pi) = f(d(P, F))$]. In general, 'function of' indicates function composition — here, $d(P, \ell) = [f \circ d](P, G)$.

The distance between a given point and a ["variable"] point of a given set may fail to have a minimum. For example, in case $B \neq C$ and $A \in -\overleftrightarrow{BC}$ the distance between A and a point $P \in \overrightarrow{BC}$ may, depending on P, be any number greater than AB. So the set of distances from A to points of $\overrightarrow{BC}$ does not have a least member. In other words, there is no point of $\overrightarrow{BC}$ which is nearest A. Although the set of distances from A to points of $\overrightarrow{BC}$ does not have a least member it does have AB as its greatest lower bound.

The increasing function referred to in Theorem 14–15 is, of course, the function f described in (2). Since f is an increasing function and since, by definition, 0 is its least argument, f has a minimum value [recall Exercise 2 on page 102] at 0. This minimum value is FP [in the case of π, or GP in the case of l]. Theorem 14–14 could be restated as:

> The distance between a given point and a point of a given line or plane has a minimum value at the foot of the perpendicular from the given point to the given line or plane.

When the set of distances between a given point and points of a given set has a minimum value, this number is called *the distance between the given point and the given set.* So, for example, we have shown that

$$d(P, \pi) = PF \text{ and } d(P, l) = PG,$$

where F and G are the feet of the perpendiculars from P to π and to l, respectively. In Exercises 3 and 4 of Part J you have seen how to compute distances between planes [or lines] and points. For example, if $l = \overleftrightarrow{A[\vec{u}]}$ where $\vec{u}$ is a unit vector and $P = A + \vec{p}$, then

$$[d(P, l)]^2 = \vec{p}\cdot\vec{p} - (\vec{p}\cdot\vec{u})^2; \tag{3}$$

and, if $\pi = \overleftrightarrow{A[\vec{u}, \vec{v}]}$ where $(\vec{u}, \vec{v})$ is orthonormal and $P = A + \vec{p}$, then

$$[d(P, \pi)]^2 = \vec{p}\cdot\vec{p} - (\vec{p}\cdot\vec{u})^2 - (\vec{p}\cdot\vec{v})^2. \tag{4}$$

The results (3) and (4) are somewhat special due to the assumption, in (3), that $\|\vec{u}\| = 1$ and, in (4), that $(\vec{u}, \vec{v})$ is orthonormal. It is not too difficult, however, to extend these results to take care of the case in which $\vec{q}$, say, is any non-$\vec{0}$ vector in $[l]$ or $(\vec{q}, \vec{r})$, say, is any basis for $[\pi]$. In the former case, $\vec{q}/\|\vec{q}\|$ is a unit vector in $[l]$ and, on substituting for '$\vec{u}$' in (3), we obtain:

$$\begin{aligned} d(P, l)^2 &= \vec{p}\cdot\vec{p} - \left(\vec{p}\cdot\frac{\vec{q}}{\|\vec{q}\|}\right)^2 \\ &= \vec{p}\cdot\vec{p} - \frac{(\vec{p}\cdot\vec{q})^2}{\vec{q}\cdot\vec{q}} \\ &= [(\vec{p}\cdot\vec{p})(\vec{q}\cdot\vec{q}) - (\vec{p}\cdot\vec{q})^2]/(\vec{q}\cdot\vec{q}) \end{aligned}$$

So,

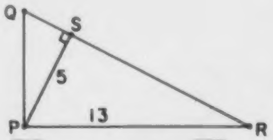
$$d(P, l)^2 = \begin{vmatrix} \vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q} \\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q} \end{vmatrix} / (\vec{q}\cdot\vec{q}).$$

Here are suggestions for the exercises of section 14.03:

(i) The discussion of Theorem 14-16 and Part A should be directed by the teacher.
(ii) Part B may be used for homework.
(iii) Part C and the discussion of Theorem 14-17 should be teacher directed.
(iv) Parts D and E (except Exercise 6 of Part E) may be assigned for homework after examples of the use of Theorem 14-17 have been presented.
(v) Exercise 6 of Part E can be a class project.

* * *

Sample Quiz

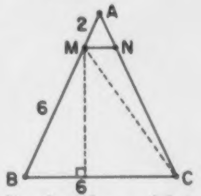


1. Given right △PQR with hypotenuse $\overline{QR}$ and altitude $\overline{PS}$ from P, suppose that PR = 13 and PS = 5. Compute each of the following.

   (a) SR (b) SQ (c) QR (d) PQ

2. Suppose that right △ABC has hypotenuse $\overline{BC}$ and altitude $\overline{AD}$ from A, and that AB = c and AC = b. Give each of the following in terms of 'b' and 'c'.

   (a) BC (b) BD (c) DC (d) AD



3. Given isosceles △ABC with $\overline{MN} \parallel \overline{BC}$, AB = 8, AM = 2, and BC = 6, as shown in the picture at the right.

   (a) Compute MN.

   (b) What is the measure of the altitude from A?

   (c) Let P be the foot of the perpendicular from M to $\overline{BC}$. Compute MP and BP.

   ☆(d) [Extra credit problem] Tell whether △BMC is equilateral and justify your answer.

Key to Sample Quiz

1.	(a) 12	(b) 25/12	(c) 169/12	(d) 65/12
2.	(a) $\sqrt{b^2 + c^2}$	(b) $c^2/\sqrt{b^2 + c^2}$	(c) $b^2/\sqrt{b^2 + c^2}$	(d) $bc/\sqrt{b^2 + c^2}$
3.	(a) 3/2	(b) $\sqrt{55}$	(c) $3\sqrt{55}/4$; 9/4	
	(d) Not equilateral, for M is not on the perpendicular bisector of $\overline{BC}$.			

Similarly, in case $[\pi] = [\vec{q}, \vec{r}]$, we obtain vectors $\vec{u}$ and $\vec{v}$ such that $[\pi] = [\vec{u}, \vec{v}]$ and $(\vec{u}, \vec{v})$ is orthonormal by choosing

(5) $\vec{u} = \vec{q}/\|\vec{q}\|$ and $\vec{v} = [\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})]/\|\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})\|$. [Explain.]

Substituting for '$\vec{u}$' and '$\vec{v}$' in (4) and simplifying leads, eventually, to a result which we state in:

> **Theorem 14–16** If $l = \overleftrightarrow{A[\vec{q}]}$ and $\pi = \overleftrightarrow{A[\vec{q}, \vec{r}]}$ then, with $\vec{p} = P - A$,
>
> **(a)** $d(P, l)^2 = \begin{vmatrix} \vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q} \\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q} \end{vmatrix} / (\vec{q}\cdot\vec{q})$, and
>
> **(b)** $d(P, \pi)^2 = \begin{vmatrix} \vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q} & \vec{p}\cdot\vec{r} \\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{r} \\ \vec{r}\cdot\vec{p} & \vec{r}\cdot\vec{q} & \vec{r}\cdot\vec{r} \end{vmatrix} \Big/ \begin{vmatrix} \vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{r} \\ \vec{r}\cdot\vec{q} & \vec{r}\cdot\vec{r} \end{vmatrix}$.

We have already proved part (a) of this theorem. Now we shall sketch a proof of part (b). You should fill in the details. We suppose given a basis $(\vec{q}, \vec{r})$ for $[\pi]$, and we suppose that $(\vec{u}, \vec{v})$ is the orthonormal basis given by (5).

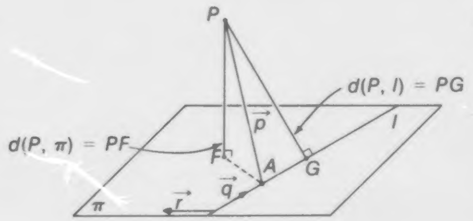


Fig. 14–8

Rather than substitute from (5) into (4) it is easier to start afresh. To do so, we note that

(6) $$\vec{p} = P - A = (F - A) + (P - F).$$

Since $(P - F) \perp (F - A)$ it follows that

$$\vec{p} \cdot (P - F) = (P - F) \cdot (P - F) = d(P, \pi)^2.$$

So, our problem is to compute '$P - F$' in terms of '$\vec{p}$', '$\vec{q}$', and '$\vec{r}$'. Since, by (6), $P - F = \vec{p} - (F - A)$, the first step is to compute '$F - A$'.

Theorem 14-16 is closely related to the corollary of Theorem 13-4 on page 126. The proof of the Theorem 14-16(b) outlined on page 169 is a simplification of the proof of part (b) of the corollary as given in the commentary on TC126.

Since $F - A \in [\pi]$ and $(\vec{u}, \vec{v})$ is an orthonormal basis for $[\pi]$ it follows that

$$F - A = \vec{u}[(F - A) \cdot \vec{u}] + \vec{v}[(F - A) \cdot \vec{v}].$$

Since $P - F$ is orthogonal to $\vec{u}$ and to $\vec{v}$ it follows by (6) that $\vec{p} \cdot \vec{u} = (F - A) \cdot \vec{u}$ and $\vec{p} \cdot \vec{v} = (F - A) \cdot \vec{v}$. Hence,

$$F - A = \vec{u}(\vec{p} \cdot \vec{u}) + \vec{v}(\vec{p} \cdot \vec{v}).$$

The next step is to compute '$\vec{v}(\vec{p} \cdot \vec{v})$' from (5). In doing so, note that $\|\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})\|^2 = \vec{r} \cdot \vec{r} - (\vec{r} \cdot \vec{u})^2$. Now, add on '$\vec{u}(\vec{p} \cdot \vec{u})$' and simplify. You should obtain:

$$F - A = \frac{\vec{u}[(\vec{p} \cdot \vec{u})(\vec{r} \cdot \vec{r}) - (\vec{p} \cdot \vec{r})(\vec{r} \cdot \vec{u})] - \vec{r}[(\vec{p} \cdot \vec{u})(\vec{r} \cdot \vec{u}) - \vec{p} \cdot \vec{r}]}{\vec{r} \cdot \vec{r} - (\vec{r} \cdot \vec{u})^2}$$

If you now multiply in numerator and denominator with '$\|\vec{q}\|^2$', recalling that $\vec{u}\|\vec{q}\| = \vec{q}$, you will obtain:

$$(7)\ F - A = \frac{\vec{q}[(\vec{p} \cdot \vec{q})(\vec{r} \cdot \vec{r}) - (\vec{p} \cdot \vec{r})(\vec{r} \cdot \vec{q})] - \vec{r}[(\vec{p} \cdot \vec{q})(\vec{r} \cdot \vec{q}) - (\vec{p} \cdot \vec{r})\|\vec{q}\|^2]}{\|\vec{q}\|^2(\vec{r} \cdot \vec{r}) - (\vec{r} \cdot \vec{q})^2}$$

Since $P - F = \vec{p} - (F - A)$, and $\|\vec{q}\|^2 = \vec{q} \cdot \vec{q}$, it follows readily that $P - F$ is

$$\frac{\vec{p}[(\vec{q} \cdot \vec{q})(\vec{r} \cdot \vec{r}) - (\vec{r} \cdot \vec{q})^2] - \vec{q}[(\vec{p} \cdot \vec{q})(\vec{r} \cdot \vec{r}) - (\vec{p} \cdot \vec{r})(\vec{r} \cdot \vec{q})] + \vec{r}[(\vec{p} \cdot \vec{q})(\vec{r} \cdot \vec{q}) - (\vec{p} \cdot \vec{r})(\vec{q} \cdot \vec{q})]}{(\vec{q} \cdot \vec{q})(\vec{r} \cdot \vec{r}) - (\vec{r} \cdot \vec{q})^2}$$

Using this last to compute '$\vec{p} \cdot (P - F)$' you will obtain a fraction whose numerator is equivalent to the third order determinant in the numerator of Theorem 14–16(b) and whose denominator is equivalent to the second order determinant in the denominator of that theorem.

Note that in proving part (b) we have also obtained, in (7), a way of locating the foot, F, of the perpendicular from P to the plane $A[\vec{q}, \vec{r}]$.

Exercises

Part A

1. Show that the two parts of Theorem 14–16 reduce to (3) and (4) on page 169 in case $\vec{q} = \vec{u}$ and $\vec{r} = \vec{v}$, where $(\vec{u}, \vec{v})$ is orthonormal.
2. What does (7) reduce to in the case described in Exercise 1?
3. What is the formula like (7) for $G - A$, where G is the foot of the perpendicular from P to the line $\overleftrightarrow{A[\vec{q}]}$?

Steps leading to (7):

$$\vec{p} \cdot \vec{v} = \vec{p} \cdot \frac{\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})}{\|\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})\|} = \frac{\vec{p} \cdot \vec{r} - (\vec{p} \cdot \vec{u})(\vec{r} \cdot \vec{u})}{\|\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})\|}$$

$$\vec{v}(\vec{p} \cdot \vec{v}) = \frac{[\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})][\vec{p} \cdot \vec{r} - (\vec{p} \cdot \vec{u})(\vec{r} \cdot \vec{u})]}{\|\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})\|^2}$$

where $\|\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})\|^2 = \vec{r} \cdot \vec{r} - (\vec{r} \cdot \vec{u})^2$.

$$\vec{u}(\vec{p} \cdot \vec{u}) + \vec{v}(\vec{p} \cdot \vec{v}) = \frac{\vec{u}(\vec{p} \cdot \vec{u})[\vec{r} \cdot \vec{r} - (\vec{r} \cdot \vec{u})^2] + [\vec{r} - \vec{u}(\vec{r} \cdot \vec{u})][\vec{p} \cdot \vec{r} - (\vec{p} \cdot \vec{u})(\vec{r} \cdot \vec{u})]}{\vec{r} \cdot \vec{r} - (\vec{r} \cdot \vec{u})^2}$$

$$= \frac{\vec{u}[(\vec{p} \cdot \vec{u})(\vec{r} \cdot \vec{r}) - (\vec{r} \cdot \vec{u})(\vec{p} \cdot \vec{r})] - \vec{r}[(\vec{p} \cdot \vec{u})(\vec{r} \cdot \vec{u}) - \vec{p} \cdot \vec{r}]}{\vec{r} \cdot \vec{r} - (\vec{r} \cdot \vec{u})^2}$$

$$= [\text{right side of (7)}]$$

The right side of (7) is obtained, as mentioned in the text, by multiplying in numerator and denominator by '$\|\vec{q}\|^2$' and simplifying. Note that each term of the numerator preceding (7), with the exception of the term '$\vec{r}(\vec{p} \cdot \vec{r})$' contains two '$\vec{u}$'s. Multiplying with '$\|\vec{q}\|^2$' yields two '$\vec{u}\|\vec{q}\|$'s, each of which is to be replaced by a '$\vec{q}$'. The result is the right side of (7).

Answers for Part A

1. (a) $d(P, \ell)^2 = \begin{vmatrix} \vec{p} \cdot \vec{p} & \vec{p} \cdot \vec{u} \\ \vec{u} \cdot \vec{p} & \vec{u} \cdot \vec{u} \end{vmatrix} / \vec{u} \cdot \vec{u} = \vec{p} \cdot \vec{p} - (\vec{p} \cdot \vec{u})(\vec{u} \cdot \vec{p}) = \vec{p} \cdot \vec{p} - (\vec{p} \cdot \vec{u})^2$

   (b) $d(P, \pi)^2 = \begin{vmatrix} \vec{p} \cdot \vec{p} & \vec{p} \cdot \vec{u} & \vec{p} \cdot \vec{v} \\ \vec{u} \cdot \vec{p} & \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{v} \\ \vec{v} \cdot \vec{p} & \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{v} \end{vmatrix} \Bigg/ \begin{vmatrix} \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{v} \\ \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{v} \end{vmatrix}$

   The denominator is $(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - 0$ or 1, whereas the numerator is $(\vec{p} \cdot \vec{p})(1 - 0) - (\vec{p} \cdot \vec{u})(\vec{u} \cdot \vec{p} - 0) + (\vec{p} \cdot \vec{v})(0 - \vec{v} \cdot \vec{p})$ or $\vec{p} \cdot \vec{p} - (\vec{p} \cdot \vec{u})^2 - (\vec{p} \cdot \vec{v})^2$.
2. $F - A = \vec{u}(\vec{p} \cdot \vec{u}) + \vec{v}(\vec{p} \cdot \vec{v})$
3. $G - A = [\vec{q}(\vec{p} \cdot \vec{q})]/(\vec{q} \cdot \vec{q})$

Part B

Suppose that, with respect to some orthonormal coordinate system, the points A, B, C, and P have coordinates:

$$A: (0, -2, -1),\ B: (1, -3, 1),\ C: (1, -1, 3), \text{ and } P: (2, -1, 1)$$

1. Find the distance between P and $\overleftrightarrow{ABC}$. [*Hint*: Use Theorem 14–16(b) with $\vec{q} = B - A$ and $\vec{r} = C - A$.]
2. Find the coordinates of the foot, F, of the perpendicular from P to $\overleftrightarrow{ABC}$. [*Hint*: Use (7), above, to find the components of $F - A$ with respect to the given orthonormal basis for $\mathscr{T}$.]
3. Check your answers for Exercises 1 and 2 by finding the distance between P and F from the coordinates of these points.
4. Find the distance between P and $\overleftrightarrow{AB}$.
5. Find the coordinates of the foot, G, of the perpendicular from P to $\overleftrightarrow{AB}$.
6. Check your answers for Exercises 4 and 5 by finding the distance between P and G from the coordinates of these points.

*

Theorem 14–16 shows how to find distances between points and lines or planes in terms of certain dot products. To obtain numerical results it is necessary to compute these dot products. These are easy to compute if you know the components of the vectors with respect to some orthonormal basis for $\mathscr{T}$.

There is another method for finding the distance between a point and a plane which, given an orthonormal basis for $\mathscr{T}$, is easier to use than is Theorem 14–16. To see what this is, note that if $A \in \pi$ and $\vec{w}$ is a unit vector in $[\pi]^{\perp}$ then

$$(8) \qquad P - F = \vec{w}[(P - A) \cdot \vec{w}]\,. \qquad \text{[Explain.]}$$

and, consequently,

$$(9) \qquad d(P, \pi) = |(P - A) \cdot \vec{w}|. \qquad \text{[Explain.]}$$

Now, if $[\pi] = [\vec{q}, \vec{r}]$, and we know the components of $\vec{q}$ and $\vec{r}$ with respect to some orthonormal basis for $\mathscr{T}$, then Theorem 13–2 tells us how to find the components of a non-$\vec{0}$ vector $\vec{m}$ in $[\pi]^{\perp}$. We can then take $\vec{w} = \vec{m}/\|\vec{m}\|$ and find $d(P, \pi)$ from (9).

Answers for Part B

1. Since $d(P, \overleftrightarrow{ABC})^2 = \begin{vmatrix} 9 & 5 & 11 \\ 5 & 6 & 8 \\ 11 & 8 & 18 \end{vmatrix} \Big/ \begin{vmatrix} 6 & 8 \\ 8 & 18 \end{vmatrix} = 100/44$, $d(P, \overleftrightarrow{ABC}) = 5/\sqrt{11}$. [$\vec{p}$, $\vec{q}$, and $\vec{r}$ have components $(2, 1, 2)$, $(1, -1, 2)$, and $(1, 1, 4)$, respectively. So, for example, by Theorem 11-12, $\vec{p} \cdot \vec{q} = 2 \cdot 1 + 1 \cdot -1 + 2 \cdot 2$.]
2. $(7/11, -16/11, 16/11)$ [By (7), $F - A = \vec{q}/22 + \vec{r}(13/22)$. So the orthonormal components of $F - A$ are $(7/11, 6/11, 27/11)$.]
3. Since $d(P, F)^2 = (7/11 - 2)^2 + (-16/11 + 1)^2 + (16/11 - 1)^2 = 275/121$, $d(P, F) = 5/\sqrt{11}$.
4. By Theorem 14-16(a), $d(P, \overleftrightarrow{AB})^2 = \begin{vmatrix} 9 & 5 \\ 5 & 6 \end{vmatrix} /6 = 29/6$. So, $d(P, \overleftrightarrow{AB}) = \sqrt{29/6} = \sqrt{174}/6$.
5. By Exercise 3 of Part A, $G - A = \vec{q}5/6$, so the components of $G - A$ are $(5/6, -5/6, 5/3)$. Since the coordinates of A are $(0, -2, -1)$, it follows that the coordinates of G are $(5/6, -17/6, 2/3)$.
6. $d(P, G) = \sqrt{(2 - 5/6)^2 + (-1 + 17/6)^2 + (1 - 2/3)^2} = \sqrt{174}/36 = \sqrt{174}/6$.

*

Explanation of (8): $P - F = \text{proj}_{[\pi]^{\perp}}(P - A)$

Explanation of (9): $d(P, \pi) = \|P - F\|$, and $\|\vec{w}\| = 1$.

*

Part C

1. Use Theorem 13-2 and (9), above, to find the distance between P and $\overleftrightarrow{ABC}$ when A, B, C, and P are as in Part B. [Be sure your answers for this exercise and Exercise 1 of Part B agree.]
2. Use (8), above, to find the components of $P - F$.
3. Use the result of Exercise 2 to find the components of $F - A$. [You found these components while doing Exercise 2 of Part B. Does your answer for the present exercise agree with these results?]
4. (a) Suppose that, with respect to a given orthonormal coordinate system, the plane π is described by the equation:

$$3(x_1 - 0) + (x_2 + 2) - (x_3 - 1) = 0$$

and that P has coordinates $(3, -1, 1)$. Find $d(P, \pi)$. [*Hint*: Recall Theorem 13-1.]

(b) Suppose that P is as in part (a) and that the plane π has the equation:

$$3x_1 + x_2 - x_3 + 3 = 0$$

Find $d(P, \pi)$. [*Hint*: In checking note that the equation given here is equivalent to the one in part (a).]

5. In each of the following you are given an equation, with respect to orthonormal coordinates, of a plane π and the coordinates of a point P. In each case, find $d(P, \pi)$.
   (a) $2(x_1 - 1) - x_2 + 2(x_3 - 1) = 0$, $(2, 1, 2)$
   (b) $2x_1 - x_2 + 2x_3 = 4$, $(2, 1, 2)$
   (c) $3x_1 + 2x_2 - 6x_3 = 14$, $(5, -3, 2)$
   (d) $3x_1 + 2x_2 - 6x_3 = 14$, $(5, 3, -2)$
6. Consider a plane σ and an orthonormal coordinate system for which σ is the third coordinate plane. An equation of the form '$a_1x_1 + a_2x_2 = b$' can be thought of either as an equation of a plane π which is perpendicular to σ or, if we ignore the coordinate x_3, as an equation of the line $\sigma \cap \pi$. [See Part B on page 131, and the text following it on page 132.] In each of the following you are given such an equation of a line $l \subseteq \sigma$ and the coordinates of a point $P \in \sigma$. Find $d(P, l)$ by a method analogous to that which you used in Exercise 5. [If you are doubtful of the method to be used, note that if $l = \sigma \cap \pi$ where $\pi \perp \sigma$, and if $P \in \sigma$, then $d(P, l) = d(P, \pi)$. Explain.]
   (a) $3x_1 + 4x_2 = 5$, $(2, -3, 0)$ (b) $5x_1 - 12x_2 = -14$, $(6, 1, 0)$
   (c) $5x_1 - 12x_2 = -14$, $(1, 2)$ (d) $5x_1 - 12x_2 = -14$, $(0, 0)$

*

Answers for Part C

1. By Theorem 13-2, the vector $\vec{m}$ with components

$$\left(\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\right)$$

is in $[\vec{q}, \vec{r}]^{\perp} = [\overleftrightarrow{ABC}]^{\perp}$. Letting $\vec{w} = \vec{m}/\|\vec{m}\|$, the components of $\vec{w}$ are $(-3/\sqrt{11}, -1/\sqrt{11}, 1/\sqrt{11})$. So, by (2), $d(P, \overleftrightarrow{ABC}) = |(P - A)\cdot\vec{w}| = |(-6 - 1 + 2)/\sqrt{11}| = 5/\sqrt{11}$.

2. From Exercise 1, $(P - A)\cdot\vec{w} = -5/\sqrt{11}$, so, by (8), $P - F = -\vec{w}5/\sqrt{11}$. Therefore, the components of $P - F$ are $(15/11, 5/11, -5/11)$.

3. $F - A = (P - A) + (F - P)$. Therefore, the components of $F - A$ are $(7/11, 6/11, 27/11)$.

4. (a) $10/\sqrt{11}$ [By Theorem 14-1, the vector with components $(3, 1, -1)$ is in $[\pi]^{\perp}$. Therefore, one unit vector, $\vec{w}$, in $[\pi]^{\perp}$ has components $(3/\sqrt{11}, 1/\sqrt{11}, -1/\sqrt{11})$. Letting A have coordinates $(0, -2, 1)$ it follows that $A \in \pi$, and that $P - A$ has components $(3, 1, 0)$. So, applying (9), we have $d(P, \pi) = |9 + 1|/\sqrt{11}$.]

   (b) $10/\sqrt{11}$ [Students should be encouraged to use values for 'A' in π other than that chosen in part (a). For example, use for A the point with coordinates $(-1, 0, 0)$.]

*

Call students attention to the fact that the work done in solving Exercise 4(a) amounts to substituting the coordinates of P for 'x_1', 'x_2', and 'x_3' in the left side of the equation of the plane, dividing the result by $\sqrt{3^2 + 1^2 + (-1)^2}$, and taking the absolute value of this quotient:

$$d(P, \pi) = |3(3 - 0) + (-1 + 2) - (1 - 1)|/\sqrt{11}$$

Similarly, since the equation in part (b) could be put in the same form as that in part (a), the same algorithm will work here:

$$d(P, \pi) = |3\cdot 3 + 1\cdot -1 - 1\cdot 1 + 3|/\sqrt{11}$$

[See the commentary following that for Exercise 6.] The parts of Exercises 5 and 6 can be solved by the same algorithm.

*

5. (a) 1 (b) 1 (c) 17/7 (d) 19/7

6. [If $P \in \sigma \perp \pi$ then the perpendicular from P to π is a subset of σ and the foot of this perpendicular belongs to $\sigma \cap \pi$ and is also the foot of the perpendicular from P to the line $\sigma \cap \pi$. So, if $P \in \sigma$ and $\ell \subseteq \sigma$ then $d(P, \ell) = d(P, \pi)$ where π is the plane through ℓ which is perpendicular to σ. If σ is one of the coordinate planes then the equation of π is the same as the equation of ℓ with respect to the related coordinate system for σ.]

   (a) 11/5 (b) 32/13 (c) 5/13 (d) 14/13

In Exercises 4 and 5 you have seen that if the equation:

$$x_1m_1 + x_2m_2 + x_3m_3 = c$$

describes a plane π with respect to some orthonormal coordinate system, and P is the point whose coordinates with respect to this coordinate system are (p_1, p_2, p_3), then

$$d(P, \pi) = |p_1m_1 + p_2m_2 + p_3m_3 - c|/\sqrt{m_1^2 + m_2^2 + m_3^2}.$$

You have also seen that $p_1m_1 + p_2m_2 + p_3m_3 - c$ may be positive, negative, or zero. Suppose, now, that F is the foot of the perpendicular from P to π and that G is the foot of the perpendicular from Q to π, where Q has coordinates (q_1, q_2, q_3). What can you say about the sense of $P - F$ and the sense of $Q - G$ if the numbers $p_1m_1 + p_2m_2 + p_3m_3 - c$ and $q_1m_1 + q_2m_2 + q_3m_3 - c$ are

(a) both positive? **(b)** both negative?

(c) one positive and the other negative? [*Hint*: Recall (8) on page 172.]

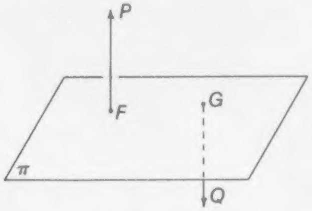


Fig. 14–9

Given a plane π, points P and Q are said to be *on the same side of* π if $P - \text{proj}_\pi(P)$ and $Q - \text{proj}_\pi(Q)$ are non-$\overrightarrow{0}$ vectors with the same sense. And P and Q are said to be *on opposite sides of* π if $P - \text{proj}_\pi(P)$ and $Q - \text{proj}_\pi(Q)$ are non-$\overrightarrow{0}$ vectors with opposite senses. By (8) on page 172 we can determine whether P and Q are on the same side of π or on opposite sides of π by computing $(P - A) \cdot \overrightarrow{m}$ and $(Q - A) \cdot \overrightarrow{m}$, where A is any chosen point of π and $\overrightarrow{m}$ is any non-$\overrightarrow{0}$ vector in $[\pi]^\perp$. P and Q are on the same side of π if these two numbers are both positive or both negative, and P and Q are on opposite sides of π if one of these numbers is positive and the other is negative. [Explain.]

We summarize some of the results obtained in Part C in the following:

Answers for (a), (b), and (c):

If (a_1, a_2, a_3) are the coordinates of any given point of π then $a_1m_1 + a_2m_2 + a_3m_3 = c$. It follows that

$$p_1m_1 + p_2m_2 + p_3m_3 - c = (p_1 - a_1)m_1 + (p_2 - a_2)m_2 + (p_3 - a_3)m_3.$$

Consequently, $p_1m_1 + p_2m_2 + p_3m_3 - c = (P - A)\cdot\vec{m}$ where $\vec{m} \in [\pi]^\perp$, and, similarly, $q_1m_1 + q_2m_2 + q_3m_3 - c = (Q - A)\cdot\vec{m}$. Since F belongs to π we may take A to be F. Since P - F and Q - G belong to the direction $[\pi]^\perp$ it follows that these vectors have the same sense if their dot products with $\vec{m}$ are both positive or both negative and that they have opposite senses if one of the dot products is positive and the other is negative. Since, as we have seen, the dot products are $p_1m_1 + p_2m_2 + p_3m_3 - c$ and $q_1m_1 + a_2m_2 + q_3m_3 - c$ it follows that in case (a) or case (b) P - F and Q - G have the same sense and that in case (c) P - F and Q - G have opposite senses.

The explanation asked for at the end of the paragraph preceding Theorem 14-17 has been given in the preceding discussion of (a), (b), and (c).

Note that an easy way to say that a and b are either both positive or both negative is to say that ab > 0.

As Exercise 6 of Part C suggests, there is a "plane analytic geometry" analogue of Theorem 14-17:

> If, with respect to some orthonormal coordinate system in the plane σ, the line ℓ is described by:
>
> $$x_1m_1 + x_2m_2 = c,$$
>
> and the point P has coordinates (p_1, p_2), then
>
> $$d(P, \ell) = |p_1m_1 + p_2m_2 - c|/\sqrt{m_1^2 + m_2^2}.$$
>
> Moreover, if Q has coordinates (q_1, q_2) then P and Q are on the same side of ℓ or on opposite sides of ℓ according as $(p_1m_1 + p_2m_2 - c)(q_1m_1 + q_2m_2 - c)$ is positive or negative.

We shall have more to say about the sides of a line — the half-planes having the line as edge — in the next chapter.

Theorem 14-17 If, with respect to some orthonormal coordinate system, the plane π is described by:

$$x_1m_1 + x_2m_2 + x_3m_3 = c,$$

and the point P has coordinates (p_1, p_2, p_3), then

$$d(P, \pi) = |p_1m_1 + p_2m_2 + p_3m_3 - c|/\sqrt{m_1^2 + m_2^2 + m_3^2}.$$

Moreover, if Q has coordinates (q_1, q_2, q_3) then P and Q are on the same side or on opposite sides of π according as $(p_1m_1 + p_2m_2 + p_3m_3 - c)(q_1m_1 + q_2m_2 + q_3m_3 - c)$ is positive or negative.

Part D

In each of the following, you are given an equation for a plane π and the coordinates of two points P and Q. Make use of Theorem 14-17 to (a) compute $d(P, \pi)$ and $d(Q, \pi)$, and (b) determine whether or not P and Q are on the same side of π.

1. $3x_1 - 2x_2 + 5x_3 = 10$ [for π]; $(6, -4, 10)$ [for P]; $(-5, 7, -3)$ [for Q]
2. $4x_1 + 3x_2 + 14x_3 = 7$ [for π]; $(1, 1, 0)$ [for P]; $(5, 5, -2)$ [for Q]
3. $9x_1 - 5x_2 + 7x_3 = 0$ [for π]; $(-5, -9, 0)$ [for P]; $(2, -7, -5)$ [for Q]
4. $-8x_1 - 7x_2 + 9x_3 = 3$ [for π]; $(-4, -3, 16)$ [for P]; $(0, 0, 0)$ [for Q]
5. $-5x_1 + 13x_2 + 12x_3 = 20$; $(6, -12, -11)$; $(11, -25, -23)$
6. $x_1 + x_2 - 2x_3 = 7$; $(1, 2, 3)$; $(0, 1, -2)$

Part E

Given an orthonormal coordinate system, suppose that π is described by the equation '$x_1 + 2x_2 - 2x_3 = 7$'.

1. Show that the point C, whose coordinates are $(3, 1, -1)$, is in π.
2. Let l be the normal to π which contains C. Compute the coordinates of the points of l each of which is at a distance 6 from C.
3. Show that the two points whose coordinates you computed in Exercise 2 are on opposite sides of π.
4. Compute the coordinates of the points of π which have first coordinate 3 *and* are at a distance of 8 from C.
5. What is the distance between the points described in Exercises 2 and 4?

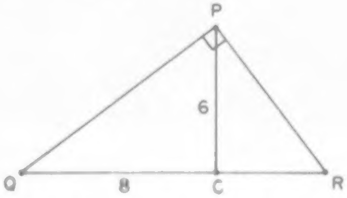
☆6. Let P be one of the points described in Exercise 2 and let Q be one of the points described in Exercise 4. Compute the coordinates of the point R of π such that PQR is a right triangle with hypotenuse $\overline{QR}$ containing C.

Answers for Part D

1. $66/\sqrt{38}$; $54/\sqrt{38}$; opposite sides
2. 0; 0; P and Q belong to π and are neither on the same side nor on opposite sides of π.
3. 0; $18/\sqrt{155}$; Since $P \in \pi$ it is neither on the same side of π with any point or on the opposite side of π from any point.
4. $\sqrt{194}$; $3/\sqrt{194}$; opposite sides
5. $337/\sqrt{338}$; $676/\sqrt{338}$; same side
6. $10/\sqrt{6}$; $2/\sqrt{6}$; same side

Answers for Part E

1. $C \in \pi$ because $1 \cdot 3 + 2 \cdot 1 - 2 \cdot -1 = 7$.
2. One of the unit translations in $[\pi]^{\perp}$ has components $(1/3, 2/3, -2/3)$. So, the points in this direction which are at a distance 6 from C have coordinates $(3 + 6/3, 1 + 12/3, -1 - 12/3)$ and $(3 - 6/3, 1 - 12/3, -1 + 12/3)$. Simplifying, the points in question have coordinates $(5, 5, -5)$ and $(1, -3, 3)$.
3. Since $(5 + 10 + 10 - 7)(1 - 6 - 6 - 7) < 0$ the points are on opposite sides of π. [This result is obvious without the use of Theorem 14-17 because the points — say, P and Q — were chosen so that P - C and Q - C have the senses of opposite unit vectors in $[\pi]^{\perp}$.]
4. Let (p_1, p_2, p_3) be the coordinates of such a point. Then $p_1 = 3$, $p_1 + 2p_2 - 2p_3 = 7$, and $\sqrt{(p_1 - 3)^2 + (p_2 - 1)^2 + (p_3 + 1)^2} = 8$. So, $p_2 - p_3 = 2$ and $(p_2 - 1)^2 + (p_3 + 1)^2 = 64$. Substituting '$p_3 + 2$' for '$p_2$' in this last equation, we have $(p_3 + 1)^2 + (p_3 + 1)^2 = 64$, so $(p_3 + 1)^2 = 32$. Therefore, $p_3 + 1 = \pm\sqrt{32}$, and $p_3 = -1 \pm \sqrt{32}$. So, the points that satisfy the given conditions have coordinates $(3, 1 + \sqrt{32}, -1 + \sqrt{32})$ and $(3, 1 - \sqrt{32}, -1 - \sqrt{32})$.
5. 10
6. Let (r_1, r_2, r_3) be the coordinates of such a point R. Since $R \in \overleftrightarrow{QC}$ and Q and C belong to the plane with equation '$x_1 = 3$', R also belongs to this plane and, so, $r_1 = 3$. Since $\overline{CP}$ is the altitude of right $\triangle QPR$ to its hypotenuse, $8 \cdot CR = 36$ and, so, $CR = 9/2$. Depending on which point from Exercise 4 we choose for Q, C - Q has components $(0, -4\sqrt{2}, -4\sqrt{2})$ or $(0, 4\sqrt{2}, 4\sqrt{2})$. So, a unit vector in the sense of C - Q has components $(0, -/\sqrt{2}, -/\sqrt{2})$ or $(0, /\sqrt{2}, /\sqrt{2})$. In the two cases the coordinates of R are $(3, 1 - 9/(2\sqrt{2}), -1 - 9/(2\sqrt{2}))$ and $(3, 1 + 9/(2\sqrt{2}), -1 + 9/(2\sqrt{2}))$, respectively.



14.04 Isometries and Congruences

Euclidean geometry has a good deal to do with the sizes and shapes of geometric figures. To clarify these ideas, note that of the four triangles pictured here, all have the same shape and three—but not all four—have the same size. If you make a tracing of one of the three congruent triangles, you can so place the tracing on the page as to exactly cover either of two of the other triangles. [Do so.]

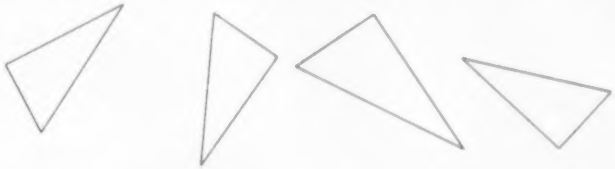


Fig. 14–10

To formalize this notion we introduce the idea of *an isometric mapping* [or: an isometry].

> Definition 14–8 f is an isometry of $\mathscr{E}$ if and only if f is a mapping of $\mathscr{E}$ onto itself such that
>
> $$\forall_X \forall_Y \, d(f(X), f(Y)) = d(X, Y).$$

In short, an isometry is a mapping of $\mathscr{E}$ onto itself which "preserves distances".

Imagine that the page holding Figure 14–10 represents a plane and hold the paper on which you traced one of the triangles so as to represent another plane. If you now place the tracing paper on the page holding Figure 14–10 you have an illustration of the effect, on points of the second plane, of an isometry which maps the second plane on the first. Your experiment with the tracing of the triangles suggests that two figures have the same size and shape if and only if there is an isometry which maps one of them onto the other. Figures which have the same size and shape are said to be *congruent* and we shall use the insight we have just obtained to define this notion:

> Definition 14–9 A first figure is congruent to a second if and only if there is an isometry of $\mathscr{E}$ which maps the first figure onto the second.

An isometry may be thought of as the mapping which results from one or more rigid motions or [see page 177] plane reflections. The introduction of isometries then justifies, intuitively, the notion of superposition. The latter has no place in conventional developments of geometry because, in these developments, there is no discussion of — and it would be difficult to discuss — isometries.

Note that an isometry is a mapping of [all of] $\mathcal{E}$ onto $\mathcal{E}$ — not merely of some geometric figure onto another.

With quite a bit more effort, we could get along with 'into' in place of 'onto' in Definition 14-8. That is, it can be proved — but, not easily — that a distance-preserving mapping of $\mathcal{E}$ into itself must map $\mathcal{E}$ onto [all of] $\mathcal{E}$.

We introduce '$\cong$' for 'is congruent to' on page 219. You may wish to introduce it earlier -- say in connection with Part C on page 179. [As a matter of fact, we do slip it in as an aside on page 177.]

Suggestions for the exercises of section 14.04

(i) So that students become properly acquainted with reflections we recommend Parts A and B as class exercises.
(ii) Parts C and D may be used for homework.
(iii) Part E and the discussion of Theorem 14-23 should be teacher directed.
(iv) Parts F and G may be used for homework. Exercise 4 of Part G will probably need further discussion, however.
(v) Parts H and I may be difficult as a homework assignment. If used as such, be sure to thoroughly discuss the exercises the following day.

As an example of the ideas formalized in Definitions 14–8 and 14–9 notice that, for any translation $\vec{a}$,

$$\begin{aligned} d(P + \vec{a}, Q + \vec{a}) &= \|(Q + \vec{a}) - (P + \vec{a})\| \\ &= \|Q - P\| = d(P, Q). \end{aligned}$$

So, we have:

‖ **Theorem 14–18** Any translation is an isometry.

In particular, if $\triangle PQR$ and $\triangle TUV$ are such that $T - P = U - Q = V - R$ then $\triangle PQR$ and $\triangle TUV$ are congruent. [For short, we write: $\triangle PQR \cong \triangle TUV$.]

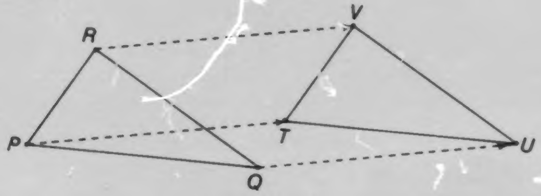


Fig. 14–11

Another kind of isometry of $\mathscr{E}$ can be illustrated intuitively by the use of flat mirror. As you must often have observed, the image of an object in such a mirror appears to have the same size and shape as the object. This suggests that there is an isometry of $\mathscr{E}$ which maps each object which is in front of the mirror onto an object behind the mirror which has the same size and shape as the mirror image of the given object. To see how to describe this isometry formally, replace the mirror by a plane π. [For simplicity, we have pictured π edge-wise in Fig. 14–12.] If M is the foot of the perpendicular from P to π then the image $f(P)$ "in π" of P will be the point $M + (M - P)$.

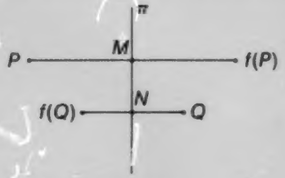


Fig. 14–12

Plane reflections are not only useful as examples of isometries. We shall prove [see Theorem 14-31] that any isometry is either the identity mapping $\vec{0}$ or a plane reflection or the resultant of at most four plane reflections. With more effort than we wish to expend in this course, one can justify replacing the word 'four' in the preceding sentence by 'three'.

The words 'in π' which occur in the text shortly preceding Definition 14-10 are double-quoted as a guard against believing that f(P) is a point in π. It is, of course, the reflection which is, in another sense of 'in', in π.

* * *

Sample Quiz

Given an orthonormal coordinate system, suppose that σ is the plane described by the equation '$3x + 2y - z = 6$' and that A and B are points whose coordinates are (2, 4, 1) and (5, −3, 5), respectively.

1. Show that neither A nor B is a point of σ.
2. Tell whether A and B are on the same side of σ or on opposite sides of σ.
3. Compute $d(A, \sigma)$ and $d(B, \sigma)$.
4. Let P and Q be the feet of the perpendiculars from A and B to σ. Show that $\overline{AB}$ and $\overline{PQ}$ have a point in common.

Key to Sample Quiz

1. Since $3 \cdot 2 + 2 \cdot 4 - 1 \neq 6$ and $3 \cdot 5 + 2 \cdot -3 - 5 \neq 6$, neither A nor B belongs to σ.
2. Since $3 \cdot 2 + 2 \cdot 4 - 1 - 6 = 7 > 0$ and $3 \cdot 5 + 2 \cdot -3 - 5 - 6 = -2 < 0$, A and B are on opposite sides of σ.
3. $d(A, \sigma) = |7|/\sqrt{9 + 4 + 1} = 7/\sqrt{14} = \sqrt{14}/2$ and
   $d(B, \sigma) = |-2|/\sqrt{14} = 2/\sqrt{14} = \sqrt{14}/7$.
4. APBQ is a trapezoid — and, so, is convex — with diagonals $\overline{AB}$ and $\overline{PQ}$. Hence, $\overline{AB}$ and $\overline{PQ}$ have a point in common. [In fact, they divide each other in the ratio 7/2.]

The corollary may be restated as:

If f is the reflection in π then $f \circ f = \vec{0}$.

— that is, each plane reflection is its own inverse or, for short, is an inversion. To see this note that, assuming that f is a plane reflection it follows from the corollary [with 'f(P)' for 'Q'] that f(f(P)) = P. Also, if f(f(P)) = P it follows [using the replacement rule for equations] that if f(P) = Q then f(Q) = P. From this last it follows that if f(Q) = P then f(P) = Q. [Merely interchange 'P' and 'Q'.]

This suggests the following definition:

> **Definition 14-10** f is the reflection in π
>
> $$\longleftrightarrow$$
>
> $$\forall_X\, f(X) = X + (M - X)2,$$
>
> where M is the foot of the perpendicular from X to π.

Using this definition it is not difficult to prove:

> **Theorem 14-19** If f is the reflection in π then
> $f(P) = Q \longleftrightarrow (Q - P \in [\pi]^{\perp}$ and the midpoint of $\overleftrightarrow{PQ} \in \pi)$.

This theorem yields the following:

> **Corollary** If f is the reflection in π then
> $f(P) = Q \longleftrightarrow f(Q) = P$.

This corollary tells us that any mapping which is the reflection in some plane is its own inverse. [Explain.] It is also useful in proving:

> **Theorem 14-20** The reflection in a plane is an isometry.

Exercises

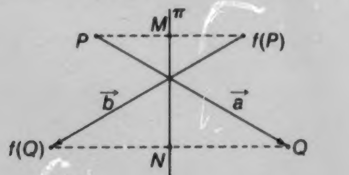
Part A

Suppose that f is the reflection in π.

1. Show that $f(P) - P \in [\pi]^{\perp}$ and that the midpoint of $\overleftrightarrow{Pf(P)} \in \pi$. [*Hint*: Use Definition 14-10.]
2. Assume that $Q - P \in [\pi]^{\perp}$ and that the midpoint of $\overleftrightarrow{PQ} \in \pi$. Show that $f(P) = Q$. [*Hint*: Let M be the midpoint of $\overleftrightarrow{PQ}$. Show that M is the foot of the perpendicular from P to π and that $Q = P + (M - P)2$.]
3. Prove Theorem 14-19.
4. Prove the corollary to Theorem 14-19. [*Hint*: Use two instances of the theorem.]

Part B

Suppose that f is the reflection of $\mathscr{E}$ in π. To prove that f is an isometry we must show that f is a mapping of $\mathscr{E}$ *onto* itself—that is,



Answers for Part A

1. By definition, $f(P) - P = (P + (M - P)2) - P = (M - P)2$, where M is the foot of the perpendicular from P to π. Since $M - P \in [\pi]^{\perp}$, it follows that $(f(P) - P) \in [\pi]^{\perp}$. The midpoint of $\overleftrightarrow{Pf(P)}$ is $P + (f(P) - P)1/2$ which is $P + (M - P)2 \cdot 1/2$ or M. Since $M \in \pi$, the midpoint of $\overleftrightarrow{Pf(P)} \in \pi$.
2. Let M be the midpoint of $\overleftrightarrow{QP}$. Since $Q - P \in [\pi]^{\perp}$, and $M \in \overleftrightarrow{QP}$, $P - M \in [\pi]^{\perp}$. We are given that the midpoint $M \in \pi$. Therefore, M is the foot of the perpendicular from P to π. Now, $M = P + (Q - P)1/2$, so $P + (M - P)2 = P + [P + (Q - P)1/2 - P]2 = P + (Q - P) = Q$. Since f is the reflection in π, $f(P) = P + (M - P)2 = Q$.
3. Suppose that f is the reflection in π. If $f(P) = Q$, then by Exercise 1, $Q - P \in [\pi]^{\perp}$ and the midpoint of $\overleftrightarrow{PQ} \in \pi$. Conversely, if $Q - P \in [\pi]^{\perp}$ and the midpoint of $\overleftrightarrow{PQ} \in \pi$, then by Exercise 2, $f(P) = Q$. Therefore, if f is the reflection in π then $f(P) = Q \Longleftrightarrow (Q - P \in [\pi]^{\perp}$ and the midpoint of $\overleftrightarrow{PQ} \in \pi.)$
4. Suppose that f is the reflection in π. If $f(P) = Q$, then by Theorem 14-19, $Q - P \in [\pi]^{\perp}$ and the midpoint of $\overleftrightarrow{PQ} \in \pi$. Therefore, $P - Q \in [\pi]^{\perp}$ and the midpoint of $\overleftrightarrow{QP} \in \pi$. So, by Theorem 14-19, $f(Q) = P$. Therefore, if f is the reflection in π then $f(P) = Q \Longrightarrow f(Q) = P$. Since P and Q are arbitrary points, the corollary follows.

TC 179 (1)

Answers for Part B

1. It follows from the corollary to Theorem 14-19. [If $Q \in \mathscr{E}$, and $f(Q) = P$, then $f(P) = Q$ so Q is in the range of f.]
2. $(\vec{b} - \vec{a}) \perp (\vec{b} + \vec{a})$ if and only if $(\vec{b} - \vec{a}) \cdot (\vec{b} + \vec{a}) = 0$ — that is, if and only if $\vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{a} = 0$. The latter is the case if and only if $\vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{a}$ — that is, if and only if $\|\vec{b}\| = \|\vec{a}\|$.
3. $\vec{b} - \vec{a} = (f(Q) - f(P)) - (Q - P) = (f(Q) - Q) - (f(P) - P)$. Now, by Exercise 1 of Part A, $f(Q) - Q \in [\pi]^{\perp}$ and $f(P) - P \in [\pi]^{\perp}$, so their difference belongs to $[\pi]^{\perp}$. Therefore, $\vec{b} - \vec{a} \in [\pi]^{\perp}$.
4. $\vec{b} + \vec{a} = (f(Q) - f(P)) + (Q - P) = (f(Q) - P) + (Q - f(P))$ $= [(f(Q) - N) + (N - M) + (M - P)] + [(Q - N) + (N - M) + (M - f(P))]$ $= [(f(Q) - N) + (Q - N)] + [(M - P) + (M - f(P))] + (N - M)2$. Now, $(f(Q) - N) + (Q - N) = \vec{0}$, and $(M - P) + (M - f(P)) = \vec{0}$. So $\vec{b} + \vec{a} = (N - M)2$. Since $N \in \pi$ and $M \in \pi$, $\vec{b} + \vec{a} \in [\pi]$.
5. We have already shown that the range of f is $\mathscr{E}$ [Exercise 1]. We must now show that $\|\vec{b}\| = \|\vec{a}\|$. By the results of Exercises 3 and 4, $(\vec{b} - \vec{a}) \perp (\vec{b} + \vec{a})$. Therefore, by Exercise 2, $\|\vec{b}\| = \|\vec{a}\|$.

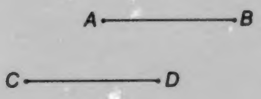
that the range of f is $\mathscr{E}$ – and that, given P and Q, $\|\vec{b}\| = \|\vec{a}\|$, where $\vec{a} = Q - P$ and $\vec{b} = f(Q) - f(P)$.

1. By definition, the domain of f is $\mathscr{E}$. From what does it follow that the range of f is $\mathscr{E}$?
2. Show that $\|\vec{b}\| = \|\vec{a}\|$ if and only if $(\vec{b} - \vec{a}) \perp (\vec{b} + \vec{a})$. [*Hint*: This is an algebraic result which you have proved before.]
3. Show that $\vec{b} - \vec{a} \in [\pi]^{\perp}$. [*Hint*: $\vec{b} - \vec{a} = (f(Q) - f(P)) - (Q - P)$.]
4. Show that $\vec{b} + \vec{a} \in [\pi]$. [*Hint*: Let M and N be the midpoints of $\overline{Pf(P)}$ and $\overline{Qf(Q)}$. Note that $f(Q) - P = (f(Q) - N) + (N - M) + (M - P)$.]
5. Complete the proof of Theorem 14–20.

Part C

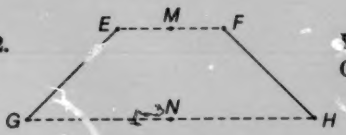
By definition, a first figure is congruent to a second if and only if there is an isometry of $\mathscr{E}$ which maps the first figure onto the second. And, we know that translations and reflections in planes are isometries. So, we are in a position to identify some pairs of congruent figures. In each of the following, you are given a pair of figures and some information about them. Try to decide which are congruent figures because there is either a translation or a plane reflection or the resultant of a translation and a plane reflection which maps one of the figures onto the other.

1.



Figures: $\overline{AB}$ and $\overline{CD}$
Given: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$; $AB = 5 = CD$; $\overleftrightarrow{AC}$ is not perpendicular to $\overleftrightarrow{AB}$

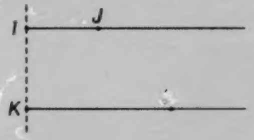
2.



Figures: $\overline{EG}$ and $\overline{FH}$
Given: M and N are the midpoints of $\overline{EF}$ and $\overline{GH}$, respectively; $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$; $\overleftrightarrow{MN} \perp \overleftrightarrow{GH}$

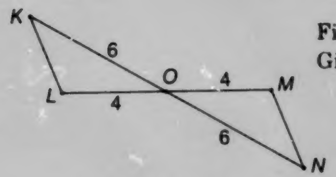
3. Figures: $\overline{EF}$ and $\overline{GH}$; same information as in Exercise 2.

4.



Figures: rays $\overrightarrow{IJ}$ and $\overrightarrow{KL}$
Given: $\overrightarrow{IJ} \parallel \overrightarrow{KL}$; $\overrightarrow{IJ}$ and $\overrightarrow{KL}$ have the same sense; $\overleftrightarrow{IK} \perp \overleftrightarrow{IJ}$
[Find *two* simple isometries which work.]

5.



Figures: $\triangle KLO$ and $\triangle NMO$
Given: $\overleftrightarrow{KN}$ and $\overleftrightarrow{LM}$ intersect in the point Q; measures of segments are indicated in picture.

Answers for Part C

[The answers for these exercises need be justified only intuitively. Later we shall prove theorems which could yield formal justifications.]

1. $\overline{AB}$ and $\overline{CD}$ are congruent for the translation C - A maps the first onto the second. [This is, at least, the case on intuitive grounds. Later we shall prove that segments — parallel or not — are congruent if and only if they have the same measure. Students should note that if, contrary to the conditions of the problem, $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$ then there is a reflection which maps $\overline{AB}$ on $\overline{CD}$.]
2. $\overline{EG}$ is congruent to $\overline{FH}$. Let π be the plane containing $\overleftrightarrow{MN}$ which is perpendicular to $\overline{EFH}$. Then the reflection in π maps $\overline{FH}$ onto $\overline{EG}$.
3. $\overline{EF}$ and $\overline{GH}$ are not congruent unless $\overleftrightarrow{EG} \parallel \overleftrightarrow{FH}$. Except in the exceptional case EF > GH, or GH > EF, and it is intuitively clear that a longer segment cannot be mapped isometrically onto a shorter one.
4. $\overrightarrow{IJ}$ and $\overrightarrow{KL}$ are congruent. The translation K - I maps the first onto the second. If π is the plane parallel to $\overleftrightarrow{IJ}$, perpendicular to $\overline{IJK}$, and containing the midpoint of $\overline{IK}$ then the plane reflection in π is another isometry which maps $\overrightarrow{IJ}$ onto $\overrightarrow{KL}$ [and $\overrightarrow{KL}$ onto $\overrightarrow{IJ}$].
5. △KLO and △NMO are congruent but not "because there is either a translation or a plane reflection or the resultant of a translation and a plane reflection which maps one figure onto the other." Students may discover that each triangle is mapped onto the other — or, at least {K, L, O} is mapped onto {N, M, O} — by a resultant of two reflections. These are the reflection in the plane π_1 through O perpendicular to $\overleftrightarrow{LM}$ and the reflection in the plane π_2 which contains $\overleftrightarrow{LM}$ and is perpendicular to the plane of the figure. Composing these two reflections in either order results in an isometry which is "the reflection in the line through O perpendicular to the plane of the figure". The resultant can also be thought of as [the result of] a half-turn about the same line as axis. [These matters are taken up in Part D on pages 189-190.]

Part D

Suppose that f is the reflection in π.

1. Show that $f(Q) = Q$ if and only if $Q \in \pi$. [*Hint*: This is easy if you use Theorem 14–19.]
2. Suppose that $f(P) \neq P$. Show that π is the perpendicular bisector of $\overline{Pf(P)}$. [*Hint*: Use Exercise 1 and Theorem 14–20 to show that each point of π is equidistant from P and $f(P)$.]
3. Show that if $Q \neq P$ and π is the perpendicular bisector of $\overline{PQ}$ then $Q = f(P)$.

Part E

Suppose that g is any isometry of $\mathscr{E}$ which leaves fixed each of three noncollinear points—say, A, B, and C.

1. Show that if $g(P) \neq P$ then the perpendicular bisector of $\overline{Pg(P)}$ is the plane $\overleftrightarrow{ABC}$. [*Hint*: By assumption, $g(A) = A$, $g(B) = B$, and $g(C) = C$. Now, argue as in Exercise 2 of Part D.]
2. Show that if $P \in \overleftrightarrow{ABC}$ then $g(P) = P$. [*Hint*: Consider the contrapositive and apply Exercise 1.]
3. Suppose that g is not the identity mapping of $\mathscr{E}$ onto itself. Show that if $g(Q) = Q$ then $Q \in \overleftrightarrow{ABC}$. [*Hint*: Since g is not the identity mapping, there is a point—say, P—such that $g(P) \neq P$. Assuming that $g(Q) = Q$, what follows concerning Q and the points P and $g(P)$?]
4. Show that g is either the identity mapping or the reflection in $\overleftrightarrow{ABC}$. [*Hint*: Suppose that g is not the identity mapping. What do Exercises 2 and 3 tell you about the points which g leaves fixed? What follows concerning P and $g(P)$ in case $P \notin \overleftrightarrow{ABC}$? Use Exercise 1 together with Exercise 3 of Part D to show that, in case $P \notin \overleftrightarrow{ABC}$, $g(P)$ is the reflection of P in $\overleftrightarrow{ABC}$. Is this also the case for a point $P \in \overleftrightarrow{ABC}$?]
5. Suppose that f is an isometry which leaves A and B fixed. Show that f leaves fixed each point of $\overleftrightarrow{AB}$. [*Hint*: If $f(P) \neq P$ then P does not belong to the perpendicular bisector of $\overline{Pf(P)}$.]

*

In Parts D and E you have proved:

> **Theorem 14–21** If f is the reflection in π then
> (a) $f(P) = P$ if and only if $P \in \pi$, and
> (b) if $f(P) \neq P$ then π is the perpendicular bisector of $\overline{Pf(P)}$.

and:

Answers for Part D

1. By Theorem 14-19, $f(Q) = Q$ if and only if $(Q - Q \in [\pi]^{\perp}$ and the midpoint of $\overline{QQ} \in \pi)$. But, in any case, $Q - Q \in [\pi]^{\perp}$, and the midpoint of $\overline{QQ}$ is Q and, so, belongs to π if and only if $Q \in \pi$.
2. If $R \in \pi$ then $f(R) = R$ and, since f is an isometry, $d(P, R) = d(f(P), f(R)) = d(f(P), R)$ — that is, R belongs to the perpendicular bisector of $\overline{Pf(P)}$. This shows that the plane π is a subset of the plane which is the perpendicular bisector of $\overline{Pf(P)}$. Consequently, π is the perpendicular bisector. [If $\pi \subseteq \sigma$ then $\pi = \sigma$. For suppose $\pi \subseteq \sigma$ and let $\{A, B, C\}$ be a noncollinear subset of π and, so, of σ. It follows that $\pi = \overleftrightarrow{ABC} = \sigma$.]
3. If π is the perpendicular bisector of $\overline{PQ}$ $[Q \neq P]$ then $Q - P \in [\pi]^{\perp}$ and the midpoint of $\overline{PQ} \in \pi$ — that is, by Theorem 14-19, $Q = f(P)$.

Answers for Part E

1. Since g is an isometry, $||A - P|| = ||g(A) - g(P)||$. But, $g(A) = A$. Therefore, $||A - P|| = ||A - g(P)||$. Similarly, $d(B, P) = d(B, g(P))$ and $d(C, P) = d(C, g(P))$. Since A, B, and C are noncollinear and are all equidistant from P and g(P), the perpendicular bisector of $\overleftrightarrow{Pg(P)}$ must be $\overleftarrow{ABC}$.

2. Suppose that $g(P) \neq P$. Then by Exercise 1, $\overleftarrow{ABC}$ is the perpendicular bisector of $\overleftrightarrow{Pg(P)}$. So, $P \notin \overleftarrow{ABC}$. Hence, if $P \in \overleftarrow{ABC}$ then $g(P) = P$.

3. Since g is not the identity mapping $\vec{0}$ there is a point — say, P — such that $g(P) \neq P$. By Exercise 1, $\overleftarrow{ABC}$ is the perpendicular bisector of $\overleftrightarrow{Pg(P)}$ and, so, contains all points equidistant from P and g(P). If $g(Q) = Q$ then Q is such a point since $d(P, Q) = d(f(P), f(Q)) = d(f(P), Q)$.

4. Suppose that g is not $\vec{0}$. It follows by Exercises 2 and 3 that g leaves fixed precisely the points of $\overleftarrow{ABC}$. In particular, if $P \notin \overleftarrow{ABC}$ then $g(P) \neq P$ and, by Exercise 1, $\overleftarrow{ABC}$ is the perpendicular bisector of $\overleftrightarrow{Pg(P)}$. Hence, by Exercise 3 of Part D, g(P) is the image of P under the reflection in $\overleftarrow{ABC}$. Since this reflection — like g — leaves each point of $\overleftarrow{ABC}$ fixed it follows that if g is not the identity mapping then g is the reflection in $\overleftarrow{ABC}$.

5. Let $P \in \overleftrightarrow{AB}$, and assume that $f(P) \neq P$. Let π be the perpendicular bisector of $\overleftrightarrow{Pf(P)}$. Then since $A = f(A)$, $d(A, P) = d(f(A), f(P)) = d(A, f(P))$. Similarly, $d(B, P) = d(B, f(P))$. Therefore, $A \in \pi$ and $B \in \pi$. So, $\overleftrightarrow{AB} \subset \pi$. But $P \in \overleftrightarrow{AB}$ and it is impossible for $P \in \pi$. Therefore, $f(P) = P$.

Since the two new concepts dealt with in this volume are those of distance and perpendicularity it might be expected that Theorem 14-6, which relates these concepts, is of major importance. That this is the case is brought out by Theorems 14-21 through 14-23, all of which depend strongly on Theorem 14-6. Recall that Theorem 14-4 is a formalization of Notion 8 on page 29. Theorem 14-9 is another formalization of the same notion.

Theorem 14–22 The only isometries of $\mathcal{E}$ which leave fixed three given noncollinear points are the identity mapping and the reflection in the plane which contains the given points.

From this and Theorem 14–21(a), we have at once the following:

Corollary The only isometry of $\mathcal{E}$ which leaves fixed four noncoplanar points is the identity mapping.

—that is, an isometry which leaves fixed each of four noncoplanar points leaves each point fixed. Also, by Exercise 5, we have:

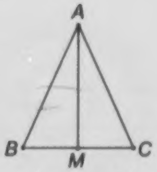
Theorem 14–23 An isometry which leaves two points fixed also leaves fixed each point of the line containing the two points.

*

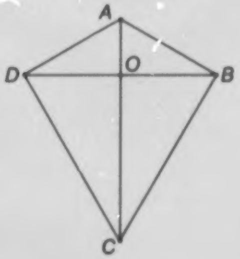
Part F

In each of the following, you are given a figure and some information about it. Make use of what you know about translations and plane reflections to help you answer questions about congruence.

1. Suppose that $\triangle ABC$ is isosceles with base $\overline{BC}$ and that M is the midpoint of $\overline{BC}$. Give reasons for believing that $\triangle ABM$ and $\triangle ACM$ are congruent. [*Hint:* Describe an isometry which appears to map one triangle on the other.]



2. Suppose that $\triangle ABD$ and $\triangle CBD$ are isosceles, each with base $\overline{BD}$, and that $\overline{AC}$ and $\overline{BD}$ intersect at O.
   (a) Give reasons for believing that $\triangle ADC$ and $\triangle ABC$ are congruent.
   (b) Under what conditions could you find an isometry which appears to map $\triangle ADB$ onto $\triangle CDB$?
   (c) Assume that $AO = 3$ and $OC = 9$ and $\triangle ADC$ is a right triangle with hypotenuse $\overline{AC}$. Compute DO, OB, AD, and BC.



Answers for Part F

[Students are not yet prepared to prove triangles congruent. The best they can do in this direction is to find an isometry which maps the vertices of one triangle on those of the other. The discovery of such an isometry is "reason for believing" the triangles to be congruent.]

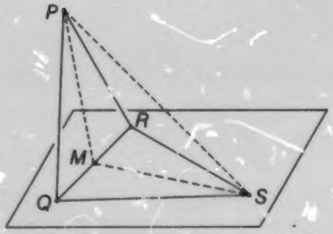
1. Let π be the plane which is perpendicular to $\overleftrightarrow{ABC}$ and contains $\overleftrightarrow{AM}$. Since, by Theorem 14-9, $\overleftrightarrow{BC}$ is perpendicular to the line $\overleftrightarrow{AM}$ in π it follows that π is the perpendicular bisector of $\overline{BC}$. The reflection in π maps each of A and M on itself and maps B on C. So, it seems likely [see, also, Theorem 14-27] that it maps $\triangle ABM$ on $\triangle ACM$. If so, $\triangle ABM$ and $\triangle ACM$ are congruent.

2. (a) Since $\triangle ABD$ and $\triangle CBD$ are isosceles, both A and C are in the perpendicular bisector, π, of $\overline{BD}$. Since O is the point of intersection of $\overline{AC}$ and $\overline{BD}$ it is also the point of intersection of π and $\overline{BD}$. So, O is the midpoint of $\overline{BD}$. Hence, just as in Exercise 1, the reflection in π maps each of A, O, and C on itself and maps D on B. Consequently, it is reasonable to guess that this reflection maps $\triangle ADC$ onto $\triangle ABC$ and, so, to guess that these triangles are congruent.
   (b) There will, seemingly, be an isometry mapping $\triangle ADB$ onto $\triangle CDB$ in case the plane which contains $\overleftrightarrow{BD}$ and is perpendicular to the plane of the figure is the perpendicular bisector of $\overline{AC}$. This will be the case if $AD = DC$ [and, consequently, $AB = BC$]. [There are other correct answers.]
   (c) $DO = 3\sqrt{3}$, $OB = 3\sqrt{3}$, $AD = 6$, $BC = 6\sqrt{3}$

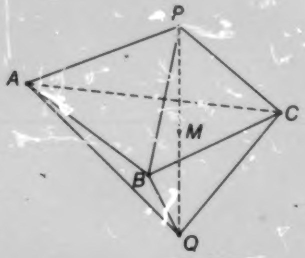
3. (a) $\overleftrightarrow{MPS}$ is a plane containing three noncollinear points equidistant from Q and R. Hence, it is the perpendicular bisector of $\overline{RQ}$.
   (b) Since the reflection in $\overleftrightarrow{MPS}$ maps Q on R and maps each of the points P and S on itself, it presumably maps $\triangle PQS$ onto $\triangle PRS$. If so, $\triangle PQS$ and $\triangle PRS$ are congruent.

4. (a) The reflection in $\overleftrightarrow{ABC}$ maps each of B and C on itself and maps P on Q so it presumably maps $\triangle PBC$ onto $\triangle QBC$. If so, the triangles are congruent.
   (b) The reflection referred to in part (a) also maps A on itself. So, it seems likely that this reflection maps each pyramid onto the other.

3. Suppose that $\triangle PQR$ and $\triangle SQR$ are isosceles, each with base $\overline{RQ}$, that $P \notin \overleftrightarrow{SQR}$, and that M is the midpoint of $\overline{RQ}$.
   (a) Show that $\overleftrightarrow{MPS}$ is the perpendicular bisector of $\overline{RQ}$.
   (b) Give reasons for believing that $\triangle PQS$ and $\triangle PRS$ are congruent.

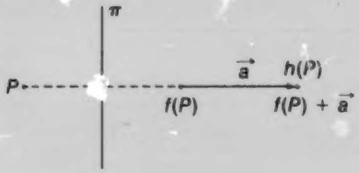


4. Suppose that M is a point in $\overleftrightarrow{ABC}$, that P and Q are two points on the normal to $\overleftrightarrow{ABC}$ through the point M, and that M is the midpoint of $\overline{PQ}$.
   (a) Give reasons for believing that $\triangle PBC$ and $\triangle QBC$ are congruent.
   (b) Give reasons for believing that the pyramids $P - ABC$ and $Q - ABC$ are congruent.



Part G

Suppose that f is the reflection in π and that $\vec{a} \in [\pi]^{\perp}$. Consider the mapping h such that, for each point X, $h(X) = f(X) + \vec{a}$ [—that is, h is the resultant of f followed by $\vec{a}$].



1. Show that, for any point P, $h(P) - P \in [\pi]^{\perp}$.
2. Show that the midpoint of $\overleftrightarrow{Ph(P)}$ is the image of the midpoint of $\overleftrightarrow{Pf(P)}$ under the translation $\vec{a}/2$.
3. Show that h is the reflection in a certain plane parallel to π.
*4. Suppose that $\vec{b}$ is any translation [while f is still the reflection in π], and let k be the resultant of f followed by $\vec{b}$. Show that k is the resultant of the reflection in a certain plane parallel to π followed by a translation $\vec{c} \in [\pi]$. [*Hint*: $\vec{b} = \vec{a} + \vec{c}$, where $\vec{a} \in [\pi]^{\perp}$ and $\vec{c} \in [\pi]$. Explain.]

*

In Exercise 3 of Part G you have established the following useful lemma:

> **Lemma** The resultant of the reflection in a plane π followed by a translation $\vec{a}$ in the direction orthogonal to π is the reflection in the plane $\pi + (\vec{a}/2)$.

Answers for Part G

1. $h(P) - P = (f(P) + \vec{a}) - P = (f(P) - P) + \vec{a}$. Since, by Theorem 14-19, $f(P) - P \in [\pi]^{\perp}$ it follows that $(f(P) - P) + \vec{a} \in [\pi]^{\perp}$. Hence, $h(P) - P \in [\pi]^{\perp}$.
2. Suppose that M is the midpoint of $\overleftrightarrow{Pf(P)}$ — that is, suppose that $M - P = f(P) - M$. What we wish to show is that $M + \vec{a}/2$ is the midpoint of $\overleftrightarrow{Ph(P)}$ — that is, that $(M + \vec{a}/2) - P = h(P) - (M + \vec{a}/2)$. It follows from our assumption that $(M + \vec{a}/2) - P = (f(P) + \vec{a}/2) - M$. And, by the definition of h, $f(P) + \vec{a}/2 = (h(P) - \vec{a}) + \vec{a}/2 = h(P) - \vec{a}/2$. So, $(M + \vec{a}/2) - P = (h(P) - \vec{a}/2) - M = h(P) - (M + \vec{a}/2)$ as we wished to show.
3. Suppose that σ is any plane parallel to π — that is, σ is any plane such that $[\sigma]^{\perp} = [\pi]^{\perp}$. To show that h is the reflection in σ we must show that, for any point P, $h(P) - P \in [\sigma]^{\perp}$ and the midpoint of $\overleftrightarrow{Ph(P)} \in \sigma$. Since, by Exercise 1, we already know that $h(P) - P \in [\pi]^{\perp}$ it follows that, for any $\sigma \parallel \pi$, $h(P) - P \in [\sigma]^{\perp}$. Our job is to find σ such that the midpoint of $\overleftrightarrow{Ph(P)} \in \sigma$. By Exercise 2, since the midpoint of $\overleftrightarrow{Pf(P)} \in \pi$, the midpoint of $\overleftrightarrow{Ph(P)} \in \pi + (\vec{a}/2)$. Since $\pi + (\vec{a}/2)$ is parallel to π, $\pi + (\vec{a}/2)$ is the plane σ we have been looking for. In short, h is the reflection in the plane $\pi + (\vec{a}/2)$.
4. Let $\vec{a} = \text{proj}_{[\pi]^{\perp}}(\vec{b})$ and $\vec{c} = \text{proj}_{[\pi]}(\vec{b})$. It follows by Theorem 12-18 that $\vec{b} = \vec{a} + \vec{c}$ and by other theorems that $\vec{a} \in [\pi]^{\perp}$ and $\vec{c} \in [\pi]$. Since, for any P, $k(P) = f(P) + \vec{b}$ it follows that $k(P) = (f(P) + \vec{a}) + \vec{c} = h(P) + \vec{c}$. Since, by Exercise 3, h is the reflection in $\pi + (\vec{a}/2)$ it follows that k is the resultant of a reflection in a plane parallel to π followed by a translation in [π]. [In a more thorough study of isometries it would be of value to know that if h is the reflection in a plane σ and $\vec{c} \in [\sigma]$ then $h(P + \vec{c}) = h(P) + \vec{c}$ — that is, that the resultant of $\vec{c}$ followed by h is the same as the resultant of h followed by $\vec{c}$. [Draw figures!] This is not difficult to prove, using Theorem 14-19. What we need to show is that $(h(P) + \vec{c}) - (P + \vec{c}) \in [\sigma]^{\perp}$ and that the midpoint of $\overleftrightarrow{(P + \vec{c})(h(P) + \vec{c})}$ belongs to π. The first is evident since $h(P) - P \in [\sigma]^{\perp}$. To prove the second, let M be the midpoint of $\overleftrightarrow{Ph(P)}$ — that is, suppose that $M - P = h(P) - M$. It follows that $(M + \vec{c}) - (P + \vec{c}) = (h(P) + \vec{c}) - (M + \vec{c})$ and, so, that $M + \vec{c}$ is the midpoint of $\overleftrightarrow{(P + \vec{c})(h(P) + \vec{c})}$. Since $M \in \sigma$ and $\vec{c} \in [\sigma]$ it follows that $M + \vec{c} \in \sigma$.]

In order to develop the consequences of this lemma and some of our previous theorems we need to recall some facts concerning function composition. Recall that if f and g are mappings of $\mathscr{E}$ onto itself then the mapping $g \circ f$ such that

$$\forall_X \, [g \circ f(X) = g(f(X))$$

is called the *resultant* of f followed by g and is, itself, a mapping of $\mathscr{E}$ onto $\mathscr{E}$. For example, the mapping h described in Part G is $\vec{a} \circ f$. [Explain.] If, continuing with this example, we let g be the reflection in the plane $\pi + (\vec{a}/2)$, the result of the lemma can be stated as:

$$\text{(1)} \qquad \vec{a} \circ f = g$$

As you learned in Chapter 1, a very important property of function composition is that it is an associative binary operation. Making use of this in connection with (1) we can conclude that

$$\text{(2)} \qquad \vec{a} \circ (f \circ f) \underset{\text{[associativity]}}{=} (\vec{a} \circ f) \circ f \underset{\text{[(1)]}}{=} g \circ f.$$

Since, by the corollary to Theorem 14–19, f is its own inverse, we know that $f \circ f = \vec{0}$ [where $\vec{0}$ is, as always, the identity mapping of $\mathscr{E}$ onto itself]. So, from (2) we obtain the result:

$$\text{(3)} \qquad \vec{a} = g \circ f$$

On recalling what f and g are, we see that we have proved:

Theorem 14–24 If π is any plane such that $\vec{a} \in [\pi]^{\perp}$ and $\sigma = \pi + (\vec{a}/2)$ then the translation $\vec{a}$ is the resultant of the reflection in π followed by the reflection in σ.

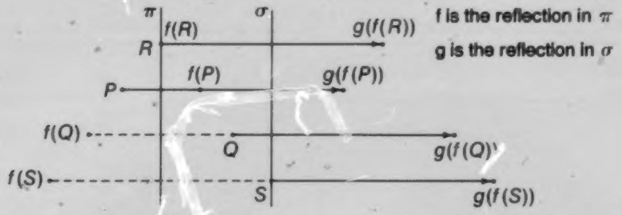


Fig. 14–13

Briefly, any translation is a resultant of reflections in parallel planes. [You can think of this as a generalization of the fact that a reflection f is its own inverse. Explain.]

The explanation called for concerning h and $\vec{a} \circ f$ is to the effect that since, by definition, $h(P) = f(P) + \vec{a}$ it follows, using ordinary function notation in place of '+', $h(P) = \vec{a}(f(P)) = [\vec{a} \circ f](P)$. Since this is the case for any point P, $h = \vec{a} \circ f$.

In the notation of Theorem 14-24, since $\pi = \sigma - (\vec{a}/2)$ it follows from the theorem that the result of the reflection in σ followed by the reflection in π is $-\vec{a}$. This shows that function composition, even when restricted to reflections in planes with a given bidirection, is not commutative. If f is the reflection in π and g the reflection in σ then the mapping $g \circ f$ is the inverse of the mapping $f \circ g$.

To say that a plane reflection f is its own image amounts to saying that $f \circ f$ is the identity mapping. Since the latter is the translation $\vec{0}$, and the plane of the reflection f is parallel to itself, it follows that the fact that any translation is the resultant of reflections in parallel planes is a generalization of the fact that any plane reflection is its own inverse.

To tidy up the theory of resultants of translations and reflections it is worth remarking here that the resultant of a translation $\vec{a} \in [\pi]^{\perp}$ and the reflection f in π is the plane reflection in $\pi - (\vec{a}/2)$. [Compare this with the lemma on page 182.] For, by the lemma, $-\vec{a} \circ f$ is the reflection in $\pi - (\vec{a}/2)$, and, since the latter is its own inverse it is $(-\vec{a} \circ f)^{-1}$. By a remark on TC 26(1), $(-\vec{a} \circ f)^{-1} = f^{-1} \circ -{-\vec{a}} = f \circ \vec{a}$. So, as claimed the resultant of $\vec{a}$ followed by f is the reflection in $\pi - (\vec{a}/2)$. [Combining this result and the lemma yields another example of the noncommutativity of function composition.]

Part H

1. Prove:

> Theorem 14–25 If $\pi \parallel \sigma$ then the resultant of the reflection in π followed by the reflection in σ is a translation.

[*Hint*: Let $P \in \pi$ and let Q be the point of intersection of $\overleftrightarrow{P[\pi]^{\perp}}$ and σ. Describe a translation $\vec{a}$ such that $\vec{a} \in [\pi]^{\perp}$ and $\sigma = \pi + (\vec{a}/2)$.]

2. Suppose that $\pi \parallel \sigma$ and let f and g be the reflections in π and in σ, respectively. From the result of Exercise 1 we know that $g \circ f$ is a translation and, also, that $f \circ g$ is a translation. Compare these translations. [Is it ever the case that $f \circ g = g \circ f$?]

Part I

1. Suppose that f, g, and h are mappings of $\mathscr{E}$ onto $\mathscr{E}$. Show that $h \circ [g \circ f] = [h \circ g] \circ f$. [*Hint*: Use the definition of function composition to compute '$[h \circ [g \circ f]](P)$' and '$[[h \circ g] \circ f](P)$'.]
2. Use a result you obtained in Part H to show that function composition is not commutative.
3. Suppose that f is any isometry of $\mathscr{E}$.
   (a) Show that f has an inverse. [*Hint*: If $P \neq Q$ can it be the case that $f(P) = f(Q)$?]
   (b) Show that f^{-1}, the inverse of f, is also an isometry.
4. Suppose that f and g are isometries of $\mathscr{E}$. Show that $g \circ f$ is also an isometry of $\mathscr{E}$. [*Hint*: Certainly, $g \circ f$ maps $\mathscr{E}$ onto itself. Does it preserve distances?]
5. Give examples illustrating Exercises 3 and 4 by citing isometries of kinds with which you are already familiar.

14.05 Some Properties of Isometries

One of our early statements of a part of what is now Postulate 4 was:

$\mathscr{T}$ is a commutative group with respect to function composition.

It results from Part I that

the set of all isometries of $\mathscr{E}$ is a group
with respect to function composition

—but [by Exercise 2, Part I], it is *not* a commutative group. By Theorem 14–18, the commutative group $\mathscr{T}$ is a *subgroup* of the group of all isometries of $\mathscr{E}$.

Answers for Part H

1. With P and Q as in the hint $\sigma = \pi + (Q - P)$ and $Q - P \in [\pi]^{\perp}$. Let $\vec{a} = (Q - P)2$ and apply Theorem 14-24.
2. $f \circ g$ is the opposite of $g \circ f$. [See the discussion preceding these answers. Alternatively note that $(g \circ f)^{-1} = f^{-1} \circ g^{-1} = f \circ g$, since each plane reflection is its own inverse.]; $f \circ g = g \circ f$ if and only if $f = g$. [For it is only in this case that $f \circ g$ turns out to be $\vec{0}$, the only translation which is its own opposite.]

Answers for Part I

1. $[h \circ [g \circ f]](P) = h([g \circ f](P)) = h(g(f(P))) = [h \circ g](f(P)) = [[h \circ g] \circ f](P)$, for any $P \in \mathcal{E}$. So $[h \circ [g \circ f]] = [[h \circ g] \circ f]$.
2. If f and g are reflections in two parallel planes then $f \circ g \neq g \circ f$.
3. (a) If $P \neq Q$ then $d(P, Q) \neq 0$. Since $d(P, Q) = d(f(P), f(Q))$ it follows that $d(f(P), f(Q)) \neq 0$ and, so, that $f(P) \neq f(Q)$. Since if $P \neq Q$ then $f(P) \neq f(Q)$ it follows that f has an inverse.
   (b) Since f is a mapping of $\mathcal{E}$ onto $\mathcal{E}$, so is f^{-1}. Moreover, $d(f^{-1}(P), f^{-1}(Q)) = d(f(f^{-1}(P)), f(f^{-1}(Q))) = d(P, Q)$. Hence, f^{-1} is an isometry of $\mathcal{E}$.
4. Since both f and g map $\mathcal{E}$ onto itself, so does $g \circ f$. Since both f and g preserve distances so does $g \circ f$. Hence, $g \circ f$ is an isometry. [In more detail, $d(g(f(P)), g(f(Q))) = d(f(P), f(Q))$ because g is an isometry and $d(f(P), f(Q)) = d(P, Q)$ because f is an isometry.]
5. A resultant of two reflections in parallel planes is a translation and, so, is an isometry; a resultant of a plane reflection and a translation normal to the plane of the reflection [in either order] is a reflection and, so, is an isometry; a resultant of two translations is a translation and, so, is an isometry.

Just as the group of isometries is related to the equivalence relation of congruence, so any group of mappings is related to an appropriate equivalence relation. This explains, in part, the importance of the notion of group.

The 'etc.' in the sentence following Theorem 14-26 includes the cases in which $\ell \parallel m$, $\ell \perp m$, $\pi \parallel \sigma$, and $\pi \perp \sigma$.

The explanation asked for amounts to noting that, since f is an isometry,

$$d(f(P), f(f^{-1}(C))) = d(P, f^{-1}(C)), \text{ and}$$
$$d(f(Q), f(f^{-1}(C))) = d(Q, f^{-1}(C))$$

and, then, substituting 'C' for '$f(f^{-1}(C))$'.

$f(P) \neq f(Q)$ because f is an isometry and $P \neq Q$. The latter is the case because $P \notin \pi$ and Q is the reflection of P in π. [See Exercise 1 of Part D on page 180.]

The fact that isometries constitute a group has important implications for the relation of congruence. In the first place, since the identity mapping is an isometry, it follows that each subset of $\mathscr{E}$ is congruent to itself—in short, congruence is a *reflexive* relation. Secondly, since the inverse of an isometry is an isometry, it follows that if a first set is congruent to a second then the second is congruent to the first—congruence is a *symmetric* relation. Finally, since the resultant of two isometries is an isometry, it follows that if a first set is congruent to a second, and the second is congruent to a third, then the first set is congruent to the third set—congruence is a *transitive* relation.

Clearly, translations have many properties which other isometries do not have. However, we do have the following basic theorem:

> **Theorem 14–26** Any isometry of $\mathscr{E}$ maps planes onto planes, maps lines onto lines, and preserves parallelism and perpendicularity.

In other words, if f is an isometry of $\mathscr{E}$ and π is any plane then the image $f(\pi)$ of π under f is also a plane; if l is a line, so is $f(l)$; if $l \parallel \pi$ then $f(l) \parallel f(\pi)$; if $l \perp \pi$ then $f(l) \perp f(\pi)$; etc. For example, consider an isometry f and a plane π. Suppose that $P \notin \pi$ and that Q is the reflection of P in π. We know, then, that π is the perpendicular bisector of $\overleftrightarrow{PQ}$ and, so, that

$$\pi = \{X : d(P, X) = d(Q, X)\}.$$

Since f has an inverse it follows that

$$\begin{aligned} C \in f(\pi) &\Longleftrightarrow f^{-1}(C) \in \pi \\ &\Longleftrightarrow d(P, f^{-1}(C)) = d(Q, f^{-1}(C)) \end{aligned}$$

Since f is an isometry and since $f(f^{-1}(C)) = C$ it follows that

$$C \in f(\pi) \Longleftrightarrow d(f(P), C) = d(f(Q), C). \qquad \text{[Explain.]}$$

So, $f(\pi)$ is the perpendicular bisector of $\overleftrightarrow{f(P)f(Q)}$. Since $f(P) \neq f(Q)$ [Why?] it follows that $f(\pi)$ is a plane.

Exercises

Part A

Suppose that f is an isometry of $\mathscr{E}$.

1. Show that if $\pi \parallel \sigma$ then $f(\pi) \parallel f(\sigma)$. [*Hint:* $\pi \parallel \sigma \Longleftrightarrow (\pi = \sigma$ or $\pi \cap \sigma = \emptyset)$ Note that if $P \in f(\pi) \cap f(\sigma)$ then $f^{-1}(P) \in \pi \cap \sigma$.]

The exercise of section 14.05 tend to be difficult for students. However, under teacher direction these difficulties can be minimized.

(i) Part A should be developed under teacher direction to insure that students understand the techniques involved. After the first few exercises, have students attempt the exercises first at their seats. You can supervise this activity to identify areas of difficulty.

(ii) Part B may be used as homework.

(iii) Part C should be carefully explained by the teacher. After this, students are usually able to proceed with Exercises 4 and 5. A tracing frequently helps to illustrate the various mappings.

(iv) Part D makes a reasonable homework assignment.

(v) Part E, and the discussion that follows should be developed in class.

(vi) Part F may be used as homework, but it may be wise to let students consult with each other.

Answers for Part A

1. Suppose that $\pi \parallel \sigma$. By the preceding discussion, $f(\pi)$ and $f(\sigma)$ are planes. In case $f(\pi) \cap f(\sigma) = \emptyset$, $f(\pi) \parallel f(\sigma)$. Suppose, then that $f(\pi) \cap f(\sigma) \neq \emptyset$ and, in particular, that $P \in f(\pi) \cap f(\sigma)$. It follows that $P \in f(\pi)$ and $P \in f(\sigma)$ and, so, that $f^{-1}(P) \in \pi \cap \sigma$. Since $\pi \parallel \sigma$ it follows that $\pi = \sigma$ and, so, that $f(\pi) = f(\sigma)$. So, whether $f(\pi) \cap f(\sigma) = \emptyset$ or $f(\pi) \cap f(\sigma) \neq \emptyset$ $f(\pi) \parallel f(\sigma)$. Hence, f being an isometry, if $\pi \parallel \sigma$ then $f(\pi) \parallel f(\sigma)$.

2. Let ℓ be a line and suppose that π and σ are planes such that $\pi \cap \sigma = \ell$. [Since $\mathcal{E}$ is three-dimensional it is easy to describe two such planes.] Now, $f(\ell) = f(\pi \cap \sigma)$ and all we need prove is that $f(\pi \cap \sigma) = f(\pi) \cap f(\sigma)$ to make sure that $f(\ell)$ is the intersection of two planes and, so, is a line. Now, for any one-to-one mapping g and any set p in the domain of g, $Q \in f(p)$ if and only if $f^{-1}(Q) \in p$. So, $Q \in f(\pi)$ if and only if $f^{-1}(Q) \in \pi$ and $Q \in f(\sigma)$ if and only if $f^{-1}(Q) \in \sigma$. Hence, $Q \in f(\pi) \cap f(\sigma)$ if and only if $f^{-1}(Q) \in \pi \cap \sigma$. And, $f^{-1}(Q) \in \pi \cap \sigma$ if and only if $Q \in f(\pi \cap \sigma)$. Since $Q \in f(\pi) \cap f(\sigma)$ if and only if $Q \in f(\pi \cap \sigma)$ it follows that $f(\pi \cap \sigma) = f(\pi) \cap f(\sigma)$.

   [Note that if g is a mapping which is not one-to-one and p and s are subsets of the domain of g then, although $f(p \cap s) \subseteq f(p) \cap f(s)$, it may happen that $g(p \cap s) \neq g(p) \cap g(s)$. For example, if $P \neq S$ and $g(S) = g(Q)$ then $\{P\}$ and $\{S\}$ are sets p and s such that $g(p \cap s) = \emptyset$ and $g(p) \cap g(s) \neq \emptyset$.]

3. Suppose that $\ell \parallel \pi$, so that $\ell \subseteq \pi$ or $\ell \cap \pi = \emptyset$. By preceding arguments, $f(\ell)$ is a line and $f(\pi)$ is a plane. If $\ell \subseteq \pi$ then, obviously, $f(\ell) \subseteq f(\pi)$. If $\ell \cap \pi = \emptyset$ then $\emptyset = f(\ell \cap \pi) = f(\ell) \cap f(\pi)$. Hence, if $\ell \parallel \pi$ then $f(\ell) \parallel f(\pi)$.

4. Suppose that $\ell \parallel m$. Let π be a plane containing ℓ and m and let σ_1 and σ_2 be parallel planes whose intersections with π are ℓ and m, respectively. The isometry f maps σ_1 and σ_2 onto parallel planes and maps ℓ and m onto the intersections of $f(\sigma_1)$ and $f(\sigma_2)$ with the plane $f(\pi)$. Since $f(\ell)$ and $f(m)$ are, then, the intersections of parallel planes with a third plane, $f(\ell)$ and $f(m)$ are parallel lines.

2. Show that, for any line l, $f(l)$ is a line. [*Hint*: Any line is the intersection of two planes, and the intersection of two intersecting planes is a line.]
3. Show that if $l \parallel \pi$ then $f(l) \parallel f(\pi)$. [*Hint*: Modify the hint for Exercise 1.]
4. Show that if $l \parallel m$ then $f(l) \parallel f(m)$.
5. Show that if $l \perp \pi$ then $f(l) \perp f(\pi)$. [*Hint*: In the proof that $f(\pi)$ is a plane, choose $P \in l$.]
6. Show that if $\pi \perp \sigma$ then $f(\pi) \perp f(\sigma)$.
7. Show that if $l \perp m$ then $f(l) \perp f(m)$.
8. Show that, for any points P and Q, $f(\overline{PQ}) = \overline{f(P)f(Q)}$. [*Hint*: $C \in f(\overline{PQ}) \longleftrightarrow f^{-1}(C) \in \overline{PQ}$. Now, use Theorem 14–3.]
9. Show that $f(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)}$. [*Hint*: Proceed as in Exercise 8, recalling that $R \in \overrightarrow{AB} \longleftrightarrow (R \in \overline{AB}$ or $B \in \overline{AR})$.]
10. Show that $f(\overline{PQ}) = \overline{f(P)f(Q)}$ and $f(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)}$.

*

In Exercises 8–10 you have proved:

> **Theorem 14–27** Any isometry of $\mathscr{E}$ maps segments onto segments and intervals onto intervals, mapping endpoints on endpoints; it maps rays onto rays and half-lines onto half-lines, mapping vertices on vertices.

Since an isometry maps the endpoints of a segment on the endpoints of its image it is clear that congruent segments must have the same measure. [Explain.] The converse is an equally important theorem. We state the two in:

> **Theorem 14–28** Segments [or: intervals] are congruent if and only if they have the same measure.

To complete the proof of this theorem all we need do is establish a lemma:

> **Lemma** If $AB = PQ$ then there is an isometry f such that $f(P) = A$ and $f(Q) = B$.

[Explain.]

The proof of the lemma is easy.

Answers for Part A [cont.]

5. Suppose that $\ell \perp \pi$ and that $P \in \ell$ and $P \notin \pi$. Let Q be the reflection of P in π. It follows that $\pi = \{X\colon d(P, X) = d(Q, X)\}$ and, so, that $f(\pi) = \{X\colon d(f(P), X) = d(f(Q), X)\}$. It follows that $f(\pi)$ is the perpendicular bisector of $\overline{f(P)f(Q)}$. Since $\ell = \overleftrightarrow{PQ}$ and $f(\ell)$ is a line it follows that $f(\ell) = \overleftrightarrow{f(P)f(Q)}$. Since, as has been shown, $f(\pi) \perp \overline{f(P)f(Q)}$, it follows that $f(\ell) \perp f(\pi)$.
6. Suppose that $\pi \perp \sigma$ and let ℓ be a line contained in σ which is perpendicular to π. By Exercise 5, $f(\ell) \perp f(\pi)$ and, since $f(\ell)$ is a line and is contained in the plane $f(\sigma)$, $f(\sigma) \perp f(\pi)$.
7. Suppose that $\ell \perp m$ and let π be the plane containing m which is perpendicular to ℓ. Since $\ell \perp \pi$, $f(\ell) \perp f(\pi)$. So, since $f(m) \subseteq f(\pi)$, $f(\ell) \perp f(m)$.
8. $C \in f(\overline{PQ})$ if and only if $f^{-1}(C) \in \overline{PQ}$. By Theorem 14-3, the latter is the case if and only if $d(P, f^{-1}(C)) + d(f^{-1}(C), Q) = d(P, Q)$. Since f is an isometry, the latter is the case if and only if $d(f(P), C) + d(C, f(Q)) = d(f(P), f(Q))$ — that is, if and only if $C \in \overline{f(P)f(Q)}$. Since $C \in f(\overline{PQ})$ if and only if $C \in \overline{f(P)f(Q)}$ it follows that $f(\overline{PQ}) = \overline{f(P)f(Q)}$. [It is only this result which is needed to change the "reasons for believing" in the answer for Part F on page 181 into proofs.]
9. $C \in f(\overrightarrow{PQ})$ if and only if $f^{-1}(C) \in \overrightarrow{PQ}$ — that is, if and only if $(f^{-1}(C) \in \overline{PQ}$ or $Q \in \overline{P f^{-1}(C)})$. By Exercise 8 the latter is the case if and only if $C \in \overline{f(P)f(Q)}$ or $f(Q) \in \overline{f(P)C}$ — that is, if and only if $C \in \overrightarrow{f(P)f(Q)}$. Hence [as in Exercise 8], $f(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)}$.
10. By Exercise 8, $f(\overline{PQ}) = \overline{f(P)f(Q)}$. Since $f(C) = f(P)$ if and only if $C = P$ and $f(C) = f(Q)$ if and only if $C = Q$ it follows that $f(\overline{PQ}) = \overline{f(P)f(Q)}$. That $f(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)}$ is proved similarly, using Exercise 9.

Explanation to follow Theorem 14-27: Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are congruent, in particular, suppose that f is an isometry such that $f(\overleftrightarrow{AB}) = \overleftrightarrow{CD}$. By Exercise 8 of Part A it follows that $\overleftrightarrow{f(A)f(B)} = \overleftrightarrow{CD}$ and, so [by Theorem 7-22(b)], $\{f(A), f(B)\} = \{C, D\}$. [This argument — and another using Theorem 7-22(a) — justifies the "mapping end points on end points" in Theorem 14-27.] It follows that either $(f(A) = C$ and $f(B) = D)$ or $(f(A) = D$ and $f(B) = C)$. In the first case $CD = d(D, C) = d(f(B), f(A)) = d(B, A) = AB$ and, so, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have the same measure. The second case can be dealt with in a similar manner.

Explanation to follow the lemma: We have just seen that segments are congruent only if they have the same measure. To justify replacing 'only if' by 'if' and, so, obtaining Theorem 14-28, we need to show that segments are congruent if they have the same measure. So, we must show that if segments have the same measure then there is an isometry mapping one onto the other. We shall know this if we can find an isometry which maps the end points of one segment on those of the other.

Note that having the same measure is often taken as the defining relation for congruence of intervals or of angles. It is a superiority of the treatment of congruence in this course that there is one definition of congruence for intervals, angles, triangles, or what have you, and that measure criteria for congruence result from theorems which are not just definitions.

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Answer for 'Why?': BA = QP by assumption; QP = Q'A because $\vec{a}$ is an isometry, $\vec{a}(Q) = Q'$, and $\vec{a}(P) = A$.

Another way of obtaining an isometry which maps P on A and Q on B is to take the resultant of the reflection g_1 in the perpendicular bisector of $\overleftrightarrow{AP}$ followed by the reflection g_2 in the perpendicular bisector of $\overleftrightarrow{Bg_2(Q)}$. That this isometry is not the same as the isometry $g \circ \vec{a}$ of the proof can be seen either by the images under each of $g \circ \vec{a}$ and $g_2 \circ g_1$ of a point not on $\overleftrightarrow{PQ}$ or by noting that $g \circ \vec{a}$ "turns the plane over" while $g_2 \circ g_1$ does not. In general, a resultant which involves an odd number of plane reflection "factors" will not give the same result as a resultant which involves an even number of such factors.

Suppose that $AB = PQ$ and let $\vec{a} = A - P$ and $Q' = Q + \vec{a}$. The mapping $\vec{a}$ is an isometry which maps P on A and Q on Q' for

$$\vec{a}(P) = P + \vec{a} = P + (A - P) = A,$$

and

$$\vec{a}(Q) = Q + \vec{a} = Q'.$$

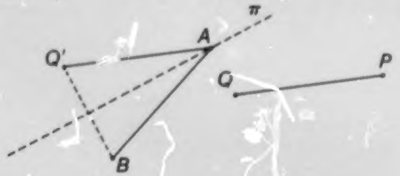


Fig. 14–14

In case $Q' = B$, $\vec{a}$ is the isometry we are after. In case $Q' \neq B$, let π be the perpendicular bisector of $\overleftrightarrow{BQ'}$ and let g be the reflection in π. Since $BA = QP = Q'A$ [Why?], it follows that $A \in \pi$. So, $g(A) = A$ and, of course, $g(Q') = B$. It follows that

$$g(\vec{a}(P)) = g(A) = A \text{ and } g(\vec{a}(Q)) = g(Q') = B.$$

So, the mapping $g \circ \vec{a}$ maps P on A and Q on B and, since both $\vec{a}$ and g are isometries, is an isometry. [Now, describe such an isometry in another way by composing reflections in two planes. Do you think this second isometry is different from the one described in the text?]

It is worth remarking that—supposing that $AB = PQ$—that there are many isometries which map P on A and Q on B. For, if f is an isometry which does this, and h is an isometry which leaves A and B fixed, then $h \circ f$ is an isometry which maps P on A and Q on B:

$$[h \circ f](P) = h(f(P)) = h(A) = A \text{ and } [h \circ f](Q) = h(f(Q)) = h(B) = B$$

Now we have already found one isometry such as f, and the reflection in any plane which contains A and B will do for h. In particular, in the case in which $Q' \neq B$ we may take for h the reflection in the plane σ which contains A, B, and Q'. If we do then

$$h \circ f = h \circ [g \circ \vec{a}] = [h \circ g] \circ \vec{a}.$$

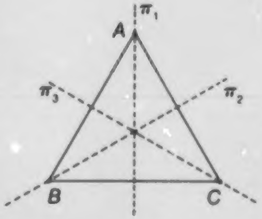
The isometries g and h are reflections in perpendicular planes π and σ whose intersection is the line containing the angle bisector of $\angle Q'AB$. As you will see in Part C of the exercises, it follows that $h \circ g$ is an isometry of a rather simple and interesting kind.

Part B

1. Show that any two rays are congruent. [*Hint*: Let A and P be the vertices of the given rays and let $\overrightarrow{u}$ and [illegible] be unit vectors in the senses of the rays. Let $B = A + \overrightarrow{u}$ and $Q = P + \overrightarrow{v}$. Now, apply the lemma and Exercise 9 of Part A.]
2. Show that any two lines are congruent. [*Hint*: You should be able to use the result of Exercise 1 to advantage.]
3. Show that any two planes are congruent. [*Hint*: Given planes π and σ, show that there are points A, B, P, and Q such that π and σ are the perpendicular bisectors of $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ and such that $AB = PQ$.]

*Part C

Consider the equilateral triangle, $\triangle ABC$, pictured at the right. Let π_1, π_2, and π_3 be the perpendicular bisectors of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Also, let f_1, f_2, and f_3 be the reflections in π_1, π_2, and π_3, respectively.



1. What are the images of A, B, and C under f_1? Under f_2? Under f_3?
2. Show that $f_1(\triangle ABC) = \triangle ACB$. What is $f_2(\triangle ABC)$? What is $f_3(\triangle ABC)$?
3. Note that f_1 maps A on A, B on C, and C on B. We abbreviate this as follows:

$$f_1\colon ABC \to ACB$$

   (a) Give similar abbreviations for what f_2 and f_3 do.
   (b) Verify that $[f_1 \circ f_3](A) = C$. What are $[f_1 \circ f_3](B)$ and $[f_1 \circ f_3](C)$? Give an abbreviation for what $f_1 \circ f_3$ does.
   (c) Give similar abbreviations for $f_3 \circ f_2$, $f_2 \circ f_3$, and for $\overrightarrow{0}$.
4. Let $f_4 = f_1 \circ f_3$ and $f_5 = f_2 \circ f_3$. Make use of your results in Exercise 3 and your knowledge of function composition to complete the following table:

$\circ$	$\overrightarrow{0}$	f_1	f_2	f_3	f_4	f_5
$\overrightarrow{0}$	$\overrightarrow{0}$	f_1	f_2	f_3	f_4	f_5
f_1	f_1	$\overrightarrow{0}$		f_4		
f_2	f_2		$\overrightarrow{0}$	f_5		
f_3	f_3		f_4	$\overrightarrow{0}$		
f_4	f_4					
f_5	f_5					

Answers for Part B

1. Let r and s be any two rays, let A and P be their vertices, and let $\vec{u}$ and $\vec{v}$ be the unit vectors in the senses of r and s, respectively. Let $B = A + \vec{u}$ and $Q = P + \vec{v}$. It follows that $r = \overrightarrow{AB}$ and $s = \overrightarrow{PQ}$. Since $\|B - A\| = 1 = \|Q - P\|$ it follows by the lemma that there is an isometry — say, f — such that $f(P) = A$ and $f(Q) = B$. By Exercise 9 of Part A, $f(\overrightarrow{PQ}) = \overrightarrow{f(P)f(Q)} = \overrightarrow{AB}$. Since there is an isometry which maps $\overrightarrow{PQ}$ onto $\overrightarrow{AB}$ it follows that $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are congruent — that is, any rays s and r are congruent.
2. Let ℓ and m be any two lines, let A and P be points of ℓ and m, and let $\vec{u}$ and $\vec{v}$ be unit vectors in the directions of ℓ and m, respectively. Let $B = A + \vec{u}$ and $Q = P + \vec{v}$. It follows that $\ell = \overleftrightarrow{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$ and $m = \overleftrightarrow{PQ} = \overrightarrow{PQ} \cup \overrightarrow{QP}$. As in Exercise 1 there is an isometry f such that $f(\overrightarrow{PQ}) = \overrightarrow{AB}$ and $f(\overrightarrow{QP}) = \overrightarrow{BA}$ and, so, $f(m) = \ell$. Hence, ℓ and m are congruent.
3. Given planes π and σ, let $\vec{m}$ and $\vec{n}$ be unit vectors in $[\pi]^{\perp}$ and $[\sigma]^{\perp}$, respectively. Let $C \in \pi$ and $R \in \sigma$ and let $A = C + \vec{m}$, $B = C - \vec{m}$, $P = R + \vec{n}$, and $Q = R - \vec{n}$. It follows that π is the perpendicular bisector of $\overleftrightarrow{AB}$ and σ is the perpendicular bisector of $\overleftrightarrow{PQ}$. Since $\|B - A\| = 2 = \|Q - P\|$ there is an isometry which maps P on A and Q on B. This isometry maps the perpendicular bisector σ of $\overleftrightarrow{PQ}$ on the perpendicular bisector π of $\overleftrightarrow{AB}$. [This, by Theorem 14-5.] Hence, π and σ are congruent.

Answers for Part C

[Before assigning these exercises be sure that students know how to interpret a "multiplication table" like that in Exercise 4.]

1. $f_1(A) = A$, $f_1(B) = C$, and $f_1(C) = B$; $f_2(A) = C$, $f_2(B) = B$, and $f_2(C) = A$; $f_3(A) = B$, $f_3(B) = A$, and $f_3(C) = C$.
2. Since $f_1(A) = A$ and $f_1(B) = C$, $f_1(\overleftrightarrow{AB}) = \overleftrightarrow{f_1(A)f_1(B)} = \overleftrightarrow{AC}$. By similar reasoning, $f_1(\overleftrightarrow{AC}) = \overleftrightarrow{AB}$ and $f_1(\overleftrightarrow{BC}) = \overleftrightarrow{CB}$. Therefore, $f_1(\triangle ABC) = \triangle ACB$. Similarly, $f_2(\triangle ABC) = \triangle CBA$, and $f_3(\triangle ABC) = \triangle BAC$. [Of course, $\triangle ABC = \triangle ACB = \triangle CBA = \triangle BAC$.]
3. (a) $f_2\colon ABC \to CBA$; $f_3\colon ABC \to BAC$.
   (b) $[f_1 \circ f_3](A) = f_1(f_3(A)) = f_1(B) = C$; $[f_1 \circ f_3](B) = A$ and $[f_1 \circ f_3](C) = B$; $f_1 \circ f_3\colon ABC \to CAB$.
   (c) $f_3 \circ f_2\colon ABC \to CAB$; $f_2 \circ f_3\colon ABC \to BCA$; $\vec{0}\colon ABC \to ABC$

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Answers for Part C [cont.]

4.

$\circ$	$\vec{0}$	f_1	f_2	f_3	f_4	f_5
$\vec{0}$	$\vec{0}$	f_1	f_2	f_3	f_4	f_5
f_1	f_1	$\vec{0}$	f_5	f_4	f_3	f_2
f_2	f_2	f_4	$\vec{0}$	f_5	f_1	f_3
f_3	f_3	f_5	f_4	$\vec{0}$	f_2	f_1
f_4	f_4	f_2	f_3	f_1	f_5	$\vec{0}$
f_5	f_5	f_3	f_1	f_2	$\vec{0}$	f_4

[Note that, as given in the text, $f_3 \circ f_2 = f_4$. For $f_3(f_2(A)) = f_3(C) = C$, $f_3(f_2(B)) = f_3(B) = A$, $f_3(f_2(C)) = f_3(A) = B$, and $f_4 = f_1 \circ f_3$ where, by Exercise 3(c), $f_1 \circ f_3$: ABC → CAB.

TC 189 (1)

5. (a) In the first place, as the table shows, the set of six mappings is closed under function composition. And, of course, $\vec{0}$ is [as the table shows] the identity element. Since '$\vec{0}$' occurs in each row and column, each of the six mappings has one of the six — perhaps itself — as its inverse. Since function composition is always associative it is bound to be associative in this situation. [However, it is worth checking a few instances. For example, $f_4 \circ (f_3 \circ f_2) = f_4 \circ f_4 = f_5$ and $(f_4 \circ f_3) \circ f_2 = f_1 \circ f_2 = f_5$.

(b) No. [For example, $f_1 \circ f_2 = f_5 \neq f_4 = f_2 \circ f_1$.]

(c) [The 2-element subgroups are those containing $\vec{0}$ and either f_1, f_2, or f_3; the only subgroup with three elements is that which contains $\vec{0}$, f_4, and f_5.]

(d) Yes.

Answers for Part D

1. Since g is the reflection in π, $g(P) - P \in [\pi]^{\perp}$ and since $\ell \subseteq \pi$, $[\pi]^{\perp} \subseteq [\ell]^{\perp}$. So, $g(P) - P \in [\ell]^{\perp}$. Since h is the reflection in σ and since $\ell \subseteq \sigma$ it follows, as above. that $h(g(P)) - g(P) \in [\ell]^{\perp}$. Since the sum of two members of the bidirection $[\ell]^{\perp}$ belongs to $[\ell]^{\perp}$ it follows that $h(g(P)) - P \in [\ell]^{\perp}$. So, since $h(g(P)) = f(P)$, $f(P) - P \in [\ell]^{\perp}$.

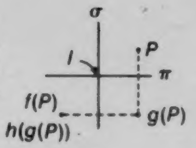
2. Following the hint we see that, since g is the reflection in π, $P + [g(P) - P]/2 \in \pi$, and that as in Exercise 1, $[f(P) - g(P)]/2 \in [\sigma]^{\perp} \subseteq [\pi]$, since $\pi \perp \sigma$. So, $M \in \pi$. Similarly, the second line of the hint shows that the midpoint M of $\overleftrightarrow{Pf(P)}$ is the image of a point of σ under a translation which belongs to $[\sigma]$ and, so, itself belongs to σ. Hence, $M \in \pi \cap \sigma = \ell$.

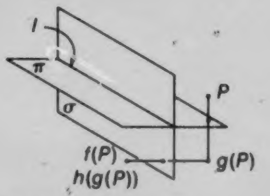
3. Isometries which, like f, are the resultants of reflections in two perpendicular planes are called line reflections; in particular, f is the reflection in the line ℓ.

5. (a) Verify from the table in Exercise 4 that $(\{\vec{0}, f_1, f_2, f_3, f_4, f_5\}, \circ)$ is a group. [Why do you not need to check for associativity?]
   (b) Is the group a commutative group? Explain.
   (c) Find at least one subgroup with two elements and one subgroup with three elements in the given group.
   (d) Are the subgroups you found in part (c) commutative groups?

Part D

Suppose that $\pi \perp \sigma$ and that g and h are the reflections in π and in σ, respectively. Let $l = \pi \cap \sigma$ and let $f = h \circ g$.





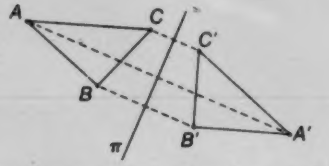
1. Show that $f(P) - P \in [l]^{\perp}$. [*Hint*: $g(P) - P \in [\pi]^{\perp}$ and $[\pi]^{\perp} \subseteq [l]^{\perp}$. Explain the preceding remarks and make a similar statement concerning $f(P) - g(P)$.]
2. Show that the midpoint of $\overleftrightarrow{Pf(P)}$ belongs to l. [*Hint*: There are many ways of doing this. One of the most straight-forward is to note that, if M is the midpoint of $\overleftrightarrow{Pf(P)}$ then

$$\begin{aligned} M &= P + [f(P) - P]/2 \\ &= (P + [g(P) - P]/2) + [f(P) - g(P)]/2, \end{aligned}$$

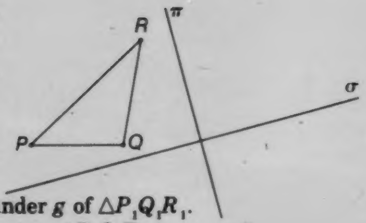
and

$$\begin{aligned} M &= f(P) + [P - f(P)]/2 \\ &= (f(P) + [g(P) - f(P)]/2) + [P - g(P)]/2. \end{aligned}$$

Use the first to show that $M \in \pi$ and the second to show that $M \in \sigma$.]
3. Compare the results in Exercises 1 and 2 with Theorem 14–19, and suggest a name for isometries like f.
4. Suppose that π_1 and σ_1 are two other perpendicular planes whose intersection is the same line l, and that g_1 and h_1 are the reflections in π_1 and σ_1. Do you think that $h_1 \circ g_1 = h \circ g$? Explain.
5. Here is a picture of $\triangle ABC$ and of its image, $\triangle A'B'C'$, under the reflection g in the plane π [viewed "end on".]

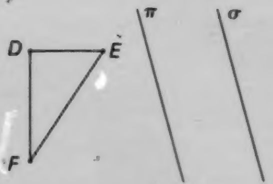


   (a) Is $\triangle ABC$ congruent to $\triangle A'B'C'$? Is $\triangle A'B'C'$ congruent to $\triangle ABC$? Explain.
   (b) Let M, N, and P be the feet of the perpendiculars from A, B, and C, respectively, to π. What can you say about $\overleftrightarrow{ABC}$ and π given that M, N, and P are collinear? Given that $\triangle MNP$ is congruent to $\triangle ABC$?
   (c) Assume that $\triangle ABC$ is congruent to $\triangle MNP$. Describe a translation which maps $\triangle ABC$ onto $\triangle MNP$. What is the image of $\triangle MNP$ under this translation?



6. Copy this picture of $\triangle PQR$ and intersecting planes π and σ [viewed "end on"] on your paper. Let f and g be the reflections in π and σ, respectively.
  - **(a)** Draw $\triangle P_1Q_1R_1$, the image, under f, of $\triangle PQR$.
  - **(b)** Draw $\triangle P_2Q_2R_2$, the image under g of $\triangle P_1Q_1R_1$.
  - **(c)** Show that $\triangle PQR$ is congruent to $\triangle P_2Q_2R_2$.
  - **(d)** Give an isometry which maps $\triangle P_2Q_2R_2$ onto $\triangle PQR$.

7. Your figure for Exercise 6 may suggest a description for $f \circ g$ other than 'the reflection in $\pi \cap \sigma$'. Try to think of one.



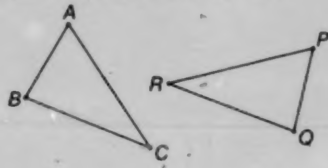
8. Copy this picture of $\triangle DEF$ and parallel planes π and σ on your paper. Let f and g be the reflections in π and σ, respectively.
  - **(a)** Draw $\triangle D_1E_1F_1$, the image under $g \circ f$ of $\triangle DEF$.
  - **(b)** Draw $\triangle D_2E_2F_2$, the image under $f \circ g$ of $\triangle DEF$.
  - **(c)** Is $g \circ f = f \circ g$? Explain your answer. What is a relation between $g \circ f$ and $f \circ g$?

Part E

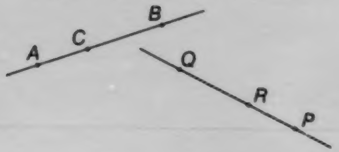
The lemma on page 186 is a first step toward proving a still more useful lemma:

> **Lemma** If $AB = PQ$, $BC = QR$, and $CA = RP$ then there is an isometry f such that $f(P) = A$, $f(Q) = B$, and $f(R) = C$.

In case $\{A, B, C\}$ is noncollinear



In case $\{A, B, C\}$ is collinear



1. Prove this lemma. [*Hint*: You can make a good start by using the lemma on page 186. According to this, since $AB = PQ$, there is an isometry—say, g—such that $g(P) = A$ and $g(Q) = B$. Suppose that $g(R) = R'$. If $R' = C$, you are done. If not, consider the reflection h in the perpendicular bisector π of $\overleftrightarrow{CR'}$. Show that $h(A) = A$ and $h(B) = B$. Conclude that $h \circ g$ is an isometry which maps P on A, Q on B, and R on C.]

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Answers for Part D [cont.]

4. Yes. As in Exercises 1 and 2, and in analogy with Theorem 14-19, $[h_1 \circ g_1](P)$ as well as $[h \circ g](P)$ will be the point Y such that $Y - P \in [\ell]^{\perp}$ and the midpoint of $\overleftrightarrow{PY}$ belongs to ℓ.

5. (a) Yes, because g is an isometry which, by Theorem 14-27, maps $\triangle ABC$ onto $\triangle A'B'C'$.

   Yes, because g is an isometry which, by Theorem 14-27, maps $\triangle A'B'C'$ onto $\triangle ABC$.

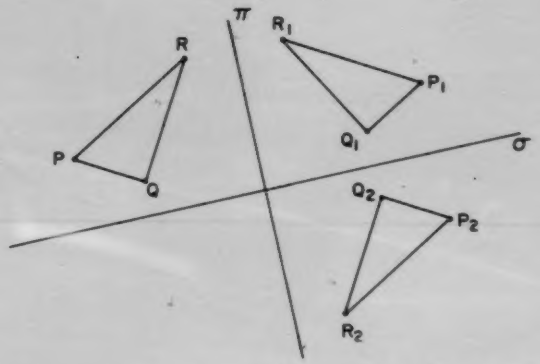
   (b) [Students need not be required to give formal justifications of their answers to these two questions.]

   If {M, N, P} is collinear then $\overleftrightarrow{ABC} \perp \pi$.; If $\triangle MNP$ is congruent to $\triangle ABC$ then $\overleftrightarrow{ABC} \parallel \pi$. [The first answer can be justified by noting that parallel lines — the lines $\overleftrightarrow{AM}$, $\overleftrightarrow{BN}$, and $\overleftrightarrow{CP}$ — through collinear points are coplanar. The plane in question contains the noncollinear points A, B, and C and, so, is $\overleftrightarrow{ABC}$. Since it contains a line perpendicular to π it is, itself, perpendicular to π. The second answer may be justified to some extent, by noting (what has not yet been proved) that an isometry which maps $\triangle ABC$ onto $\triangle MNP$ must map vertices on vertices. It seems reasonable to assume, in this case, that such an isometry maps A on M, B on N, and C on P. So, it follows that $\|B - A\| = \|N - M\|$. Then use the reasonable appearing result that if $\|\text{proj}_{[\pi]}(\vec{b})\| = \|\vec{b}\|$ then $\vec{b} \in [\pi]$ to show that $\overleftrightarrow{AB} \parallel \pi$. Similarly, $\overleftrightarrow{BC} \parallel \pi$ and, so, $\overline{ABC} \parallel \pi$.]

   (c) M - A [or: N - B, or: P - C]; $\triangle A'B'C'$

TC 190 (1)

6. (a), (b)



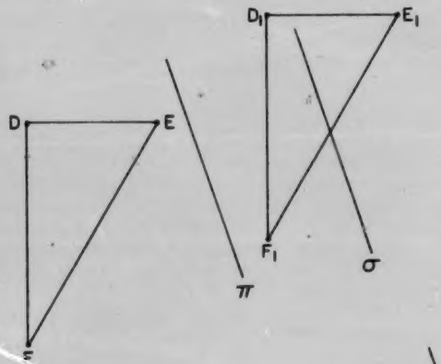
   (c) $g \circ f$ is an isometry which maps $\triangle PQR$ onto $\triangle P_2Q_2R_2$. [See Theorem 14-27.]

   (d) $f \circ g$ $[(g \circ f)^{-1} = f^{-1} \circ g^{-1} = f \circ g]$

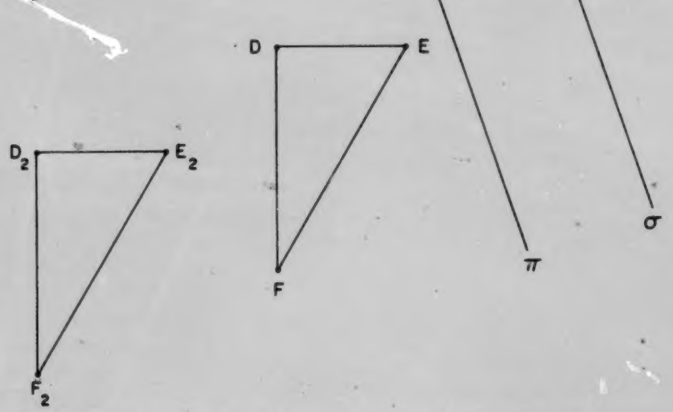
Answers for Part D [cont.]

7. The line reflection in ℓ is also called 'the half-turn about the axis ℓ'. [This is an example of the fact that a resultant of reflections in two intersecting planes is a rotation about their line of intersection.]

8. (a)



(b)



(c) No, $g \circ f \neq f \circ g$. They are inverses [or, since they are translations, opposites].

Answers for Part E

1. Proceeding as in the hint we see that $R'A = g(R)g(P) = RP = CA$ and, so, that A belongs to the perpendicular bisector π of $\overline{CR'}$. Similarly, $B \in \pi$. So, the reflection h in π maps A on A and B on B and, of course, R′ on C. Hence $h \circ g$ maps P on A, Q on B, and R on C.

2. Suppose that, in addition to points A, B, C, P, Q, and R related as in the lemma of Exercise 1, we are given points D and S such that $AD = PS, BD = QS$, and $CD = RS$. Show that there is an isometry f which maps P on A, Q on B, R on C and S on D. [*Hint*: Build on the lemma of Exercise 1 just as, when proving that lemma, you built on the lemma on page 186.]
3. Given that A, B, C, P, Q, and R are related as in the lemma of Exercise 1 and that D and S are points such that $AD = PS$ and $BD = QS$, draw a picture in which $CD \neq RS$.

*

In Exercise 2 you have proved an important existence theorem for isometries:

> **Theorem 14–29** Given points A, B, C, and D and points P, Q, R, and S, if $AB = PQ, AC = PR, AD = PS, BC = QR, BD = QS$, and $CD = RS$, then there exists an isometry which maps P on A, Q on B, R on C, and S on D.

Notice that the case of the theorem for which $D = C$ and $S = R$ gives us the lemma proved in Exercise 1. [Explain.] Does the theorem also include the lemma on page 186?

In discussing the lemma on page 186 we noticed that, assuming that $AB = PQ$, there are many isometries which map P on A and Q on B. Once we have mapped P on A and Q on B, by means of some isometry, we may go on to perform any number of reflections in planes containing $\overleftrightarrow{AB}$ and the resultant will be an isometry which still maps P on A and Q on B.

In the situation dealt with in the lemma of Exercise 1, Part E, there are, again, many isometries which map P on A, Q on B, and R on C *in case* $\{A, B, C\}$ *is collinear*. But, the case in which $\{A, B, C\}$ is noncollinear is very different. To see how different, suppose that f_1 and f_2 are isometries each of which maps P on A, Q on B, and R on C. It follows that their inverses, f_1^{-1} and f_2^{-1} are isometries and that each of these maps A on P, B on Q, and C on R. In particular,

$$f_1^{-1}(A) = f_2^{-1}(A),\ f_1^{-1}(B) = f_2^{-1}(B), \text{ and } f_1^{-1}(C) = f_2^{-1}(C).$$

From the first of these it follows that $f_2(f_1^{-1}(A)) = f_2(f_2^{-1}(A)) = A$. So, we see that

$$[f_2 \circ f_1^{-1}](A) = A,\ [f_2 \circ f_1^{-1}](B) = B, \text{ and } [f_2 \circ f_1^{-1}](C) = C.$$

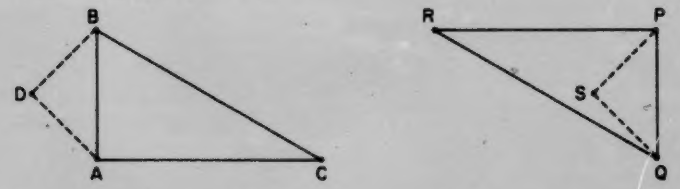
Since f_2 and f_1^{-1} are isometries, so is $f_2 \circ f_1^{-1}$. So, recalling Theorem 14–22, it follows [in case $\{A, B, C\}$ is noncollinear] that $f_2 \circ f_1^{-1}$ either is the identity mapping $\overrightarrow{0}$ of $\mathscr{E}$ into itself or is the reflection—say, g—in the plane $\overleftrightarrow{ABC}$.

Answers for Part E [cont.]

2. By the lemma of Exercise 1 there is an isometry — say, f — such that f(P) = A, f(Q) = B, and f(R) = C. Suppose that f(S) = S′. If S′ = D then f is an isometry of the kind sought for. If not, let h be the reflection in the plane π which is the perpendicular bisector of $\overleftrightarrow{DS'}$. As in Exercise 1, A, B, and C belong to π. [For example, S′A = f(S)f(P) = SP = DA, so $A \in \pi$.] It follows that h leaves each of A, B, and C fixed. Since h(S′) = D it follows that $h \circ f$ is an isometry which maps P on A, Q on B, R on C, and S on D.

[Obviously we could continue on the path beginning with the lemma on page 186 and continuing through the lemma of Exercise 1 and the result (Theorem 14-29) just established in Exercise 2. The "catch" is the corollary to Theorem 14-30 — knowing the images of four noncoplanar points under an isometry determines the isometry. If, in extension of Exercise 2, we consider points E and T such that the distances from E to A, B, C, and D are the same, respectively, as those from T to P, Q, R, and S then, if A, B, C, and D are noncoplanar, the isometry of Exercise 2 will necessarily map T on E.]

3.



AD=PS, BD=QS, but CD≠RS

*

Explanation asked for after Theorem 14-29: If, in Theorem 14-29 we replace 'D' by 'C' and 'S' by 'R', the result is merely a redundant form of the lemma of Exercise 1. If we replace both 'C' and 'D' by 'B' and both 'R' and 'S' by 'Q', we have a redundant form of the lemma on page 186.

That is,

$$f_2 \circ f_1^{-1} = \vec{0} \text{ or } f_2 \circ f_1^{-1} = g.$$

From this it follows that

$$[f_2 \circ f_1^{-1}] \circ f_1 = \vec{0} \circ f_1 = f_1 \text{ or } [f_2 \circ f_1^{-1}] \circ f_1 = g \circ f_1.$$

But, since function composition is associative,

$$[f_2 \circ f_1^{-1}] \circ f_1 = f_2 \circ [f_1^{-1} \circ f_1] = f_2 \circ \vec{0} = f_2.$$

So, either $f_2 = f_1$ or f_2 is the resultant of f_1 followed by the reflection in $\overleftrightarrow{ABC}$.

In short, in case $\{A, B, C\}$ is noncollinear, there are only two choices for the mapping f referred to in the lemma.

The preceding result leads to a useful theorem. To state this theorem in a convenient way, let us say that a mapping f and a mapping g *agree at P* if and only if $f(P) = g(P)$.

Theorem 14–30 If two isometries agree at each of three noncollinear points then they agree at each point of the plane which contains these three points; moreover, each of the given isometries is the resultant of the other followed by the reflection in the image of this plane.

Corollary No two isometries agree at each of four noncoplanar points.

Part F

1. Prove Theorem 14–30. [*Hint*: Suppose that f_1 and f_2 are *two* isometries which agree at each of the noncollinear points P, Q, and R. Show that $f_1(P)$, $f_1(Q)$, and $f_1(R)$ are noncollinear.]
2. Prove the corollary to Theorem 14–30.

*

The discussion of isometries has probably convinced you of the basic importance of reflections in planes. All the isometries we have made use of have been compounded out of isometries each of which was either a translation or a reflection. And, as we have seen, each translation is the resultant of two reflections. This might suggest that

> each isometry of $\mathscr{E}$ is either the identity mapping, or the reflection in some plane, or the resultant of two or more such reflections.

Answers for Part F

1. The only difference between Theorem 14-30 and the result proved in the argument which precedes it is, in the notation of the argument, that Theorem 14-30 assumes that $\{P, Q, R\}$ is noncollinear while the argument assumes that $\{A, B, C\}$ is noncollinear. So, to make use of the argument to prove Theorem 14-30 we need to show that if $\{P, Q, R\}$ is noncollinear then so is $\{f_1(P), f_1(Q), f_1(R)\}$. That is, we need to show that the isometry f_1 maps noncollinear points on noncollinear points. This should be obvious since the isometry f_1^{-1} maps lines onto lines [Theorem 14-26].
2. Suppose that $\{P, Q, R, S\}$ is noncoplanar and that f is an isometry. By Theorem 14-30, if g is any isometry which agrees with f on P, Q, and R then either $g = f$ or $g = h \circ f$ where h is the reflection in the image, $f(\overleftrightarrow{PQR})$, of $\overleftrightarrow{PQR}$ under f. Since $S \notin \overleftrightarrow{PQR}$ and f is one-to-one, $f(S) \notin f(\overleftrightarrow{PQR})$ and, so, $[h \circ f](S) \neq f(S)$. Hence, if g agrees with f on S — as well as on P, Q, and R — then $g \neq h \circ f$. Consequently, if g agrees with f in P, Q, R, and S then $g = f$.

As a matter of fact, we can quite easily prove a somewhat stronger theorem.

To see what this theorem is, suppose that g is any isometry of $\mathscr{E}$ and that $\{P, Q, R, S\}$ is noncoplanar. By the corollary to Theorem 14–30 we know that any isometry which agrees with g at each of the points P, Q, R, and S must be g. Now, in Exercise 2 of Part E we have seen how to find an isometry f such that $f(P) = g(P)$, $f(Q) = g(Q)$, $f(R) = g(R)$, and $f(S) = g(S)$. [Explain. What should be taken for the points A, B, C, and D of Exercise 2?] It follows, then, that g must be the mapping f constructed in Exercise 2. Let's recall how f was obtained. Taking $A = g(P)$, $B = g(Q)$, $C = g(R)$, and $D = g(S)$, the first step was to choose a translation f_1 such that $f_1(P) = A$. [In case $P = A$, $f_1 = \vec{0}$.] We then found an isometry f_2 such that $f_2(A) = A$ and $f_2(f_1(Q)) = B$. [In case $f_1(Q) = B$ we could take $f_2 = \vec{0}$. In case $f_1(Q) \neq B$ we could take f_2 to be the reflection in the perpendicular bisector of $\overline{Bf_1(Q)}$.] It then followed that $f_2 \circ f_1$ is an isometry which maps P on A and maps Q on B. The third step was to find an isometry f_3 such that $f_3(A) = A$, $f_3(B) = B$, and $f_3([f_2 \circ f_1](R)) = C$. [Again, f_3 could be taken either as $\vec{0}$ or as the reflection in a plane.] It followed that $f_3 \circ [f_2 \circ f_1]$ is an isometry which maps P on A, Q on B, and R on C. In a fourth step—just like the second and third—we found an isometry f_4—either $\vec{0}$ or the reflection in a plane—such that $f_4 \circ [f_3 \circ [f_2 \circ f_1]]$ maps P on A, Q on B, R on C, and S on D. This resultant is the mapping f which we know must be the given mapping g. Since the translation f_1 is either $\vec{0}$ or the resultant of two reflections, it is clear that g is the resultant of at most five reflections [if it is not merely $\vec{0}$ or a plane reflection]. We can obtain a slight improvement by noticing that the only purpose of the translation f_1 was to map P on A. In case $P \neq A$ we can do this by the reflection in the perpendicular bisector of $\overline{AP}$. With this different beginning we shall need different choices for f_2, f_3, and f_4, but, as before, each will be either $\vec{0}$ or the reflection in a properly chosen plane. As a result we have:

> **Theorem 14–31** Each isometry of $\mathscr{E}$ is either the identity mapping, or the reflection in some plane, or the resultant of two, three, or four such reflections.

[Of course, the resultant of five or more reflections is an isometry. What the theorem tells us is that such an isometry is, also, the resultant of four or fewer reflections. For example, the resultant of three translations is the resultant of six reflections in appropriate planes and is also the resultant of two such reflections. Explain.]

As remarked earlier in this commentary, the ', three, or four' in Theorem 14-31 can be replaced by 'or three'. For our purposes, however, it is sufficient to know that any isometry can be "put together" out of plane reflections.

Since each translation is the resultant of two reflections a resultant of any number of translations is also the resultant of twice as many reflections. However, a resultant of translations is, we know, a translation and, so, is the resultant of just two reflections.

14.06 Chapter Summary

Vocabulary Summary

- triangle
- distance function
- equidistant
- isosceles triangle
- right triangle
- Pythagorean Theorem
- triangle inequality
- reflection in a plane
- max (a, b)
- distance from P to Q
- bisect
- equilateral triangle
- hypotenuse
- mean proportional
- isometry
- resultant
- min (a, b)
- increasing function
- perpendicular
  - bisector
  - from point to line
  - from point to plane
- foot of a perpendicular
- congruent figures

Definitions

14-1. $d(P, Q) = \|Q - P\|$

14-2. The perpendicular bisector of $\overleftrightarrow{AB}$ is $\{X: X - M \in [B - A]^{\perp}$, where M is the midpoint of $\overleftrightarrow{AB}\}$.

14-3. (a) $\overline{PQ}$ is the perpendicular from P to l $\longleftrightarrow (Q \in l \text{ and } P - Q \in [l]^{\perp})$

(b) $\overline{PQ}$ is the perpendicular from P to π $\longleftrightarrow (Q \in \pi \text{ and } P - Q \in [\pi]^{\perp})$

14-4. The altitude of a triangle from one of its vertices [or, to the opposite side of the triangle] is the perpendicular from that vertex to the line containing the opposite side.

14-5. $PQ = d(P, Q)$

14-6. (a) $\triangle ABC$ is an isosceles triangle with base $\overline{AB} \longleftrightarrow BC = CA$

(b) $\triangle ABC$ is an equilateral triangle $\longleftrightarrow BC = CA = AB$

14-7. $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB} \longleftrightarrow \overleftrightarrow{CA} \perp \overleftrightarrow{BC}$

14-8. f is an isometry of $\mathscr{E}$ if and only if f is a mapping of $\mathscr{E}$ onto itself such that $\forall_X \forall_Y\, d(f(X), f(Y)) = d(X, Y)$.

14-9. A first figure is congruent to a second if and only if there is an isometry of $\mathscr{E}$ which maps the first figure onto the second.

14-10. f is a reflection in $\pi \longleftrightarrow \forall_X\, f(X) = X + (M - X)2$, where M is the foot of the perpendicular from X to π.

Other Theorems

Lemma. $\vec{a} \cdot \vec{b} \le \|\vec{a}\|\,\|\vec{b}\|$, and $\vec{a} \cdot \vec{b} = \|\vec{a}\|\,\|\vec{b}\| \longleftrightarrow (\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have the same sense)

14-1. [The Triangle Inequality] $\big|\|\vec{a}\| - \|\vec{b}\|\big| \le \|\vec{a} + \vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$, with equality on the left if and only if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have opposite senses, and on the right if and only if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ have the same sense.

14-2. (a) $d(P, Q) \ge 0$ (b) $d(P, Q) = 0 \longleftrightarrow Q = P$

(c) $d(Q, P) = d(P, Q)$ (d) $d(P, R) \le d(P, Q) + d(Q, R)$

14-3. $d(P, R) = d(P, Q) + d(Q, R) \longleftrightarrow Q \in \overline{PR}$

14-4. For $A \ne B$, the perpendicular bisector of $\overleftrightarrow{AB}$ is the plane which contains the midpoint of $\overleftrightarrow{AB}$ and is perpendicular to $\overleftrightarrow{AB}$.

14-5. The perpendicular bisector of $\overleftrightarrow{AB}$ is $\{X: d(A, X) = d(B, X)\}$.

14-6. A point P is equidistant from two points A and B if and only if it belongs to the plane which is perpendicular to $\overleftrightarrow{AB}$ at the midpoint of $\overleftrightarrow{AB}$.

Corollary. There is one and only one point of $\overleftrightarrow{AB}$ which is equidistant from A and B, and this point is the midpoint of $\overleftrightarrow{AB}$.

14-7. The intersection of the three perpendicular bisectors of the sides of a triangle is a line which is perpendicular to the plane of the triangle.

Corollary 1. The perpendicular bisectors of the sides of a triangle intersect the plane of the triangle in three concurrent lines.

Corollary 2. The lines containing the altitudes of a triangle are concurrent.

14-8. In $\triangle ABC$, (a) $|BC - CA| < AB < BC + CA$ and (b) $D \in \overline{AB} \longleftrightarrow CD < \max(BC, CA)$.

14-9. $\triangle ABC$ is an isosceles triangle with base $\overline{AB}$ if and only if its median from C is its altitude from C.

14-10. [The Pythagorean Theorem] $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$ if and only if $AB^2 = BC^2 + CA^2$.

Corollary 1. The ratio of the measure of the hypotenuse of an isosceles right triangle to the measure of either of its legs is $\sqrt{2}$.

Corollary 2. The ratio of the measure of an altitude of an equilateral triangle to the measure of any of its sides is $\sqrt{3}/2$.

14-11. A triangle is a right triangle with a given side as hypotenuse if and only if the measure of the given side is twice the measure of the median to that side.

14-12. (a) The altitude to the hypotenuse of a right triangle is the mean proportional between the measures of the intervals into which its foot divides the hypotenuse.

(b) Either leg of a right triangle is the mean proportional between the hypotenuse and the measure of that one of the two intervals, into which the foot of the altitude divides the hypotenuse, which is adjacent to the given leg.

14-13. In any triangle, the product of any side by the altitude to it is the same as the product of any other side by the altitude to it.

Corollary. In a right triangle, the product of the hypotenuse by the altitude to it is the same as the product of the legs.

14-14. The perpendicular from a given point to a given line or plane is the shortest of the intervals whose endpoints are the given point and a point of the given line or plane.

14-15. The distance between a given point and a point of a given line or plane is an increasing function of the distance between the second point and the foot of the perpendicular to the given line or plane.

14-16. If $l = \overleftrightarrow{A[\vec{q}]}$ and $\pi = \overrightarrow{A[\vec{q}, \vec{r}]}$ then, with $\vec{p} = P - A$,

(a)
$$d(P, l)^2 = \begin{vmatrix} \vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q} \\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q} \end{vmatrix} / (\vec{q}\cdot\vec{q}), \text{ and}$$

(b)
$$d(P, \pi)^2 = \begin{vmatrix} \vec{p}\cdot\vec{p} & \vec{p}\cdot\vec{q} & \vec{p}\cdot\vec{r} \\ \vec{q}\cdot\vec{p} & \vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{r} \\ \vec{r}\cdot\vec{p} & \vec{r}\cdot\vec{q} & \vec{r}\cdot\vec{r} \end{vmatrix} \Bigg/ \begin{vmatrix} \vec{q}\cdot\vec{q} & \vec{q}\cdot\vec{r} \\ \vec{r}\cdot\vec{q} & \vec{r}\cdot\vec{r} \end{vmatrix}$$

14-17. If, with respect to some orthonormal coordinate system, the plane π is described by:

$$x_1m_1 + x_2m_2 + x_3m_3 = c,$$

and the point P has coordinates (p_1, p_2, p_3), then

$$d(P, \pi) = |p_1m_1 + p_2m_2 + p_3m_3 - c| / \sqrt{m_1^2 + m_2^2 + m_3^2}.$$

Moreover, if Q has coordinates (q_1, q_2, q_3) then P and Q are on the same side or on opposite sides of π according as $(p_1m_1 + p_2m_2 + p_3m_3 - c)\,(q_1m_1 + q_2m_2 + q_3m_3 - c)$ is positive or negative.

14-18. Any translation is an isometry.

14-19. If f is the reflection in π then $f(P) = Q$
$\longleftrightarrow$ $(Q - P \in [\pi]^{\perp}$ and the midpoint of $\overleftrightarrow{PQ} \in \pi)$.

Corollary. If f is the reflection in π then $f(P) = Q \longleftrightarrow f(Q) = P$.

14-20. The reflection in a plane is an isometry.

14-21. If f is the reflection in π then (a) $f(P) = P$ if and only if $P \in \pi$, and (b) if $f(P) \neq P$ then π is the perpendicular bisector of $\overleftrightarrow{Pf(P)}$.

14-22. The only isometries of $\mathscr{E}$ which leave fixed three given noncollinear points are the identity mapping and the reflection in the plane which contains the given points.

Corollary. The only isometry of $\mathscr{E}$ which leaves fixed four noncoplanar points is the identity mapping.

14-23. An isometry which leaves two points fixed also leaves fixed each point of the line containing the two points.

Lemma. The resultant of the reflection in a plane π followed by a translation $\vec{a}$ in the direction orthogonal to π is the reflection in the plane $\pi + (\vec{a}/2)$.

14-24. If π is any plane such that $\vec{a} \in [\pi]^{\perp}$ and $\sigma = \pi + (\vec{a}/2)$ then the translation $\vec{a}$ is the resultant of the reflection in π followed by reflection in σ.

14-25. If $\pi \parallel \sigma$ then the resultant of the reflection in π followed by the reflection in σ is a translation.

14-26. Any isometry of $\mathscr{E}$ maps planes onto planes, maps lines onto lines, and preserves parallelism and perpendicularity.

14-27. Any isometry of $\mathscr{E}$ maps segments onto segments and intervals onto intervals, mapping endpoints on endpoints; it maps rays onto rays and half-lines onto half-lines, mapping vertices on vertices.

14-28. Segments [or: intervals] are congruent if and only if they have the same measure.

Lemma. If $AB = PQ$ then there is an isometry f such that $f(P) = A$ and $f(Q) = B$.

Lemma. If $AB = PQ$, $BC = QR$, and $CA = RP$ then there is an isometry f such that $f(P) = A$, $f(Q) = B$, and $f(R) = C$.

14-29. Given A, B, C, and D and points P, Q, R, and S, if $AB = PQ$, $AC = PR$, $AD = PS$, $BC = QR$, $BD = QS$, and $CD = RS$, then there exists an isometry which maps P on A, Q on B, R on C, and S on D.

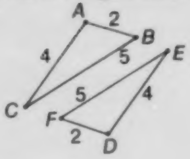
14-30. If two isometries agree at each of three noncollinear points then they agree at each point of the plane which contains these three points; moreover, each of the given isometries is the resultant of the other followed by the reflection in the image of this plane.

Corollary. No two isometries agree at each of four noncoplanar points.

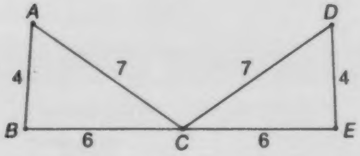
14-31. Each isometry of $\mathscr{E}$ is either the identity mapping, or the reflection in some plane, or the resultant of two, three, or four such reflections.

Chapter Test

1. Suppose that ABC is a triangle and that $AB = 6$ and $BC = 2$.
   - (a) What is the range of values for AC, the length of the third side of $\triangle ABC$?
   - (b) Given that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AC}$, compute AC.
   - (c) Given that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$, compute AC.
   - (d) Given that $\triangle ABC$ is isosceles with base $\overline{BC}$, compute the length of its median to $\overline{BC}$.
2. Suppose that $\triangle DEF$ is an equilateral triangle.
   - (a) Given that $DE = 6$, compute the length of the median of $\triangle DEF$ from F.
   - (b) Given that G is the foot of the perpendicular from D to $\overleftrightarrow{EF}$ and that $DG = 6$, compute both DE and EG.
   - (c) Given that $\overline{EH}$ and $\overline{DG}$ are medians of $\triangle DEF$, that O is their point of intersection, and that $DE = 8$, compute DO and OG.
   - (d) With the information given in part (c), compute the ratios $\overline{DH} : \overline{DO}$ and $\overline{GO} : \overline{GE}$.
3. Suppose that $\|\vec{a}\| = 3$, $\|\vec{b}\| = 6$, $\text{comp}_{\vec{b}}(\vec{a}) = \frac{1}{2}$, and that $A - O = \vec{a}$ and $B - O = \vec{b}$.
   - (a) Compute $\vec{a} \cdot \vec{b}$. What does this tell you about $(\vec{a}, \vec{b})$?
   - (b) Compute AB. What does this tell you about O, A, and B?
4. Given that a plane π is described by the equation '$3x_1 - 4x_2 + 5x_3 = 6$' and that points P and Q have coordinates $(3, 2, -1)$ and $(2, -2, 1)$, respectively, with respect to an orthonormal coordinate system.
   - (a) Compute $d(P, \pi)$ and $d(Q, \pi)$.
   - (b) Determine whether P and Q are on the same side, or on opposite sides of π.
5. In each of the following, you are given two triangles and some information about them. Make use of what you know about isometries to help you answer the questions about congruence.
   - (a) Suppose that $\overline{AB} \parallel \overline{DF}$, $\overline{AC} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{FE}$, and that the measures of the intervals are as given. Show that $\triangle ABC$ and $\triangle DEF$ are congruent.



   - (b) Suppose that the measures of the sides are as indicated. Show that $\triangle ABC$ and $\triangle DEC$ are congruent.



Answers for Chapter Test

1. (a) $4 < AC < 8$

   (b) $AC = \sqrt{36 + 4} = 2\sqrt{10}$

   (c) $36 = 4 + AC^2$, so $AC = 4\sqrt{2}$.

   (d) Let M be the midpoint of $\overleftrightarrow{BC}$. Then $AM^2 + 1 = 36$ [since $\overleftrightarrow{AM} \perp \overleftrightarrow{BC}$], so $AM = \sqrt{35}$.

2. (a) Let M be the midpoint of $\overleftrightarrow{DE}$. Then $FM^2 + 9 = 36$, so $FM = 3\sqrt{3}$.

   (b) Let $x = EG$, then $2x = DE$ and $36 + x^2 = 4x^2$, so $x = 2\sqrt{3}$. Therefore, $EG = 2\sqrt{3}$ and $DE = 4\sqrt{3}$.

   (c) Since $ED = 8$, $DG = 4\sqrt{3}$. Since $GO = HO$, we have $DO^2 = 16 + GO^2 = 16 + (4\sqrt{3} - DO)^2$. Thus $DO = 8\sqrt{3}/3$, and $GO = 4\sqrt{3}/3$.

   (d) $DH:DO = \sqrt{3}/2$, $GO:GE = \sqrt{3}/3$.

3. (a) $\vec{a} \cdot \vec{b} = \text{comp}_{\vec{b}}(\vec{a})\|\vec{b}\|^2 = 6^2/2 = 18$; Since $\vec{a} \cdot \vec{b} = \|\vec{a}\|\,\|\vec{b}\|$, $(\vec{a}, \vec{b})$ is linearly dependent.

   (b) $AB = \|B - A\| = \|\vec{b} - \vec{a}\|$, and $\|\vec{b} - \vec{a}\|^2 = 6^2 + 3^2 - 2 \cdot 18 = 9$. So, $AB = 3$. Since $AB + OA = 3 + 3 = 6 = OB$, O, A, and B are collinear. In fact, $A \in \overline{OB}$.

4. (a) By Theorem 14-17, $d(P, \pi) = \sqrt{2}$, and $d(Q, \pi) = 13\sqrt{2}/10$.

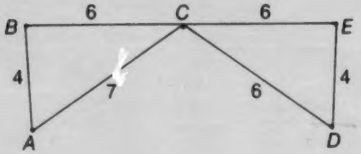
   (b) P and Q are on opposite sides of π. [$9 - 8 - 5 - 6 < 0$ and $6 + 8 + 5 - 6 > 0$]

5. (a) By Theorem 14-29 there is an isometry — say, f — which maps A on D, B on F, and C on E. It follows by Theorem 14-27 that f maps $\overleftrightarrow{AB}$ onto $\overleftrightarrow{DF}$, $\overleftrightarrow{BC}$ onto $\overleftrightarrow{FE}$, and $\overleftrightarrow{CA}$ onto $\overleftrightarrow{ED}$. So, f maps $\triangle ABC$ onto $\triangle DFE$. Hence, $\triangle ABC$ is congruent to $\triangle DFE$. [Note that the assumptions of parallelism are irrelevant. Note, also that this is a preview of the proof of the side-side-side congruence theorem.]

   (b) [Exactly the same as answer for part (a), but with 'C' for 'E' and 'E' for 'F'.]

   (c) $\triangle DEC$ is an isosceles triangle and, so, any isometry will map it onto an isosceles triangle. [This depends on Theorem 14-27.] $\triangle ABC$ is not isosceles and, so, is not the image of $\triangle DEC$ under any isometry. Hence, $\triangle ABC$ and $\triangle DEC$ are not congruent.

(c) Suppose that the measures of the sides are as indicated. Show that $\triangle ABC$ and $\triangle DEC$ are not congruent.



Background Topic

We have taken your knowledge of the real numbers for granted, reviewing and perhaps adding to it in Chapter 4 and in some of the background topics at the ends of earlier chapters. Like the latter exercises, these exercises will review some things you know and, perhaps, add to your knowledge of the real numbers. We shall be particularly concerned with the nonnegative integers, 0, 1, 2, 3, etc., and with clarifying what the 'etc.' means. We shall use '*Nn*' as a name for the set of all nonnegative integers.

To begin with, let's consider the notion of the *powers* of a number — say, the number 2. As you know, 2^2 — the second power of 2 — is 4, the third power of 2 is 8, the fourth is 16, etc. You have probably learned something like "To find the fourth power of 2, multiply 2 by itself four [or, maybe, three] times." or "The fourth power of 2 is four twos multiplied together." Neither of these is very satisfactory, and we wish to have something better. One better way of describing the powers of 2 is by what is called a *recursive definition*. Such a definition for the powers of 2 is:

(1)
$$2^0 = 1$$
$$2^{a+1} = 2^a \cdot 2, \text{ for } a \in Nn$$

To see in what way this is a definition of the powers of 2 let's use it to compute the value of '2^4'. To begin with, since $4 = 3 + 1$ and $3 \in Nn$, we have that

$$2^4 = 2^{3+1} = 2^3 \cdot 2.$$

Similarly, since $3 = 2 + 1$ and $2 \in Nn$, we have that

$$2^3 \cdot 2 = 2^{2+1} \cdot 2 = (2^2 \cdot 2) \cdot 2.$$

Again, since $2 = 1 + 1$ and $1 \in Nn$, we have that

$$2^2 \cdot 2 \cdot 2 = 2^{1+1} \cdot 2 \cdot 2 = (2^1 \cdot 2) \cdot 2 \cdot 2.$$

Finally, since $1 = 0 + 1$ and $0 \in Nn$, we have that

$$2^1 \cdot 2 \cdot 2 \cdot 2 = 2^{0+1} \cdot 2 \cdot 2 \cdot 2 = (2^0 \cdot 2) \cdot 2 \cdot 2 \cdot 2 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2.$$

and, so $2^4 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2$.

It is desirable that students have some knowledge of proof by mathematical induction and of some results concerning integers whose proofs require mathematical induction. Hopefully, students will already have had some experience with mathematical induction. Since this will not certainly be the case, the treatment given here in the text is, though brief, self-contained. For a more extended treatment, see Beberman and Vaughan, High School Mathematics, Course 3, Heath (1966).

Through some of the exercises, the treatment of mathematical induction serves as a vehicle for some work on nonnegative integral exponents.

Our choice of 'Nn' as a name for the set of nonnegative integers is not standard, but we are at a loss for a better.

The unsatisfactoriness of the first of the quoted descriptions of the fourth power of 2 is fairly obvious. [One oscillates between 'four' and 'three'.] That of the second arises from the fact that, since there is only one integer 2, you can't find four of them to multiply together. The recursive definition (1) suggests a better description: For any nonnegative integer a, 2^a is the result of starting with 1 and multiplying by 2 a times in succession. The satisfactoriness of this description in the cases in which a = 0 and a = 1 is part of the motivation for accepting the definition '$2^0 = 1$'. [See answers for Exercise 1 of Part A which follows.]

The reason for calling (1) a recursive definition is illustrated in the text by the five "backward steps" taken in using it to compute the fourth power of 2. Definitions like (1) are sometimes called 'inductive definitions'. A careful inspection of proofs by mathematical induction shows that each such proof is based on one or more such definitions.

This agrees with our original idea of the meaning of '2^4' and suggests that if we wish to state a rule describing how to compute the value of '2^4' we might say "Start with 1 and multiply by 2 four times in succession." Evidently we can use (1) to compute, a step at a time, any nonnegative integral power of 2 that we wish. And an equivalent rule to (1) is:

> For any $a \in Nn$, 2^a is the result of starting with 1 and multiplying by 2 a times in succession.

Part A

1. Write correct statements of the form '2^a is the result of starting with 1 and multiplying by 2 a times in succession' for $a = 2$, 1, and 0.
2. Make a table of powers of 2 starting with 2^0 and ending with 2^{10}:

a	0	1	2	...
2^a	1	2	4	

3. Consider the following sentences, some of which we have left for you to write [if you need to] and some of which we have left incomplete. Write sentences from the list until you are sure you know how to complete the last one.

$$
\begin{aligned}
2^0 &= 1\\
2^0 + 2^1 &= 3\\
2^0 + 2^1 + 2^2 &= 7\\
&\vdots\\
2^0 + \ldots + 2^5 &=\\
&\vdots\\
2^0 + \ldots + 2^{69} &=
\end{aligned}
$$

[*Hint*: If, after writing out, say, six of the sentences you don't see how to complete the last one, compare the right sides of your sentences with some of the entries in your table of powers of 2.]

4. Try to find a way of increasing your certainty in the correctness of the guess you made in answering Exercise 3. [*Hint*: How can you use the result stated in one sentence to find out how to complete the next?]

*

In doing Exercise 3 you must have found that $2^0 + \ldots + 2^5 = 63$ and comparing the various sentences should have called it to your attention that $63 = 64 - 1 = 2^6 - 1$. This, and study of some of the other sentences, might have led you to guess that $2^0 + \ldots + 2^{69} = 2^{70} - 1$. You could strengthen the grounds of your faith in this result by noticing that, since $2^0 + \ldots + 2^5 = 2^6 - 1$,

Answers for Part A

1. 2^2 is the result of starting with 1 and multiplying by 2 twice.
   2^1 is the result of starting with 1 and multiplying by 2 once.
   2^0 is the result of starting with 1 and multiplying by 2 zero times.
2. The successive powers of 2 through the tenth power are 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, and 1024. [Incidentally, fairly good estimates of higher powers of 2 can be obtained by using the fact that 2^{10} is approximately 1000. For example, $2^{16} = 2^6 \cdot 2^{10} \doteq 64000$. Actually, $2^{16} = 65536$.]
3. $2^{70} - 1$ [You will recognize this exercise as having to do with sums of geometric progressions. What is nearly the general case is taken up in Exercise 1 of Part B.]
4. Once we see, for example, that $2^0 + \ldots + 2^5 = 2^6 - 1$ it is easy to see that $2^0 + \ldots + 2^6 = (2^6 - 1) + 2^6 = 2^6 \cdot 2 - 1 = 2^{6+1} - 1 = 2^7 - 1$. Proceeding stepwise in this way from one sentence to the next, we can be certain that $2^0 + \ldots + 2^{69} = 2^{70} - 1$. [Note the use of the recursive definition (1) in justifying the replacement of '$2^6 \cdot 2$' by '2^{6+1}'.]

$$\begin{aligned} 2^0 + \ldots + 2^6 &= (2^0 + \ldots + 2^5) + 2^6 \\ &= (2^6 - 1) + 2^6 \\ &= 2^6 \cdot 2 - 1 \\ &= 2^7 - 1 \end{aligned}$$

by the recursive definition (1). More generally, you might note that if $2^0 + \ldots + 2^b = 2^{b+1} - 1$ then

$$\begin{aligned} 2^0 + \ldots + 2^{b+1} &= (2^0 + \ldots + 2^b) + 2^{b+1} \\ &= (2^{b+1} - 1) + 2^{b+1} \\ &= 2^{b+1} \cdot 2 - 1 \\ &= 2^{b+2} - 1 \end{aligned}$$

This gives us a general procedure for getting each result [except the first] from the preceding one. Clearly, if we start with '$2^0 = 2^1 - 1$', which is equivalent to our first result, then by applying this general procedure 69 times we will end up with '$2^0 + \ldots + 2^{69} = 2^{70} - 1$'. Having such a general procedure would seem to justify our acceptance of the theorem:

$$\text{If } a \in Nn \text{ then } 2^0 + \ldots + 2^a = 2^{a+1} - 1.$$

Part B

1. Use the fact that, for any real number b, $b^0 = 1$, and, if $a \in Nn$, $b^{a+1} = b^a \cdot b$, to show that

   $$\text{If } a \in Nn \text{ then, for } b \neq 1,\ b^0 + \ldots + b^a = (b^{a+1} - 1)/(b - 1).$$

   [*Hint*: Show that the equation holds for $a = 0$ and that if it holds when a is a given number $c \in Nn$ then it must also hold when a is $c + 1$.]

2. Consider these sentences:

   $$\begin{aligned} 1 &= 1 \\ 1 + 3 &= 4 \\ 1 + 3 + 5 &= 9 \\ &\vdots \\ 1 + \ldots + 11 &= \\ &\vdots \\ 1 + \ldots + 69 &= \end{aligned}$$

   and answer questions like those in Exercises 3 and 4 of Part A. [*Hint*: The first three sentences should suggest to you that '$1 + \ldots + 11$' is to be taken as short for '$1 + 3 + 5 + 7 + 9 + 11$' and, so, to indicate the sum of the first six odd numbers. Similarly, the left side of the last equation indicates the sum of the first thirty-five odd numbers.]

3. Show that if a is a positive integer then $1 + \ldots + (2a - 1) = a^2$. [*Hint*: Show that the equation holds for $a = 1$ and show that if it holds for $a = c$, where c is a positive integer, then it must hold for $a = c + 1$.]

TC 201

Note that in Exercise 1 of Part B we adopt the recursive definition:

$$b^0 = 1$$
$$b^{a+1} = b^a \cdot b \quad [a \in Nn]$$

The congruence that $0^0 = 1$ is intended. Although it is customary to leave '0^0' undefined it turns out to be natural, as well as extremely convenient, to define '0^0' to be a numeral for 1. [On this, see the teacher's edition of Beberman and Vaughan, Course 3; in particular, see pages T 412(1) - T 412(3).]

Answers for Part B

1. [See the text immediately following these exercises. Students should not be expected to do "very well" on this exercise. It's principal purpose is to lead them to try and, so, to be ready to appreciate the discussion in the text.]

2. [Exercise 3] $1 + \ldots + 69 = 35^2$ [Assuming that the '...'s are interpreted as suggested by the earlier sentences.]

   [Exercise 4] Once we see, for example, that the sum of the first six odd integers is 6^2 and note that the seventh is $2 \cdot 6 + 1$ then it is clear that the sum of the first seven odd integers is $6^2 + 2 \cdot 6 + 1$ — that is, is $(6 + 1)^2$ and, so, 7^2. Proceeding stepwise we would arrive at the fact that the sum of the first thirty-five odd numbers is 35^2.

3. [Proof which students might very roughly approximate is given in the text immediately preceding Part C.]

TC 202

The '...'-notation, as used in (2) and earlier, is often vague, and students are likely to have difficulty in handling it. [They have, however, met it, or something very like it, in the discussion of sequences in Chapter 6 of volume 1.] In a more extended account of induction one would introduce "Σ-notation" instead. [See Beberman and Vaughan, Course 3, Chapter 10.]

TC 203

The use of the quantifier 'for each x' near the end of the proof of (2) is motivated by the need for '$\forall_x$' in the statement of (Nn_3) which follows. Recall that the use of a "free" variable in the antecedent of a conditional $[(Nn_3)]$ amounts to the use of an existential quantifier. So, in stating (Nn_3) we must use the quantifier.

*

In the exercises of Part A and Part B we have considered theorems concerning all nonnegative integers [or in Exercise 3 of Part B, all positive integers]. The hints for Exercises 1 and 3 of Part B have indicated a way in which such theorems can be proved. This method of proof is called *proof by mathematical induction* and we shall illustrate it by giving such a proof for the example of Exercise 1.

What we wish to prove is:

$$(2)\quad a \in Nn \longrightarrow b^0 + \ldots + b^a = (b^{a+1} - 1)/(b - 1) \qquad [b \neq 1]$$

The expression '$b^0 + \ldots + b^a$' has a value, for any value of 'a' in Nn, which may be described by saying "Start with b^0 and add b^1, b^2, etc., ending with b^a." So, for example, '$b^0 + \ldots + b^1$' is equivalent to '$b^0 + b^1$', and '$b^0 + \ldots + b^0$' is equivalent to 'b^0'.

Ignoring for a moment the restriction on 'b' which is needed for the algebra involving '$/(b - 1)$', this theorem is of the form:

$$a \in Nn \longrightarrow Fa$$

where 'Fa' stands in place of the sentence '$b^0 + \ldots + b^a = (b^{a+1} - 1)/(b - 1)$'. The proof amounts to showing, first, that, for $b \neq 1$, $F0$ – that is, that

$$\text{(i)}\qquad b^0 + \ldots + b^0 = (b^{0+1} - 1)/(b - 1) \qquad [b \neq 1]$$

and, second, that, for $c \in Nn$, $Fc \longrightarrow F(c + 1)$ – that is

$$\text{(ii)}\qquad \begin{array}{l} b^0 + \ldots + b^c = (b^{c+1} - 1)/(b - 1) \longrightarrow \\ b^0 + \ldots + b^{c+1} = (b^{[c+1]+1} - 1)/(b - 1) \qquad [b \neq 1] \end{array}$$

Statement (i) gives our proof a foundation by asserting that the first nonnegative integer, 0, satisfies the sentence Fa. Statement (ii) asserts that, for any number $c \in Nn$, if c satisfies Fa then so does the next number, $c + 1$ of Nn. If we can justify each of these two assertions we are justified – intuitively, at least – in accepting (2) as a theorem. We can now give a proof of (2).

(i) Since, by definition, $b^0 + \ldots + b^0 = b^0 = 1$ and $(b^{0+1} - 1)/(b - 1) = (b^1 - 1)/(b - 1) = (b - 1)/(b - 1) = 1$, for $b \neq 1$, it follows that $b^0 + \ldots + b^0 = (b^{0+1} - 1)/(b - 1)$.

(ii) Suppose, for a given $c \in Nn$, that $b^0 + \ldots + b^c = (b^{c+1} - 1)/(b - 1)$. Since $b^0 + \ldots + b^{c+1} = (b^0 + \ldots + b^c) + b^{c+1}$ it follows that

$$\begin{aligned}(b^0 + \ldots + b^c) + b^{c+1} &= \frac{b^{c+1} - 1}{b - 1} + b^{c+1} \\ &= \frac{b^{c+1} - 1 + b^{c+1} \cdot b - b^{c+1}}{b - 1} \\ &= \frac{b^{c+1} \cdot b - 1}{b - 1} \\ &= \frac{b^{[c+1]+1} - 1}{b - 1}\end{aligned}$$

In short, for each x,

if $x \in Nn$ and $b^0 + \ldots + b^x = \dfrac{b^{x+1} - 1}{b - 1}$

then $b^0 + \ldots + b^{x+1} = \dfrac{b^{(x+1)+1} - 1}{b - 1}$.

By (i) and (ii) it follows by mathematical induction that

$$a \in Nn \longrightarrow b^0 + \ldots + b^a = (b^{a+1} - 1)/(b - 1). \qquad [b \neq 1]$$

To justify formally proofs like the preceding, as well as other proofs concerning properties of the nonnegative integers, we need three postulates which serve to characterize Nn:

(Nn_1) $0 \in Nn$
(Nn_2) $a \in Nn \longrightarrow a + 1 \in Nn$
(Nn_3) $(F0 \text{ and } \forall_x [(x \in Nn \text{ and } Fx) \longrightarrow F(x + 1)]) \longrightarrow [a \in Nn \longrightarrow Fa]$

If you were to try to tell someone what the nonnegative integers are you might start by telling him that 0 is one of them [(Nn_1)] and that $0 + 1$ – or, 1 – is another, $1 + 1$ – or, 2 – is another, etc. [(Nn_2)], and that you will get all of them by continuing in this way [(Nn_3)].

Note that, in the preceding proof, it is (Nn_3) which is referred to by the phrase 'by mathematical induction'.

Sometimes, as in Exercise 3 of Part B, we wish to prove theorems about all the positive integers, 1, 2, 3, etc. If we use 'I^+' as a name for the set of all positive integers it is possible, using (Nn_1)–(Nn_3) to justify another principle like (Nn_3):

$$(F1 \text{ and } \forall_x [(x \in I^+ \text{ and } Fx) \longrightarrow F(x + 1)]) \longrightarrow [a \in I^+ \longrightarrow Fa]$$

We shall not take time to justify this and other similar principles of mathematical induction. But you may use it and indicate its use by writing 'by mathematical induction'.

For example, a proof for Exercise 3 of Part B would run as follows:

(i) Since $1 + \ldots + (2 \cdot 1 - 1) = 1$ and $1^2 = 1$ it follows that $1 + \ldots + (2 \cdot 1 - 1) = 1^2$. [This is $F1$ where Fa is the sentence '$1 + \ldots + (2a - 1) = a^2$'.]

(ii) Suppose, for a given $c \in I^+$, that $1 + \ldots + (2c - 1) = c^2$. Since $1 + \ldots + (2c - 1) + [2(c + 1) - 1] = [1 + \ldots + (2c - 1)] + (2c + 1)$ it follows that $1 + \ldots + [2(c + 1) - 1] = c^2 + (2c + 1) = (c + 1)^2$. In short, for each x,

if $x \in I^+$ and $1 + \ldots + (2x - 1) = x^2$ then $1 + \ldots + [2(x + 1) - 1] = (x + 1)^2$. By (i) and (ii) it follows by mathematical induction that

$$a \in I^+ \longrightarrow 1 + \ldots + (2a - 1) = a^2.$$

*

*Part C

1. Use (Nn_2) and (Nn_3) to show that Nn is closed with respect to addition. [*Hint*: Prove that $b \in Nn \longrightarrow a + b \in Nn$, subject to the restrictions '$a \in Nn$'. In other words, take for 'Fb' the sentence '$a + b \in Nn \quad [a \in Nn]$'.
2. Prove: $b \in Nn \longrightarrow 2^{a+b} = 2^a \cdot 2^b \; [a \in Nn]$. [*Hint*: Use the recursive definition (1) on page 199. In part (ii) of the proof you will need the result of Exercise 1.]

*

Later in the course we shall need a number of theorems about integers and we shall begin by proving two of them now. Each is intuitively rather trivial, but it is not trivial that they follow from $(Nn_1)-(Nn_3)$. The proofs will give you further examples of proof by mathematical induction. The first theorem says that each number in Nn is nonnegative:

$$a \in Nn \longrightarrow a \geq 0 \tag{3}$$

The second says that there is no nonnegative integer between 0 and 1:

$$a \in Nn \longrightarrow [a > 0 \longrightarrow a \geq 1] \tag{4}$$

In proving both (3) and (4) we shall need to use a lemma:

$$a + 1 > a$$

The exercises of Part C are optional since we are not concerned at this point with students mastering the construction of inductive proofs. Here, and in the text following the exercises we are mostly concerned with understanding inductive proofs. Some of your more capable students may wish to attempt these exercises, and you may wish to spend a small amount of class time in discussing their results.

Answers for Part C

1. (i) Since $a + 0 = a$ it follows that, for a in Nn, $a + 0 \in$ Nn.

   (ii) Suppose, for a given number $c \in$ Nn, that $a + c \in$ Nn. Since $a + (c + 1) = (a + c) + 1$ it follows, by (Nn_2) that $a + (c + 1) \in$ Nn. By (i) and (ii) it follows by mathematical induction that, for $a \in$ Nn, $b \in \text{Nn} \Longrightarrow a + b \in$ Nn.

   [The form of the conclusion just reached is suited to bring out the different roles played by 'a' and 'b' in the proof. Equivalently, and more naturally, we might go on to conclude (by importation) that if $a \in$ Nn and $b \in$ Nn then $a + b \in$ Nn.]

2. (i) $2^{a+0} = 2^a = 2^a \cdot 1 = 2^a \cdot 2^0$

   (ii) Suppose, for a given $c \in$ Nn, that $2^{a+c} = 2^a \cdot 2^c$. Then $2^{a+(c+1)} = 2^{(a+c)+1}$ and since, for $a \in$ Nn, $a + c \in$ Nn [Exercise 1] it follows that $2^{(a+c)+1} = 2^{a+c} \cdot 2$. So, since $2^{a+c} = 2^a \cdot 2^c$, $2^{a+(c+1)} = (2^a \cdot 2^c) \cdot 2 = 2^a(2^c \cdot 2) = 2^a \cdot 2^{c+1}$. Hence, for each $x \in$ Nn, if $c \in$ Nn and $2^{a+x} = 2^a \cdot 2^x$ then $2^{a+(x+1)} = 2^a \cdot 2^{x+1}$ [$a \in$ Nn].

   By (i) and (ii) it follows by mathematical induction that, for $a \in$ Nn, $b \in \text{Nn} \Longrightarrow 2^{a+b} = 2^a \cdot 2^b$ [or, equivalently, if $a \in$ Nn and $b \in$ Nn then $2^{a+b} = 2^a \cdot 2^b$].

*

Theorems (3) and (4) form the basis for some important theorems concerning integers which will be dealt with at the end of Chapter 15. It is to be hoped that students can follow the proofs given here. The irreducible minimum is that they should understand the theorems.

This is a theorem about all real numbers and you can prove it by first proving '$1 > 0$'. [*Hint*: Use the theorem '$a^2 > 0\ [a \neq 0]$' and a couple of parts of Postulate 5'.]

Proof of (3): In the first place, since $0 = 0$, $0 \geq 0$. Suppose, now [for a given number b] that $b \in Nn$ and $b \geq 0$. Since $b + 1 > b$ it follows that $b + 1 \geq 0$. Hence, for each x, if $x \in Nn$ and $x \geq 0$ then $x + 1 \geq 0$. Since, also, $0 \geq 0$ it follows by mathematical induction that if $a \in Nn$ then $a \geq 0$.

Proof of (4): In the first place, since $0 \not> 0$ it follows that $0 \not\geq 1 \longrightarrow 0 \not> 0$ and, so, that $0 > 0 \longrightarrow 0 \geq 1$. Suppose, now, that $b \in Nn$ and that $b > 0 \longrightarrow b \geq 1$. Since we wish to infer that $b + 1 > 0 \longrightarrow b + 1 \geq 1$, we shall also assume that $b + 1 > 0$. By (3) it is sufficient to consider two cases—that in which $b > 0$ and that in which $b = 0$. In the first case it follows from one of our assumptions that $b \geq 1$ and, so, that $b + 1 \geq 1 + 1 > 1$. [Note the use of our lemma.] So, in this case, if $b + 1 > 0$ then $b + 1 \geq 1$. In the second case $b + 1 = 0 + 1 \geq 1$ and, so, in this case as well, if $b + 1 > 0$ then $b + 1 \geq 1$. Hence, for each x, if $x \in Nn$ and $[x > 0 \longrightarrow x \geq 1]$ then $[x + 1 > 0 \longrightarrow x + 1 \geq 1]$. Since, also, $0 > 0 \longrightarrow 0 \geq 1$ it follows that if $a \in Nn$ then, if $a > 0$, $a \geq 1$.

Chapter Fifteen
Angles

15.01 Introduction

The word 'angle' is used in many ways. In this chapter we shall use the following definition:

> Definition 15-1 An angle is a set of points which is the union of two noncollinear rays with the same vertex.

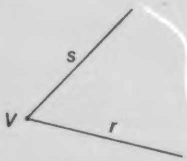


Fig. 15-1

So, for example, the set $r \cup s$ pictured in Figure 15-1 is an angle. The point V which is the common vertex of the rays r and s is called *the vertex of* $r \cup s$. Each of the two half-lines whose vertex is V and whose sense is that of r or s is called *a side of* $r \cup s$. [To justify speaking of *the* vertex of $r \cup s$ one should show that $r \cup s$ is not the union of two other rays with a different vertex. This is easy to prove but, since it is intuitively obvious we shall not do so.]

When $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are noncollinear rays it is convenient to use '$\angle AVB$' [read this as: angle A, V, B] to refer to the angle $\overrightarrow{VA} \cup \overrightarrow{VB}$. As in the case of '$\triangle AVB$', use of the symbol '$\angle AVB$' is justified only when we have proved—or are assuming—that $\{A, V, B\}$ is noncollinear. Note that in view of the remark at the end of the preceding paragraph, $\angle DEF = \angle ABC$ if and only if $E = B$ and each of D and F belongs to a different one of the sides, $\overrightarrow{BA}$ and $\overrightarrow{BC}$, of $\angle ABC$. [Explain.]

According to the notion of angles introduced here, an angle is a set of points. In a later chapter we shall introduce a notion of sensed angles according to which a sensed angle is an ordered pair of rays — collinear or not — with the same vertex. Both these kinds of angle have their uses. The word 'angle' is also used — but not in this text — where we would use 'measure of [the] angle', and is sometimes used for either the set of points which is the interior — in our sense — of an angle or for the set of all rays which have the vertex of one of our angles and are [except for their vertex] interior to it. Each of these notions has its own advantages, but it is not necessary to deal with all of them in the same course.

The reason for requiring that the rays whose union is an angle be noncollinear is to make it possible to speak unambiguously of the vertex and the sides of an angle. A union of two collinear rays with the same vertex is a straight line and any point of the line has an equal right to be called a vertex of such a "straight angle".

The proof that an angle has a unique vertex and a unique pair of sides is easy once one has shown that if r and s are noncollinear rays with the same vertex then any ray $t \subseteq r \cup s$ is a subset of r or of s. To see this note that since a ray t contains more than two points then [if $t \subseteq r \cup s$] it must be the case either that at least two points of t are in r or at least two points of t are in s. It follows that t is contained either in the line containing r or in the line containing s. In the former case t can contain no point of s other than the common vertex of r and s. [This is because r and s are noncollinear.] In the latter case t can contain no point of r other than the common vertex. So, in any case, $t \subseteq r$ or $t \subseteq s$.

To show that an angle has a unique vertex and a unique pair of sides, suppose that $r \cup s = t \cup u$ where t and u, like r and s, are noncollinear rays with a common vertex. By the result of the preceding paragraph, each of t and u is a subset of one of the rays r and s and each of r and s is a subset of t or of u. Canvassing the possibilities for equality, one easily sees that either $u = r$ and $t = s$ or $u = s$ and $t = r$.

Each angle is contained in a unique plane [Why?] The plane is called *the plane of the angle.* The lines containing the sides of an angle also contain the sides of three other angles which have the same vertex as the given angle and are contained in the same plane. For example, the opposites of the half-lines which are the sides of a given angle are the sides of another angle. Two such angles are called *vertical angles.* Either may be described as being the other's vertical angle. In Figure 15–2, $\angle AVB$ and $\angle CVD$ are vertical angles. Name two other vertical angles in Figure 15–2.

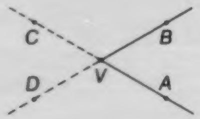


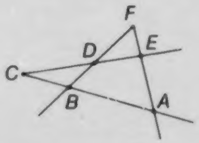
Fig. 15–2

When two angles share a side, and their other sides are opposite half-lines, the angles are called *adjacent supplementary angles.* Either may be described as being an adjacent supplement of the other. In Figure 15–2, $\angle AVB$ and $\angle CVB$ are adjacent supplementary angles. Does $\angle AVB$ have another adjacent supplement? What are the adjacent supplements of $\angle CVD$?

Exercises

Part A

The picture at the right shows two angles, with vertices C and F, whose sides intersect at points A, B, D, and E. It also shows a number of other angles. Since the figure shows only one angle with vertex C we may, in discussing this figure, refer to that angle as '$\angle C$'.



1. $\angle C = \angle BCE$. Use the figure to obtain seven other three letter names for $\angle C$.
2. Give the four angles which have D as vertex. Which of these are vertical angles? Which are adjacent supplementary angles?
3. Give two adjacent supplementary angles neither of which has D as its vertex.

*

Answer to 'Why?': Each angle is contained in the union of some two intersecting lines, and each two intersecting lines is contained in exactly one plane.

*

We introduce the notion of adjacent supplementary angles at this point in order to have something to talk about. Later in this chapter we define 'adjacent angles' and 'supplementary angles' and show that what we are here calling adjacent supplementary angles are, indeed, just those which are both adjacent and supplementary. [See page 220.]

*

Answers to questions: Another adjacent supplement of $\angle AVB$ is $\angle DVA$. The adjacent supplements of $\angle CVD$ are $\angle CVB$ and $\angle CVA$.

*

Parts A and B may be used either in class or as homework.

Answers for Part A

1. $\angle BCD$, $\angle ACD$, $\angle ACE$, $\angle ECA$, $\angle DCA$, $\angle DCB$, $\angle ECB$
2. $\angle CDB$, $\angle CDF$, $\angle BDE$, and $\angle EDF$ [Since each of the four angles has two three-letter names, there are other correct answers.]; ($\angle BDC$, $\angle EDF$) and ($\angle CDF$, $\angle BDE$) are pairs of vertical angles.; ($\angle CDF$, $\angle EDF$), ($\angle EDF$, $\angle BDE$), ($\angle EDB$, $\angle CDB$), and ($\angle BDC$, $\angle CDF$) are pairs of adjacent angles.
3. $\angle FED$ and $\angle AED$ [or: $\angle CBD$ and $\angle ABD$]

Any angle determines two regions in its plane—the *interior* of the angle in question and the *exterior* of this angle. These are introduced in:

Definition 15-2

(a) C is interior to $\angle AVB$ if and only if there exist points X and Y on $\overrightarrow{VA}$ and $\overrightarrow{VB}$, respectively, such that $C - V = (X - V) + (Y - V)$.

(b) C is exterior to $\angle AVB$ if and only if $C \in \overline{A\overset{\leftrightarrow}{V}B}$ but belongs neither to $\angle AVB$ nor to the interior of $\angle AVB$.

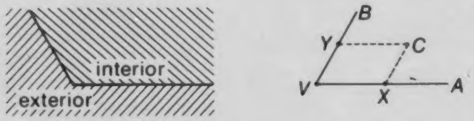


Fig. 15-3

Part B

In these exercises we shall discuss questions concerning the interior of $\angle AVB$. As the definition suggests, it will be natural to use position vectors with respect to V. We shall follow our usual convention in such cases:

$$\vec{a} = A - V, \vec{b} = B - V, \vec{c} = C - V, \ldots, \vec{p} = P - V, \ldots.$$

Recall that $C \in \overline{A\overset{\leftrightarrow}{V}B}$ if and only if there are numbers x and y such that $\vec{c} = \vec{a}x + \vec{b}y$. [Explain.] Show each of the following.

1. C is interior to $\angle AVB$ if and only if $\vec{c} = \vec{a}a + \vec{b}b$ where $a > 0$ and $b > 0$.
2. If C is interior to $\angle AVB$ then so is each point of $\overrightarrow{VC}$.
3. Each point of $\overline{AB}$ is interior to $\angle AVB$. [*Hint*: Recall that $P \in \overline{AB}$ if and only if $\vec{p} = \vec{a}(1 - r) + \vec{b}r$, where $0 < r < 1$.]
4. If both C and D are interior to $\angle AVB$ then so is each point of $\overline{CD}$.
5. C is interior to $\angle AVB$ if and only if $\overline{AB}$ intersects $\overrightarrow{VC}$. [*Hint*: Your work in Exercises 2 and 3 should have reminded you that there is a point of $\overline{AB}$ which belongs to $\overrightarrow{VC}$ if and only if there are numbers —say, r and s—such that $\vec{a}(1 - r) + \vec{b}r = \vec{c}s$, $0 < r < 1$, and $s > 0$. Given such numbers r and s, show how to find numbers a and b satisfying the condition in Exercise 1. And, given such numbers a and b, show how to find corresponding numbers r and s.]

*

A perhaps more familiar definition of 'interior of $\angle$AVB' is that C is interior to $\angle$AVB if it is on the same side of $\overleftrightarrow{VA}$ as is B and is on the same side of $\overleftrightarrow{VB}$ as is A. That our definition is formally equivalent to this will follow when, later in this chapter, we study "sides" of lines. [See Exercise 4 of Part C on page 212.]

*

Answers for Part B

[Explanation: $\overline{AVB} = \{X: \exists_x \exists_y\ X - V = (A - V)x + (B - V)y\}$ Note that we are essentially introducing cartesian coordinates in $\overline{AVB}$. Exercise 1 points out that the interior of $\angle$AVB consists of the points which are in the "first quadrant" with respect to this coordinate system.]

1. For each X and Y, X $\in \overrightarrow{VA}$ and Y $\in \overrightarrow{VB}$ if and only if there exist positive numbers x and y such that X - V = (A - V)x and Y - V = (B - V)y. Comparing with Definition 15-2(a) and introducing the notation in the preamble to these exercises, we see that C is interior to $\angle$AVB if and only if there are positive numbers — say, a and b — such that $\vec{c} = \vec{a}a + \vec{b}b$.
2. D $\in \overrightarrow{VC}$ if and only if there is a number — say, d — such that d > 0 and D - V = (C - V)d. If C is interior to $\angle$AVB then C - V = (A - V)a + (B - V)b where a > 0 and b > 0. In this case, D - V = (A - V)(ad) + (B - V)(bd) where, since a, b, and d are positive, ad > 0 and bd > 0. So, D is interior to $\angle$AVB. Hence, if C belongs to the interior of $\angle$AVB then $\overrightarrow{VC}$ is a subset of the interior of $\angle$AVB.

Answers for Part B [cont.]

3. $P \in \overline{AB}$ if and only if there is a number — say, r — such that $0 < r < 1$ and $P = A + (B - A)r$. Since, in this case, and only then, $P - V = (A - V) + [(B - V) - (A - V)]r = (A - V)(1 - r) + (B - V)r$ it follows that $P \in \overline{AB}$ if and only if $\vec{p} = \vec{a}(1 - r) + \vec{b}r$, where $0 < r < 1$. All that remains to showing that $\overline{AB}$ is contained in the interior of $\angle AVB$ is to note that, for $0 < r < 1$, $1 - r > 0$ and $r > 0$. Then, apply Exercise 1.

4. If C and D are both interior to $\angle AVB$, then there exist positive numbers c_1, c_2, d_1, and d_2 such that $\vec{c} = \vec{a}c_1 + \vec{b}c_2$ and $\vec{d} = \vec{a}d_1 + \vec{b}d_2$. Let $P \in \overline{CD}$. Then $\vec{p} = \vec{c}(1 - r) + \vec{d}r$, where $0 < r < 1$. But $\vec{c}(1 - r) + \vec{d}r = \vec{a}[c_1(1 - r) + d_1r] + \vec{b}[c_2(1 - r) + d_2r]$. Since the coefficients of $\vec{a}$ and $\vec{b}$ are positive, it follows that P is interior to $\angle AVB$. [The result just obtained may be reformulated as 'The interior of $\angle AVB$ is a convèx set of points.' Our definition of 'convex' in connection with quadrilaterals (intersecting diagonals) is framed in such a way that a quadrilateral is convex if and only if its "interior" — a word we have not defined in connection with quadrilaterals — is a convex set.]

5. Suppose that C is interior to $\angle AVB$. Then there are positive numbers a and b such that $\vec{c} = \vec{a}a + \vec{b}b$. Let $\vec{p} = \vec{c}/(a + b)$. Then $P \in \overrightarrow{VC}$ and since $\vec{p} = \vec{a}[a/(a + b)] + \vec{b}[b/(a + b)]$, $P \in \overline{AB}$. [Let $r = b/(a + b)$.] Conversely, suppose that $\overline{AB}$ intersects $\overrightarrow{VC}$. Let $P \in \overline{AB} \cap \overrightarrow{VC}$. Then there are positive numbers c and r, $r < 1$, such that $\vec{p} = \vec{a}(1 - r) + \vec{b}r$ and $\vec{p} = \vec{c}c$. Therefore, $\vec{c} = \vec{a}((1 - r)/c) + \vec{b}(r/c)$. Since the coefficients of $\vec{a}$ and $\vec{b}$ are positive, it follows that C is interior to $\angle AVB$.

TC 209

Suggestions for use of the exercises of section 15.02:
(i) Parts A and B may be used as homework.
(ii) Parts C and D should be teacher directed.
(iii) Part E may be used as homework.

The results of the preceding exercises are conveniently summarized in two theorems:

Theorem 15-1 Each segment whose endpoints are interior to an angle is a subset of the interior of that angle.

Theorem 15-2
(a) Each point of $\overline{AB}$ is interior to $\angle AVB$.
(b) If C is interior to $\angle AVB$ then so is each point of $\overrightarrow{VC}$.
(c) C is interior to $\angle AVB$ if and only if $\overline{AB}$ intersects $\overrightarrow{VC}$.

[Note that in part (c), since $\{A, V, B\}$ is noncollinear, $\overline{AB} \cap \overrightarrow{VC}$ consists of a single point.] Theorem 15-2 has the following corollary:

Corollary If, in $\triangle ABC$, $D \in \overline{BC}$ and $E \in \overline{CA}$ then $\overline{AD}$ and $\overline{BE}$ intersect.

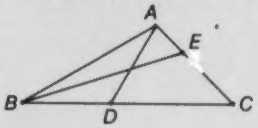


Fig. 15-4

15.02 Sides of Lines, and Adjacent Angles

Theorem 15-2(c) suggests investigating what can happen if $\overline{AB}$ intersects the line $\overleftrightarrow{VC}$. The first two pictures in Figure 15-5 show what may happen if $\{A, V, B\}$ is noncollinear. The third illustrates the case in which $\{A, V, B\}$ is collinear. This third case should be familiar; how would you describe $\angle AVC$ and $\angle BVC$? In all three cases $\angle AVC$ and $\angle BVC$ are called *adjacent angles*.

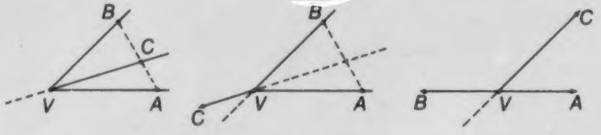


Fig. 15-5

Before giving a formal definition of 'adjacent angles' it is worthwhile to forget about the angles for a moment and consider mainly the points A and B and the line $\overleftrightarrow{VC}$. It seems intuitively reasonable to interpret the fact that $\overline{AB}$ intersects $\overleftrightarrow{VC}$ by saying that A and B *are on opposite sides of* $\overleftrightarrow{VC}$. More generally, given any line l, if P and Q are points which are not on l, but which belong to a plane π containing l, we shall say that P and Q are on opposite sides of l if and only if $\overline{PQ}$ intersects l.

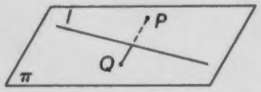


Fig. 15-6

And, we shall say that such points P and Q are on *the same side of* l if and only if $\overline{PQ}$ does not intersect l. Note that two points may be neither on the same side of l nor on opposite sides of l. For either to be the case it is necessary that both points belong to some plane which contains l and that neither point belongs to l. We shall adopt:

Definition 15-3
(a) P and Q are on opposite sides of l if and only if neither P nor Q belongs to l but $\overline{PQ} \cap l \neq \varnothing$.
(b) P and Q are on the same side of l if and only if P and Q are [together] coplanar with l but $\overset{\bullet\!\!\leftrightarrow\!\!\bullet}{PQ} \cap l = \varnothing$.

As a fairly immediate consequence of this definition we have:

Theorem 15-3 If $R \in l$, $P \notin l$, and $Q \in \overleftrightarrow{RP}$ then P and Q are on the same side of l if and only if $Q \in \overrightarrow{RP}$, and are on opposite sides of l if and only if Q belongs to the opposite of $\overrightarrow{RP}$.

We can now define 'adjacent angles':

Definition 15-4 Two angles are adjacent if and only if they have a common side and their other sides are on opposite sides of the line containing their common side.

And, we have:

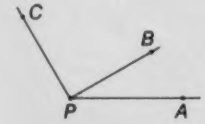
Theorem 15-4 $\angle AVC$ and $\angle BVC$ are adjacent if and only if $\overline{AB}$ intersects $\overleftrightarrow{VC}$.

Exercises

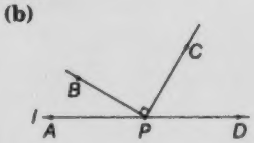
Part A

1. Suppose that $l \subseteq \pi$ and that A, B, and C are noncollinear points of π such that both A and B, and A and C, are on opposite sides of l.
   (a) Draw a picture for these conditions.
   (b) Show that there is a point in $l \cap \overline{AB}$. Is there more than one such point?
   (c) Is there a point in $l \cap \overline{AC}$? In $l \cap \overset{\bullet\!\!\leftrightarrow\!\!\bullet}{AC}$? Explain.
   (d) Is there a point in $l \cap \overline{BC}$? In $l \cap \overset{\bullet\!\!\leftrightarrow\!\!\bullet}{BC}$? Explain.
2. Given $l \subseteq \pi$ and noncollinear points A, B, and C of π, none of which belongs to l. In each of the following, draw a picture for the conditions given and answer the question.
   (a) Assume that neither $\overline{AB}$ nor $\overline{BC}$ intersects l. Are A and C on the same side or on opposite sides of l?
   (b) Assume that $\overline{AB}$ intersects l and that $\overline{BC}$ does not intersect l. Are A and C on the same side or on opposite sides of l?
3. In each of the following, you are given pictures of coplanar rays with a common vertex. In each case, give at least one pair of adjacent angles with that vertex, and at least one pair of angles with that vertex which are not adjacent.

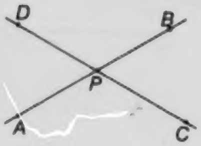
(a)



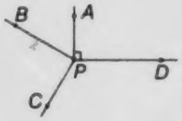
(b)



(c)



(d)



Part B

Prove each of the following.

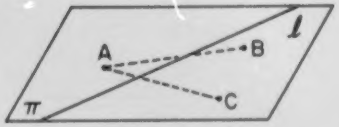
1. The corollary to Theorem 15-2
2. Theorem 15-3
3. Theorem 15-4

Part C

It will be to our advantage to obtain an algebraic criterion for determining when two points are on the same or opposite sides of a given line. We shall obtain one such criterion by re-examining the situation discussed in Part B of page 208. There we dealt with an

Answers for Part A

1. (a)

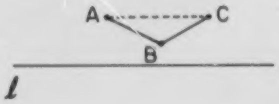


(b) By Definition 15-3(a), $\overline{AB} \cap \ell \neq \emptyset$. Since $A \notin \ell$, $\overleftrightarrow{AB} \neq \ell$. So, $\ell \cap \overline{AB}$ consists of a single point.

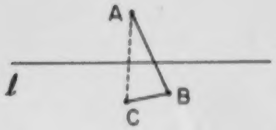
(c) Yes, $\ell \cap \overline{AC} \neq \emptyset$ by Definition 15-3(a).; $\ell \cap \overleftrightarrow{AC} \neq \emptyset$, since $\overline{AC} \subset \overleftrightarrow{AC}$.

(d) No.; No. B and C are on the same side of ℓ, so $\ell \cap \overleftrightarrow{BC} = \emptyset$. $\ell \cap \overline{BC} = \emptyset$ because $\overline{BC} \subseteq \overleftrightarrow{BC}$. [It is intuitively obvious that, when A and B are on opposite sides of ℓ and A and C are on opposite sides of ℓ, B and C are on the same side of ℓ. This, and similar theorems pointed out in Exercise 2 can be proved on the basis of our postulates and definitions. We shall not, however, take up these proofs.]

2. (a) same side



(b) opposite sides



3. (a) $\angle APB$ and $\angle BPC$ are adjacent; $\angle APC$ and $\angle BPC$ are not.

(b) $\angle DPC$ and $\angle CPA$ are adjacent; $\angle DPB$ and $\angle CPB$ are not.

(c) $\angle CPB$ and $\angle BPD$ are adjacent; $\angle CPA$ and $\angle BPD$ are not.

(d) $\angle DPB$ and $\angle BPC$ are adjacent; $\angle CPD$ and $\angle BPA$ are not.

[There is more than one choice for most of the preceding answers.]

Answers for Part B

1. E is interior to $\angle ABC$ since $E \in \overline{CA}$. Since $\angle ABC = \angle ABD$, it follows that $\overline{AD}$ intersects $\overrightarrow{BE}$ [Theorem 15-2(c)]. Similarly, D is interior to $\angle BAC$ since $D \in \overline{BC}$. Since $\angle BAC = \angle BAE$, it follows that $\overrightarrow{AD}$ intersects $\overline{BE}$. Since $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ are different intersecting lines, they intersect at only one point, so that point must be in $\overline{AD} \cap \overline{BE}$.

2. Suppose that $R \in \ell$, $P \notin \ell$, and $Q \in \overleftrightarrow{RP}$, so that ℓ, P, and Q are coplanar, and $\overleftrightarrow{RP} \cap \ell = \{R\}$. Then, Q belongs to the opposite of $\overrightarrow{RP}$ if and only if $R \in \overline{QP}$ — that is, if and only if $\overline{QP}$ intersects ℓ. So, $Q \in -\overrightarrow{RP}$ if and only if P and Q are on opposite sides of ℓ. Q belongs to $\overrightarrow{RP}$ if and only if $R \notin \overline{QP}$ and $Q \neq R$ — that is, since $P \notin \ell$, if and only if $\overleftrightarrow{PQ} \cap \ell = \emptyset$. Hence, $Q \in \overrightarrow{RP}$ if and only if P and Q are on the same side of ℓ.

3. Suppose that $\angle AVC$ and $\angle BVC$ are adjacent. Since their common side is $\overrightarrow{VC}$, their other sides are on opposite sides of $\overleftrightarrow{VC}$. In particular, A and B are on opposite sides of $\overleftrightarrow{VC}$, so that $\overline{AB}$ intersects $\overleftrightarrow{VC}$. Conversely, suppose that $\overline{AB}$ intersects $\overleftrightarrow{VC}$ — that is, suppose that A and B are on opposite sides of $\overleftrightarrow{VC}$. It follows by Theorem 15-3 that all points of $\overrightarrow{VA}$ are on the same side of $\overleftrightarrow{VC}$ as is A and all points of $\overrightarrow{VB}$ are on the same side of $\overleftrightarrow{VC}$ as is B. So, $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are on opposite sides of $\overleftrightarrow{VC}$ — that is, $\angle AVC$ and $\angle BVC$ are adjacent.

The results obtained in Exercises 1 - 3 of Part C are intuitively reasonable. Thinking of the coordinate system based on V and $[\vec{a}, \vec{b}]$, Exercises 1 and 2 say that C is on the opposite side of $\overleftrightarrow{VA}$ from B if and only if its $\vec{b}$-coordinate is negative. Exercise 3 says that C is on the same side of $\overleftrightarrow{VA}$ with B if and only if its $\vec{b}$-coordinate is positive.

angle, $\angle AVB$, and discovered a means for telling whether or not a point C is interior to this angle. Recall that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are the position vectors of A, B, and C with respect to V. Show each of the following.

1. If B and C are on opposite sides of $\overleftrightarrow{VA}$ then $\vec{c} = \vec{a}a + \vec{b}b$, where $b < 0$. [*Hint*: As in Exercise 5 on page 208, there is a point common to $\overline{BC}$ and $\overleftrightarrow{VA}$ if and only if there are numbers – say, r and s – such that $\vec{a}s = \vec{b}(1 - r) + \vec{c}r$ and $0 < r < 1$.]
2. If $\vec{c} = \vec{a}a + \vec{b}b$ with $b < 0$ then B and C are on opposite sides of $\overleftrightarrow{VA}$. [*Hint*: Given a and given $b < 0$, find numbers r and s such that $\vec{a}s = \vec{b}(1 - r) + \vec{c}r$ and $0 < r < 1$.]
3. B and C are on the same side of $\overleftrightarrow{VA}$ if and only if $\vec{c} = \vec{a}a + \vec{b}b$ where $b > 0$. [*Hint*: See Definition 15–3.]
4. C is interior to $\angle AVB$ if and only if B and C are on the same side of $\overleftrightarrow{VA}$ and A and C are on the same side of $\overleftrightarrow{VB}$.

Part D

Continuing with the situation dealt with in Part C, let $\vec{u}$ be the unit vector in the sense of $\overrightarrow{VA}$. Assume that $\vec{c} = \vec{a}a + \vec{b}b$.

1. Show that $\vec{c} = \vec{u}u + \vec{b}b$, where $u = a\|\vec{a}\|$.
2. Show that $\vec{c} - \vec{u}(\vec{c} \cdot \vec{u}) = [\vec{b} - \vec{u}(\vec{b} \cdot \vec{u})]b$. [*Hint*: Use Exercise 1 to compute '$\vec{c} \cdot \vec{u}$'.]
3. Conclude that B and C are on the same or opposite sides of $\overleftrightarrow{VA}$ according as the vectors $\vec{b} - \vec{u}(\vec{b} \cdot \vec{u})$ and $\vec{c} - \vec{u}(\vec{c} \cdot \vec{u})$ have the same or opposite sense.
4. Give geometrical interpretations of the vectors referred to in Exercise 3. [*Hint*: How might you describe the point $B - [\vec{b} - \vec{u}(\vec{b} \cdot \vec{u})]$ with reference to B and $\overleftrightarrow{VA}$?]
5. Compute the norms of the vectors referred to in Exercise 3.
6. Use the results of Exercises 2, 3, and 5 to calculate the value of 'b' in case B and C are on the same side of $\overleftrightarrow{VA}$. In case B and C are on opposite sides of $\overleftrightarrow{VA}$.
7. Show that C is interior to $\angle AVB$ if and only if B and C are on the same side of $\overleftrightarrow{VA}$ and

$$\vec{c} \cdot \vec{u} > \frac{\vec{b} \cdot \vec{u}\sqrt{\|\vec{c}\|^2 - (\vec{c} \cdot \vec{u})^2}}{\sqrt{\|\vec{b}\|^2 - (\vec{b} \cdot \vec{u})^2}}.$$

[*Hint*: By Exercise 2, $\vec{c} = \vec{u}[(\vec{c} \cdot \vec{u}) - (\vec{b} \cdot \vec{u})b] + \vec{b}b$. Now, use the result of Exercise 6.]

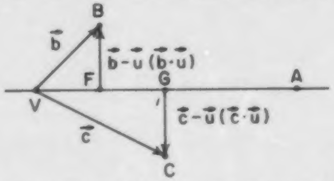
Answers for Part C

[As indicated in the preamble, we are dealing with $\angle AVB$ — in particular, $A \neq V$ and $B \notin \overleftrightarrow{VA}$. So, with $\vec{c} = \vec{a}a + \vec{b}b$, $C \in \overleftrightarrow{VA}$ if and only if $b = 0$. Note, for future applications, that $A \neq V$ and $B \notin \overleftrightarrow{VA}$ if and only if $(\vec{a}, \vec{b})$ is linearly independent.]

1. Suppose that B and C are on opposite sides of $\overleftrightarrow{VA}$. Let $\overline{BC} \cap \overleftrightarrow{VA} = \{P\}$. Then $\vec{p} = \vec{a}s$ for some $s > 0$, and $\vec{p} = \vec{b}(1 - r) + \vec{c}r$, where $0 < r < 1$. Therefore, $\vec{a}s = \vec{b}(1 - r) + \vec{c}r$ — that is, $\vec{c} = \vec{a}(s/r) + \vec{b}[(r - 1)/r]$. Since $0 < r < 1$, $(r - 1)/r < 0$.
2. Suppose that $\vec{c} = \vec{a}a + \vec{b}b$, where $b < 0$. Since, by the preamble to these exercises, $(\vec{a}, \vec{b})$ is linearly independent, it follows that $B \notin \overleftrightarrow{VA}$ and, since $b \neq 0$, that $C \notin \overleftrightarrow{VA}$. So, to show that B and C are on opposite sides of $\overleftrightarrow{VA}$ it is sufficient to find a point P which belongs to $\overline{BC} \cap \overleftrightarrow{VA}$. This amounts to finding numbers s and r, with $0 < r < 1$, such that $\vec{a}s = \vec{b}(1 - r) + \vec{c}r$. It follows from our assumption concerning $\vec{c}$ that $\vec{a}a = \vec{c} + \vec{b} \cdot -b$ and, since $1 - b \neq 0$, that $\vec{a}[a/(1 - b)] = \vec{c} \cdot /(1 - b) + \vec{b}[-b/(1 - b)]$. Taking $r = /(1 - b)$, it follows that $1 - r = -b/(1 - b) > 0$.
3. By Definition 15-3, B and C are on the same side of $\overleftrightarrow{VA}$ if and only if they are coplanar with $\overleftrightarrow{VA}$, neither belongs to $\overleftrightarrow{VA}$, and they are not on opposite sides of $\overleftrightarrow{VA}$. Since, by assumption in these exercises, $B \notin \overleftrightarrow{VA}$, it follows that B and C are on the same side of $\overleftrightarrow{VA}$ if and only if $\vec{c} = \vec{a}a + \vec{b}b$, where $b \neq 0$ [to make sure that $C \notin \overleftrightarrow{VA}$] and, by Exercises 1 and 2, $b \not< 0$ [to make sure that C and B are not on opposite sides of $\overleftrightarrow{VA}$]. Hence, B and C are on the same side of $\overleftrightarrow{VA}$ if and only if $\vec{c} = \vec{a}a + \vec{b}b$ with $b > 0$.
4. B and C are on the same side of $\overleftrightarrow{VA}$ if and only if $\vec{c} = \vec{a}a + \vec{b}b$, where $b > 0$. Similarly, A and C are on the same side of $\overleftrightarrow{VB}$ if and only if $\vec{c} = \vec{a}a + \vec{b}b$, where $a > 0$. [The tacit assumption, in each application of Exercise 3 is that $\{A, V, B\}$ is noncollinear.] Compare, now, with Exercise 1 of Part B on page 208.

Answers for Part D

1. $\vec{u} = \vec{a}/||\vec{a}||$ and, so, $\vec{a}a = \vec{u}(a||\vec{a}||)$.
2. By Exercise 1, $\vec{c}\cdot\vec{u} = (\vec{u}u + \vec{b}b)\cdot\vec{u} = u + (\vec{b}\cdot\vec{u})b$. So, $\vec{c} - \vec{u}(\vec{c}\cdot\vec{u})$
$= \vec{c} - \vec{u}[u + (\vec{b}\cdot\vec{u})b] = (\vec{c} - \vec{u}u) - \vec{u}(\vec{b}\cdot\vec{u})b = \vec{b}b - \vec{u}(\vec{b}\cdot\vec{u})b$
$= [\vec{b} - \vec{u}(\vec{b}\cdot\vec{u})]b$.
3. By Part C, since $\vec{c} = \vec{a}a + \vec{b}b$, B and C are on the same side or on opposite sides of $\overleftrightarrow{VA}$ according as $b > 0$ or $b < 0$. By Exercise 2, $\vec{b} - \vec{u}(\vec{b}\cdot\vec{u})$ and $\vec{c} - \vec{u}(\vec{c}\cdot\vec{u})$ have the same sense or opposite senses according as $b > 0$ or $b < 0$. [Note that this criterion for points to be on the same side or on opposite sides of a line is entirely analogous to the criterion given in Theorem 14-17 for points to be on the same side or on opposite sides of a plane.]
4. The point $B - [\vec{b} - \vec{u}(\vec{b}\cdot\vec{u})]$ is the foot, F, of the perpendicular from B to $\overleftrightarrow{VA}$. So, $\vec{b} - \vec{u}(\vec{b}\cdot\vec{u})$ is the translation from F to B. Similarly, $\vec{c} - \vec{u}(\vec{c}\cdot\vec{u})$ is the translation from G to C, where G is the foot of the perpendicular from C to $\overleftrightarrow{VA}$. According to Exercise 3, B and C are on the same side of $\overleftrightarrow{VA}$ if and only if these translations have the same sense.


B
$\vec{b}$
$\vec{b} - \vec{u}(\vec{b}\cdot\vec{u})$
F
G
A
V
$\vec{c} - \vec{u}(\vec{c}\cdot\vec{u})$
$\vec{c}$
C

[Note that the sense of each of these translations is a subset of $[\overleftrightarrow{VA}]^{\perp}$ and that each sense which is a subset of this bidirection "determines" a side of $\overleftrightarrow{VA}$.]

5. $\sqrt{||\vec{b}||^2 - (\vec{b}\cdot\vec{u})^2}$, $\sqrt{||\vec{c}||^2 - (\vec{c}\cdot\vec{u})^2}$
6. From Exercise 2, $|b|$ is the ratio of the norms computed in Exercise 5. By Exercise 3, $b = \sqrt{||\vec{c}||^2 - (\vec{c}\cdot\vec{u})^2}/\sqrt{||\vec{b}||^2 - (\vec{b}\cdot\vec{u})^2}$ in case B and C are on the same side of $\overleftrightarrow{VA}$ and b has the opposite value in case B and C are on opposite sides of $\overleftrightarrow{VA}$.
7. With $\vec{c} = \vec{a}a + \vec{b}b$, C is interior to $\angle AVB$ if and only if $a > 0$ and $b > 0$. Now, $b > 0$ if and only if B and C are on the same side of $\overleftrightarrow{VA}$ and, by Exercise 6, this is the case if and only if $b = \sqrt{||\vec{c}||^2 - (\vec{c}\cdot\vec{u})^2}/\sqrt{||\vec{b}||^2 - (\vec{b}\cdot\vec{u})^2}$. By the hint it follows that with this value for 'b', $a > 0$ if and only if $\vec{c}\cdot\vec{u} > \dfrac{\vec{b}\cdot\vec{u}\sqrt{||\vec{c}||^2 - (\vec{c}\cdot\vec{u})^2}}{\sqrt{||\vec{b}||^2 - (\vec{b}\cdot\vec{u})^2}}$.

TC 213

8. (a) By the hint, $\sqrt{||\vec{c}||^2 - (\vec{c}\cdot\vec{u})^2} = ||\vec{c}||\sqrt{1 - (\vec{w}\cdot\vec{u})^2}$ and $\sqrt{||\vec{b}||^2 - (\vec{b}\cdot\vec{u})^2} = ||\vec{b}||\sqrt{1 - (\vec{v}\cdot\vec{u})^2}$. Since $\vec{w} = \vec{c}/||\vec{c}||$ and $\vec{v} = \vec{b}/||\vec{b}||$, the desired result is readily obtained from Exercise 7.
   (b) [The hint tells all.]

8. Let $\vec{v}$ and $\vec{w}$ be the unit vectors in the senses of $B - V$ and $C - V$, and suppose that $C \notin \overleftrightarrow{VA}$. Show that

   (a) C is interior to $\angle AVB$ if and only if B and C are on the same side of $\overleftrightarrow{VA}$ and $\dfrac{\vec{w}\cdot\vec{u}}{\sqrt{1 - (\vec{w}\cdot\vec{u})^2}} > \dfrac{\vec{v}\cdot\vec{u}}{\sqrt{1 - (\vec{v}\cdot\vec{u})^2}}$.

   (b) $C \in \overrightarrow{VB}$ if and only if B and C are on the same side of $\overleftrightarrow{VA}$ and

$$\frac{\vec{w}\cdot\vec{u}}{\sqrt{1 - (\vec{w}\cdot\vec{u})^2}} = \frac{\vec{v}\cdot\vec{u}}{\sqrt{1 - (\vec{v}\cdot\vec{u})^2}}$$

[*Hint*: For part (a), note that $||\vec{c}||^2 - (\vec{c}\cdot\vec{u})^2 = ||\vec{c}||^2[1 - (\vec{w}\cdot\vec{u})^2]$. Then use Exercise 7. For part (b) note that, by the hint for Exercise 7, $\vec{c} \in [\vec{b}]^+$ if and only if $\vec{c}\cdot\vec{u} = (\vec{b}\cdot\vec{u})b$, where $b > 0$. Then proceed as in part (a).]

*

In Exercise 3 of Part C you showed that, given a line l, a point $V \in l$, and a point $B \notin l$, B and C are on the same side of l if and only if $C - V$ is the sum of a vector in the *direction* of l and a vector in the *sense* of $B - V$. In Exercises 3 and 4 of Part D you showed that if $B \notin l$, and F and G are the feet of the perpendicular from B and C to l, then B and C are on the same side of l if and only if $B - F$ and $C - G$ have the same sense.

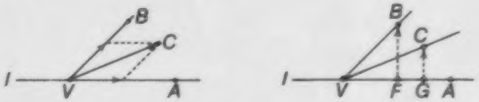


Fig. 15–7

In the following exercises you will see how to simplify the results obtained in Exercise 8. This will lead to a new interpretation of dot products of unit vectors which will be of fundamental importance in all our later work.

Part E

Consider the function f which is defined for all x such that $-1 < x < 1$ by:

$$f(x) = \frac{x}{\sqrt{1 - x^2}}$$

1. (a) Compute the value of f at $\frac{1}{2}$ and at $\frac{1}{3}$. Which value is greater?
   (b) Suppose that a and b are two arguments of f such that $a > b$. Make a conjecture concerning $f(a)$ and $f(b)$.
2. Suppose that $1 > a > b \geq 0$. Show that $a\sqrt{1 - b^2} > b\sqrt{1 - a^2}$. [*Hint*: You have handled problems like this in Exercise 8 on page 8.]
3. It follows from Exercise 2 that if a and b are arguments of f such that $a > b \geq 0$ then $f(a) > f(b)$. [Explain.] Suppose, now, that a and b are arguments of f such that $0 \geq a > b$. What can you conclude about $f(a)$ and $f(b)$? [*Hint*: If $0 \geq a > b > -1$ then $1 > -b > -a \geq 0$.]
4. You have shown that in two of three cases in which $a > b$, $f(a) > f(b)$. What is the third case? Is $f(a) > f(b)$ in this case also?
5. Show that, for any arguments a and b of f, if $f(a) > f(b)$ then $a > b$. [*Hint*: Prove the contrapositive.]

*

We can now use the results obtained in Part E to simplify those in Exercise 8 of Part D. These results are concerned with a line $\overleftrightarrow{VA}$ and points B and C which belong to a plane containing this line but do not belong to the line. The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are unit vectors in the senses of $A - V$, $B - V$, and $C - V$. From Exercise 8(a) and your work in Part E it follows that

(1) C is interior to $\angle AVB$ if and only if B and C are on the same side of $\overleftrightarrow{VA}$ and $\vec{w} \cdot \vec{u} > \vec{v} \cdot \vec{u}$

and, similarly, that

(2) B is interior to $\angle AVC$ if and only if B and C are on the same side of $\overleftrightarrow{VA}$ and $\vec{w} \cdot \vec{u} < \vec{v} \cdot \vec{u}$. [Explain.]

From Exercise 8(b) and your work in Part E it follows that

(3) $\angle AVC = \angle AVB$ if and only if B and C are on the same side of $\overleftrightarrow{VA}$ and $\vec{w} \cdot \vec{u} = \vec{v} \cdot \vec{u}$.

TC 214

Answers for Part E

1. (a) $f(1/2) = 1/\sqrt{3}$, $f(1/3) = 1/\sqrt{8}$; $f(1/2) > f(1/3)$
   (b) $f(a) > f(b)$
2. Suppose that $1 > a > b \geq 0$. It follows that $a^2 > b^2$ and that $0 < 1 - a^2 < 1 - b^2$. So, $\sqrt{1 - a^2} < \sqrt{1 - b^2}$ and, since $b < a$, $b\sqrt{1 - a^2} < a\sqrt{1 - b^2}$.
3. [If a and b are arguments of f such that $a > b \geq 0$ then $1 > a > b \geq 0$. By Exercise 2, if $1 > a > b \geq 0$ then $a/\sqrt{1 - a^2} > b/\sqrt{1 - b^2}$ and, so, $f(a) > f(b)$.] Suppose that if a and b are arguments of f such that $0 \geq a > b$ then $0 \geq a > b > -1$ and, so, $1 > -b > -a \geq 0$. So, by Exercise 2, $-a\sqrt{1 - (-b)^2} < -b\sqrt{1 - (-a)^2}$ and, hence, $b\sqrt{1 - a^2} < a\sqrt{1 - b^2}$. As in the preceding bracketed explanation it follows that $f(a) > f(b)$.
4. The third case is that in which $a > 0 > b$. In this case, $f(a) > 0 > f(b)$.
5. We have seen that, for any arguments a and b of f, if $a > b$ then $f(a) > f(b)$. It follows that if $f(a) \not> f(b)$ then $a \not> b$ — that is, if $f(a) \leq f(b)$ then $a \leq b$. So, interchanging 'a' and 'b' it follows that if $f(a) > f(b)$ then $a \geq b$. [Note that if $f(a) > f(b)$ then $f(a) \geq f(b)$.] However, if $a = b$ then $f(a) = f(b)$ and, so, $f(a) \not> f(b)$. Hence, if $f(a) > f(b)$ then $a > b$.

   [Note that we now know that, for any arguments a and b of f, $f(a) > f(b)$ if and only if $a > b$.]

*

Results (1) and (3) follow from (a) and (b) of Exercise 8 of Part D by virtue of the results of Part E. [See the immediately preceding bracketed remark.] The result (2) is merely a restatement of (1) obtained by interchanging 'B' and 'C', and '$\vec{v}$' and '$\vec{w}$'.

Theorem 15-5 follows from (1) - (3) because, in any case, $\vec{w} \cdot \vec{u}$ is greater than, equal to, or less than $\vec{v} \cdot \vec{u}$.

TC 215 (1)

Explanation called for in the text: Theorem 15-5 follows directly from (1) - (3) and the fact that, for the numbers $\vec{w} \cdot \vec{u}$ and $\vec{v} \cdot \vec{u}$, it must be the case that either $\vec{w} \cdot \vec{u} > \vec{v} \cdot \vec{u}$ or $\vec{w} \cdot \vec{u} = \vec{v} \cdot \vec{u}$ or $\vec{w} \cdot \vec{u} < \vec{v} \cdot \vec{u}$.

*

In case B and C are on the same side of $\overleftrightarrow{VA}$ it is natural to say that $\angle AVC$ is smaller than $\angle AVB$ if and only if C is interior to $\angle AVB$. By (1) on page 214, the latter is the case if and only if $\vec{w} \cdot \vec{u}$ is greater than $\vec{v} \cdot \vec{u}$. So, $\vec{w} \cdot \vec{u}$ appears to give information as to the size of $\angle AVC$. The greater the number $\vec{w} \cdot \vec{u}$ is, the smaller is $\angle AVC$. [It is also almost obvious that congruent angles — which should be considered as having the same size — have the same cosine.]

One consequence of (1)–(3) which is worth recording is:

> **Theorem 15–5** If B and C are on the same side of $\overleftrightarrow{VA}$ then either C is interior to $\angle AVB$, or $C \in \overrightarrow{VB}$, or B is interior to $\angle AVC$.

[Explain.]

15.03 Cosines of Angles

The results of (1)–(3) also suggest that the dot product of unit vectors in the senses of the sides of an angle can be used to give us an idea of the "size" of the angle. [Explain.] So, it may well be worthwhile to introduce a word to use in referring to such dot products. The usual word is *cosine*, which we introduce in:

> **Definition 15–5** The cosine of an angle is the dot product of the unit vectors in the senses of the sides of the angle.

What is the cosine of an angle whose sides are contained in perpendicular lines? What can you say about any angle whose cosine is 0? What theorem tells you that the absolute value of the cosine of any angle is less than 1?

It is customary to abbreviate the phrase 'the cosine of' to 'cos'. So, for example:

> **Theorem 15–6** $\cos \angle ABC = (\vec{a} \cdot \vec{b})/(\|\vec{a}\|\,\|\vec{b}\|)$, where $\vec{a} = A - C$ and $\vec{b} = B - C$.

Exercises

Part A

By an earlier definition, $\vec{v} \cdot \vec{u} = \text{comp}_{\vec{u}}(\vec{v})$, for any unit vectors $\vec{u}$ and $\vec{v}$. So, $\cos \angle ACB = \text{comp}_{\vec{u}}(\vec{v})$, where $\vec{u}$ and $\vec{v}$ are the unit vectors in the senses of $A - C$ and $B - C$. Using this, we can link up the notion of the cosine of an angle with notions concerning orthogonal projections. Show that:

1. $\vec{b} \notin [l] \longrightarrow \text{proj}_{[l]}(\vec{b}) = \vec{u}\,(\|\vec{b}\| \cos \angle ACB)$, where $\vec{u}$ is either of the unit vectors in $[l]$, $A = C + \vec{u}$, and $B = C + \vec{b}$.
2. $B \notin \overleftrightarrow{C[l]} \longrightarrow \text{proj}_l(B) - \text{proj}_l(C) = \vec{u}(CB \cos \angle ACB)$, where $\vec{u}$ and A are as in Exercise 1.
3. $B - C \notin [l] \longrightarrow d(\text{proj}_l(C), \text{proj}_l(B)) = d(C, B)|\cos \angle ACB|$, where $\vec{0} \neq A - C \in [l]$.

*

The cosine of an angle whose sides are contained in perpendicular lines is 0. The Schwarz Inequality [Theorem 11-8] tells us that the absolute value of the cosine of any angle is less than 1. [Note that unit vectors in the senses of the sides of an angle are linearly independent.]

Note that cos is a function whose domain is the set of all angles in $\mathcal{E}$. The construction procedure illustrated in Figure 15-10, together with the Schwarz Inequality, shows that the range of cos is $\{x\colon |x| < 1\}$. Later we shall have a cosine function whose domain is the set of sensed angles [see TC 206] and whose range is $\{x\colon |x| \leq 1\}$. We shall also have a cosine function with the same range whose domain is $\mathcal{R}$. Although these three functions will have to be distinguished from one another [and from still other cosine functions] it would probably be ultimately more confusing than not if we invented different names for them. So, we shall follow mathematical usage and call each of them 'cos'.

The proof for Theorem 15-6 is trivial since $\vec{a}/\|\vec{a}\|$ and $\vec{b}/\|\vec{b}\|$ are the unit vectors in the senses of $\overrightarrow{VA}$ and $\overrightarrow{VB}$, respectively, and since $(\vec{a}/\|\vec{a}\|)\cdot(\vec{b}/\|\vec{b}\|) = (\vec{a}\cdot\vec{b})/(\|\vec{a}\|\,\|\vec{b}\|)$.

Suggestions for the exercises of section 15.03:

(i) Part A and the discussion of Theorem 15-7 should be developed in class.
(ii) After appropriate examples, Part B may be used as homework.
(iii) Part C may be used as homework (either with Part B or as a separate assignment.)
(iv) Part D should be teacher directed so that students realize the importance of the result.
(v) Part E may be used for homework.

Answers for Part A

1. Suppose that $\vec{b} \notin [\ell]$. It follows from the choice that {A, B, C} is noncollinear and that $\cos \angle ACB = \vec{u}\cdot\vec{b}/\|\vec{b}\|$. Since $\text{proj}_{[\ell]}(\vec{b}) = \vec{u}(\vec{u}\cdot\vec{b})$ it follows that $\text{proj}_{[\ell]}(\vec{b}) = \vec{u}(\|\vec{b}\| \cos \angle ACB)$.

   [Note that the two possible choices for $\vec{u}$ are equally effective. They result in different choices for A and in angles whose cosines are opposites. Since the choices for $\vec{u}$ are also opposites things work out as they should.]
2. Suppose that $B \notin \overleftrightarrow{C[\ell]}$. It follows that $B - C \notin [\ell]$ and so, by Exercise 1, that $\text{proj}_{[\ell]}(B - C) = \vec{u}(\|B - C\| \cos \angle ACB)$. The desired result follows from the fact that $\text{proj}_{[\ell]}(B - C) = \text{proj}_\ell(B) - \text{proj}_\ell(C)$ [Theorem 12-21(a)] and the fact that $\|B - C\| = CB$.
3. [This results from Exercise 2 by taking norms.]

TC 216

First explanation asked for in connection with Figure 15-9: In this case $CB = \|\vec{v}\| = 1$.

Second explanation: $\vec{u}\cdot(B - A') = \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{u}\cos\angle C = \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} = 0$; $\|B - A'\|^2 = \|\vec{v} - \vec{u}\cos\angle C\|^2 = 1 + (\cos\angle C)^2 - 2(\vec{u}\cdot\vec{v})\cos\angle C = 1 - (\cos\angle C)^2$, for $\vec{u}\cdot\vec{v} = \cos\angle C$.

Note that, by Exercise 3, the ratio in which an interval $\overline{BC}$ is foreshortened when it is projected orthogonally onto a line l is the absolute value of the cosine of "the angle between the lines $\overleftrightarrow{BC}$ and l". Despite the vagueness of the latter phrase, the preceding statement helps to give a notion of the significance of cosines.

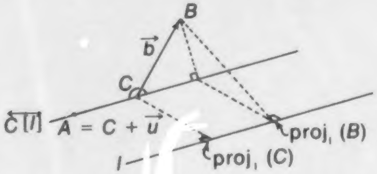


Fig. 15–8

More information can be obtained from Exercise 2. Suppose given an angle, $\angle C$, with vertex C, and suppose that $\vec{u}$ and $\vec{v}$ are the unit vectors in the senses of its sides. Suppose that $A = C + \vec{u}$ and $B = C + \vec{v}$. Let $l = \overleftrightarrow{CA}$ and let $A' = \text{proj}_l\,(B)$. By Exercise 2, $A' - C = \vec{u} \cos \angle ACB = \vec{u} \cos \angle C$. [Explain.]

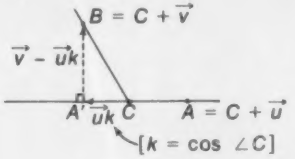


Fig. 15–9

This interpretation of $\cos \angle C$—which amounts to recognizing that $\cos \angle C = \text{comp}_{\vec{u}}\,(\vec{v})$—can be used to suggest a method for obtaining angles with a given cosine. To see how this comes about, note that $B - A' = \vec{v} - \vec{u} \cos \angle C$. It follows that $B - A'$ is a vector in $[\vec{u}]^{\perp}$ and that the square of its norm is $1 - (\cos \angle C)^2$. [Explain.]

By reversing the procedure we went through in the preceding paragraph we can obtain descriptions of angles which have a given cosine. Specifically, given a number k such that $|k| < 1$ it is easy to find points A, B, and C such that $\cos \angle ACB = k$. To do so, let C be any point, $\vec{u}$ any unit vector, and $\vec{n}$ any unit vector in $[\vec{u}]^{\perp}$. Use these to locate points A and B such that $\cos \angle ACB = k$. For example, to draw an angle $\angle ACB$ whose cosine is $\frac{3}{5}$, we note that, in Fig. 15–9, $k = \frac{3}{5}$ so that $A' - C = \vec{u}(\frac{3}{5})$ and $B - A' = \vec{v} - \vec{u}(\frac{3}{5})$. Thus, $A'C = \frac{3}{5}$ and $BA' = \sqrt{1 - k^2} = \frac{4}{5}$. So, $B - A' = \vec{n}(\frac{4}{5})$ and a picture of the required angle looks like this:

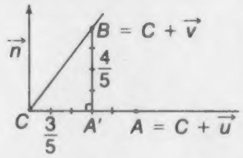


Fig. 15–10

On your paper, you should follow this procedure and draw an angle, $\angle ACB$, whose cosine is $\frac{3}{5}$, and angle, $\angle PQR$, whose cosine is $-\frac{4}{5}$. And, be prepared to justify your results.

The argument given in the preceding paragraph shows that, for any real number k such that $|k| < 1$, there are many angles which have k as cosine. To describe the situation more explicitly, let's see what our freedom in choosing the point C, and the unit vectors $\vec{u}$ and $\vec{n}$ amounts to. In choosing C, you were choosing the vertex of the angle in question. Your choice of $\vec{u}$ then completed the determination of one side, $\overrightarrow{CA}$, of the angle [since, presumably, you chose A to be $C + \vec{u}$]. Your choice of $\vec{n}$ determined a side of the line $\overleftrightarrow{CA}$ which [assuming you chose B to be $C + \vec{u}k + \vec{n}\sqrt{1 - k^2}$] contained all points of the angle's other side, $\overrightarrow{CB}$. [Explain.] Consequently, you have proved "half" of the following:

> **Theorem 15–7** Given a number k such that $|k| < 1$, and given a half-line r, there is one and only one angle whose cosine is k, which has r as one of its sides, and whose other side is contained in a given side of the line containing r.

How do you know that, as the theorem asserts, there is at most one such angle?

Part B

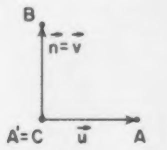
1. Use the procedure suggested by Fig. 15–10 to draw angles whose cosines are as follows:
   (a) 0 (b) $\frac{4}{5}$ (c) $-\frac{4}{5}$ (d) $\frac{12}{13}$ (e) $-\frac{5}{13}$ (f) $\sqrt{2}/2$
   [*Hint*: For part (f), note that $\sqrt{2}/2 = 1/\sqrt{2}$ (Why?), and that $\sqrt{2}/2$ is approximately 0.7.]
2. Complete the argument, outlined in the text, which shows that there exists an angle such as described in Theorem 15–7.
3. What result, noted a few pages back, tells you that there is at most one angle such as described in Theorem 15–7?

Explanation preceding Theorem 15-7: $B - C = \vec{u}k + \vec{n}\sqrt{1 - k^2}$ so, since $\vec{n} \in [A - C]^{\perp}$ and $\sqrt{1 - k^2} > 0$, all points of $\overrightarrow{CB}$ are contained in the '$\vec{n}$-side' of $\overleftrightarrow{CA}$. [See, if necessary, answer for Exercise 4 of Part C on page 212.]

We know by (3) on page 214 that there is at most one angle of the sort specified in Theorem 15-7.

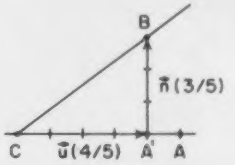
Answers for Part B

1. (a)



$\cos \angle ACB = 0$ $[\vec{u} \cdot \vec{v} = 0]$

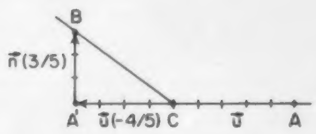
(b)



$\cos \angle ACB = 4/5$

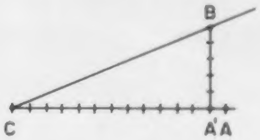
$[\sqrt{1 - (4/5)^2} = 3/5]$

(c)



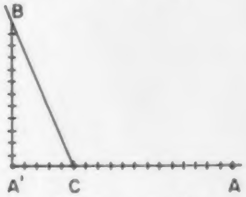
$\cos \angle ACB = -4/5$

(d) $[\sqrt{1 - (12/13)^2} = 5/13]$



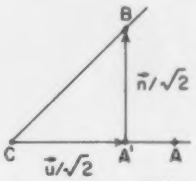
$\cos \angle ACB = 12/13$

(e)



$\cos \angle ACB = -5/13$

(f)



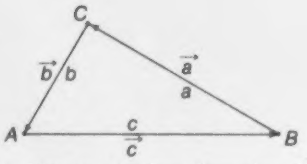
$\cos \angle ACB = \sqrt{2}/2$

[Note that for part (f) we might locate $B' \in \overrightarrow{CB}$ such that $B' - A = \vec{n}$.]

2. Let C be the vertex of r and let $\vec{u}$ be the unit vector in the sense of r. Let $\vec{n}$ be the unit vector in $[\vec{u}]^{\perp}$ such that P belongs to the given side of the line containing r if and only if $P = \vec{u}a + \vec{n}b$ with $b > 0$. Now, let $A = C + \vec{u}$ and $B = C + \vec{u}k + \vec{n}\sqrt{1 - k^2}$. It follows that $||B - C|| = 1$ and $(B - C) \cdot (A - C) = k$. So, since $||A - C|| = 1$, $\cos \angle ACB = k$.

3. Result (3) on page 214 tells us that there is at most one angle of the sort specified in Theorem 15-7.

Part C



Consider $\triangle ABC$, with $C - B = \vec{a}$, $A - C = \vec{b}$, and $B - A = \vec{c}$, and $\|\vec{a}\| = a$, $\|\vec{b}\| = b$, and $\|\vec{c}\| = c$. Note that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

1. Show that $\cos \angle A = (-\vec{b} \cdot \vec{c})/(bc)$. Obtain similar formulas for $\cos \angle B$ and $\cos \angle C$.
2. Show that $\cos \angle A = (\vec{a} \cdot \vec{c} + c^2)/(bc)$. Obtain similar formulas for $\cos \angle B$ and $\cos \angle C$.
3. Obtain three other formulas, similar to those of Exercise 2, for the cosines of $\angle A$, $\angle B$, and $\angle C$.
4. Use results from Exercises 2 or 3 to show that
$$a \cos \angle B + b \cos \angle A = c.$$
Interpret this result in terms of projections.
5. In each of the following you are given some information about a triangle. Make a sketch of the triangle in question and use the results of Exercises 1–4 to help in answering the questions about these triangles.
   (a) Given $\triangle ABC$, with $AC = 5$, $AB = 7$, and $\cos \angle A =$ [illegible]. What is BC? $\cos \angle B$? $\cos \angle C$?
   (b) Given $\triangle ABC$, with $AC = 12$, $AB = 9$, and $BC = 5$, compute the cosines of $\angle A$, $\angle B$, and $\angle C$.
   (c) Given $\triangle ABC$, with $AB = 8$, $BC = 4$, and $CA = 3$, compute the cosines of $\angle A$, $\angle B$, and $\angle C$.
   (d) Given $\triangle PQR$, with $PQ = 6 = QR$ and $PR = 5$. What are the cosines of $\angle P$, $\angle Q$, and $\angle R$?
   (e) Given that $\triangle ABC$ is equilateral, compute the cosines of $\angle A$, $\angle B$, and $\angle C$.
6. Suppose that $\triangle DEF$ is isosceles with base $\overline{EF}$. Show that $\cos \angle E = \cos \angle F$.
7. Use results from Exercises 1–3 to show that if $\cos \angle C \leq 0$ then $\cos \angle A > 0$ and $\cos \angle B > 0$.
8. Suppose that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$.
   (a) What is $\cos \angle C$?
   (b) Find formulas for $\cos \angle A$ and $\cos \angle B$ in terms of the measures of the sides of $\triangle ABC$.

Part D

1. Show that, for any vectors $\vec{a}$ and $\vec{b}$,
$$\|\vec{b} - \vec{a}\|^2 = \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2(\vec{a} \cdot \vec{b}).$$
2. (a) Show that
$$\cos \angle ACB = \frac{\|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{b} - \vec{a}\|^2}{2\|\vec{a}\|\,\|\vec{b}\|},$$
where $\vec{a} = A - C$ and $\vec{b} = B - C$.

TC 218 (1)

Answers for Part C

1. Since $\angle A = \angle CAB$ and since $C - A = -\vec{b}$ and $B - A = \vec{c}$ it follows from Theorem 15-6 that $\cos\angle A = (-\vec{b}\cdot\vec{c})/(\|-\vec{b}\|\,\|\vec{c}\|)$. Since $\|-\vec{b}\| = \|\vec{b}\| = b$ and $\|\vec{c}\| = c$ it follows that $\cos\angle A = (-\vec{b}\cdot\vec{c})/(bc)$. Similarly, $\cos\angle B = (-\vec{c}\cdot\vec{a})/(ca)$ and $\cos\angle C = (-\vec{a}\cdot\vec{b})/(ab)$.
2. Since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $-\vec{b} = \vec{a} + \vec{c}$. So, by Exercise 1, $\cos\angle A = (\vec{a}\cdot\vec{c} + c^2)/(bc)$. [Note that $\vec{c}\cdot\vec{c} = \|\vec{c}\|^2 = c^2$.] Similarly, $\cos\angle B = (\vec{b}\cdot\vec{a} + a^2)/(ca)$ and $\cos\angle C = (\vec{c}\cdot\vec{b} + b^2)/(ab)$.
3. Since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $\vec{c} = -(\vec{a} + \vec{b})$ and $-\vec{b}\cdot\vec{c} = \vec{b}\cdot\vec{a} + b^2$. So, by Exercise 1, $\cos\angle A = (\vec{b}\cdot\vec{a} + b^2)/(bc)$. Similarly, $\cos\angle B = (\vec{c}\cdot\vec{b} + c^2)/(ca)$ and $\cos\angle C = (\vec{a}\cdot\vec{c} + a^2)/(ab)$.
4. $a\cos\angle B + b\cos\angle A = (\vec{b}\cdot\vec{a}+a^2)/c + (\vec{a}\cdot\vec{c} + c^2)/c = [(\vec{a} + \vec{b} + \vec{c})\cdot\vec{a} + c^2]/c = c$; By Exercise 1 of Part A, assuming that $\vec{u}$ is the unit vector in $[\vec{c}\,]^+$, $\mathrm{proj}_{[\vec{c}\,]}(\vec{a}) = -\vec{u}(a\cos\angle B)$ and $\mathrm{proj}_{[\vec{c}\,]}(\vec{b}) = -\vec{u}(b\cos\angle A)$. So, $\mathrm{proj}_{[\vec{c}\,]}(\vec{a} + \vec{b}) = -\vec{u}(a\cos\angle B + b\cos\angle A)$. Since $\vec{a} + \vec{b} = -\vec{c}$, and since $\vec{c} = \vec{u}c$, it follows that $-\vec{u}c = -\vec{c} = \mathrm{proj}_{[\vec{c}\,]}(-\vec{c}) = -\vec{u}(a\cos\angle B + b\cos\angle A)$. Hence, $c = a\cos\angle B + b\cos\angle A$. [This is, of course, an alternative proof of the conclusion.]
5. [These problems are somewhat difficult at this stage. They are inserted here to give students the feeling that the formulas can be used for calculating numerical results.]
   - (a) $BC = 3\sqrt{2}$; $\cos\angle B = \sqrt{2}/2$; $\cos\angle C = -\sqrt{2}/10$ [Using the formulas for '$\cos\angle A$' in Exercises 1 - 3 and the given values for '$\cos\angle A$', 'b', and 'c', we find that $\vec{b}\cdot\vec{c} = -28$, $\vec{a}\cdot\vec{c} = -21$, and $\vec{b}\cdot\vec{a} = 3$. Comparing the values given for '$\cos\angle B$' in Exercises 1 and 2 shows that $a = 3\sqrt{2}$ and, then, $\cos\angle B = \sqrt{2}/2$. We can now use the formula for '$\cos\angle C$' from Exercise 1 to find that $\cos\angle C = -\sqrt{2}/10$.]
   - (b) $\cos\angle A = 25/27$; $\cos\angle B = -19/45$; $\cos\angle C = 11/15$ [Use the result of Exercise 4 and two similar results to obtain three equations in '$\cos\angle A$', '$\cos\angle B$', and '$\cos\angle C$'.]
   - (c) There is no such triangle. [If one fails to notice this and applies the method which was successful in part (b) one obtains the equations: $\cos\angle A = 19/16$, $\cos\angle B = 71/16$, $\cos\angle C = -13/8$. Since the absolute values of the values of cos are less than 1, these equations cannot be satisfied.]
   - (d) $\cos\angle P = 5/12 = \cos\angle R$; $\cos\angle Q = 47/72$
   - (e) $\cos\angle A = \cos\angle B = \cos\angle C = 1/2$
6. Using two equations like that in Exercise 4 we have $d\cos\angle F + f\cos\angle D = e$ and $d\cos\angle E + e\cos\angle D = f$. Since $e = f$ and $d \neq 0$ it follows that $\cos\angle F = \cos\angle E$.

TC 218 (2)

Answers for Part C [cont.]

7. By Exercises 2 and 3, if $\cos\angle C \leq 0$ then $\vec{c}\cdot\vec{b} + b^2 \leq 0$ and $\vec{a}\cdot\vec{c} + a^2 \leq 0$. Since, by Exercise 1, $\cos\angle A = (-\vec{b}\cdot\vec{c})/(bc)$ and $\cos\angle B = (-\vec{c}\cdot\vec{a})/(ca)$ it follows that if $\cos\angle C \leq 0$ then $\cos\angle A \geq b/c$ and $\cos\angle B \geq a/c$. Hence, if $\cos\angle C \leq 0$ then $\cos\angle A > 0$ and $\cos\angle B > 0$. [In the terminology to be introduced later — in particular, see Exercise 1 of Part C on page 236 — the result of this exercise can be expressed by saying that each triangle has at least two acute angles.]
8. (a) $\cos\angle C = 0$ [by Exercise 1]
   - (b) $\cos\angle A = b/c$ and $\cos\angle B = a/c$ [by Exercises 3 and 2]

     [Students might now reconsider Exercises 5(d), 5(e), and 6 making use of the fact that the altitude to the base of an isosceles triangle is a median.]

Answers for Part D

1. $\|\vec{b} - \vec{a}\|^2 = (\vec{b} - \vec{a})\cdot(\vec{b} - \vec{a}) = \vec{b}\cdot\vec{b} + \vec{a}\cdot\vec{a} - 2(\vec{a}\cdot\vec{b}) = \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2(\vec{a}\cdot\vec{b})$
2. (a) $\cos\angle ACB = (\vec{a}\cdot\vec{b})/(\|\vec{a}\|\,\|\vec{b}\|)$ and, by Exercise 1, $\vec{a}\cdot\vec{b} = [\|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{b} - \vec{a}\|^2]/2$. [The result of Exercise 2 is, of course, the Cosine Law, and will be identified as such in the next chapter. Students will appreciate its utility if they are required to re-do Exercises 5 and 6 using this result.]

TC 219 (1)

(b) $\cos\angle PQR = \dfrac{8^2 + 9^2 - 11^2}{2\cdot 8\cdot 9} = \dfrac{1}{6}$; $\cos\angle QRP = \dfrac{23}{33}$; $\cos\angle QPR = \dfrac{13}{22}$.

3. (a) Since, by Theorem 14-26, an isometry maps lines on lines it follows that if $\{A, B, C\}$ were collinear then, since f^{-1} is an isometry, $\{P, Q, R\}$ would be collinear. Since this is not the case $\{A, B, C\}$ is noncollinear.
   - (b) Since, by Theorem 14-27, f maps rays onto rays, preserving vertices, f maps $\overrightarrow{RP}$ onto $\overrightarrow{CA}$ and $\overrightarrow{RQ}$ onto $\overrightarrow{CB}$. So, f maps $\angle PRQ$ onto $\angle ACB$.
   - (c) Let $P - R = \vec{p}$, $Q - R = \vec{q}$, $A - C = \vec{a}$, and $B - C = \vec{b}$. It follows that $Q - P = \vec{q} - \vec{p}$ and $B - A = \vec{b} - \vec{a}$. Since f is an isometry which maps P on A, Q on B, and R on C it follows that $\|\vec{p}\| = \|\vec{a}\|$, $\|\vec{q}\| = \|\vec{b}\|$, and $\|\vec{q} - \vec{p}\| = \|\vec{b} - \vec{a}\|$. Hence, by Exercise 2,

$$\cos\angle PRQ = \frac{\|\vec{p}\|^2 + \|\vec{q}\|^2 - \|\vec{q} - \vec{p}\|^2}{2\|\vec{p}\|\,\|\vec{q}\|}$$

$$= \frac{\|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{b} - \vec{a}\|^2}{2\|\vec{a}\|\,\|\vec{b}\|} = \cos\angle ACB.$$

(b) Suppose that $\triangle PQR$ is such that $PQ = 8$, $QR = 9$, and $RP = 11$. Make use of the result in (a) to compute $\cos \angle PQR$, $\cos \angle QRP$, and $\cos \angle QPR$.

3. Given $\angle PRQ$ and an isometry f which maps P on A, Q on B, and R on C, show that
   (a) $\{A, B, C\}$ is noncollinear,
   (b) f maps $\angle PRQ$ onto $\angle ACB$, and
   (c) $\cos \angle ACB = \cos \angle PRQ$.

*

In Exercise 3 of Part D you have shown that congruent angles have the same cosine. This suggests trying to show that angles which have the same cosine are congruent. As in Exercise 3, this can be done by using the result obtained in Exercise 2.

Suppose that $\angle C$ and $\angle R$ are angles which have the same cosine. We wish to show that there is an isometry which maps $\angle R$ onto $\angle C$. Recall that we have already shown that, given points A, B, C, P, Q, and R such that $AB = PQ$, $BC = QR$, and $CA = RP$, there is an isometry which maps P on A, Q on B, and R on C. If, in addition, $\{P, Q, R\}$ is noncollinear, it follows as in Exercises 3(a) and (b) that $\{A, B, C\}$ is noncollinear and that the isometry in question maps $\angle PRQ$ on $\angle ACB$. So, returning to our given angles, $\angle C$ and $\angle R$, all we need do is find points A, B, P, and Q such that $\angle C$ is $\angle ACB$, $\angle R$ is $\angle PRQ$, $AB = PQ$, $BC = QR$, and $CA = RP$. This is easy enough to do. For we may choose A and B on the sides of $\angle C$ and choose P and Q on the sides of $\angle R$ in such a way that the last two equations are satisfied. [For example, we can choose all four points to be the images of C and R, respectively, under the unit translations in the senses of the sides of the respective angles.] Then, using the result of Exercise 2 and the assumption that $\cos \angle C = \cos \angle R$, it follows at once that $AB = PQ$. Consequently, we have:

Theorem 15–8 Angles are congruent if and only if they have the same cosine.

This theorem is analogous to an earlier one which told us that intervals are congruent if and only if they have the same measure. We shall use '$\cong$' as an abbreviation for 'is congruent to'. So, for example, Theorem 15–8 and the analogous theorem concerning intervals can be restated as:

$$\angle C \cong \angle R \longleftrightarrow \cos \angle C = \cos \angle R,$$
$$\overline{AB} \cong \overline{PQ} \longleftrightarrow AB = PQ$$

In view of Theorem 15-8 it is reasonable to think of the cosine of an angle as a measure of its size. For, according to the theorem and our intuitive notion of the meaning of 'congruence', angles have the same size [and shape] if and only if they have the same cosine. This "cosine-measure" for angles has some pecularities and disadvantages. For one thing, the values of cos are real numbers between -1 and 1; for another, the greater the cosine, the smaller the angle; for a third, there is no way of "adding angles" so that the cosine-measure of their sum is the sum of their cosine-measures. When we come to deal with circles we shall be able to introduce a more satisfactory angle-measure. Until then we can do more than one might expect with cosine-measures.

It is a thought which might be worth a teacher's brooding over that, while we measure sizes of intervals, we seem to measure shapes of angles.

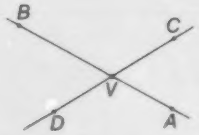
TC 220

Answers for Part E

1. Let $\vec{u}$ and $\vec{v}$ be unit vectors in the senses of $B - V$ and $D - V$, respectively. Then $\cos \angle BVD = \vec{u} \cdot \vec{v} = (-\vec{u}) \cdot (-\vec{v}) = \cos \angle AVC$. Therefore, $\angle AVC \cong \angle BVD$.
2. $\cos \angle AVC + \cos \angle BVC = (-\vec{u}) \cdot (-\vec{v}) + \vec{u} \cdot (-\vec{v}) = 0$, where $\vec{u}$, $\vec{v}$ are as defined in the answer to Exercise 1.
3. (a) They are congruent.

   (b) Their cosines are opposites.
4. Since $\angle AVC$ and $\angle BVC$ are adjacent angles there are numbers — say, s and r — such that $0 < r < 1$ and $\vec{w}s = \vec{u}(1 - r) + \vec{v}r$. [See hint for Exercise 1 of Part C on page 212.] Since $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are not opposites and [because $\angle AVC$ and $\angle BVC$ are adjacent] are not the same, $(\vec{u}, \vec{v})$ is linearly independent and, so, $s \neq 0$. Let $a = (1 - r)/s$ and $b = r/s$. Since $1 - r$ and r are both positive and $s \neq 0$ it follows that a and b are both positive or a and b are both negative. Since $\vec{w} = \vec{u}a + \vec{v}b$, $\vec{u} \cdot \vec{w} = a + (\vec{u} \cdot \vec{v})b$ and $\vec{v} \cdot \vec{w} = (\vec{u} \cdot \vec{v})a + b$. Hence, $\cos \angle AVC + \cos \angle BVC = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = (a + b)(1 + \vec{u} \cdot \vec{v}) \neq 0$. [Since a and b are both positive or both negative, $a + b \neq 0$; since $(\vec{u}, \vec{v})$ is linearly independent, $1 + \vec{u} \cdot \vec{v} \neq 0$.]
5. By Theorem 15-8, an angle is congruent to $\angle R$ if and only if its cosine is $\cos \angle R$ and, by Definition 15-4, it is adjacent to $\angle AVC$ if and only if its side other than $\overrightarrow{VC}$ is on the opposite side of $\overrightarrow{VC}$ from A. By Theorem 15-7 there is one and only one angle which satisfies these conditions.

Part E



Referring to the figure, show that

1. $\angle AVC \cong \angle BVD$
2. $\cos \angle AVC + \cos \angle BVC = 0$
3. What may you conclude about
   (a) vertical angles? (b) adjacent supplementary angles?
4. Suppose that [contrary to the figure] $\angle AVC$ and $\angle BVC$ are adjacent angles whose sides $\overrightarrow{VA}$ and $\overrightarrow{VB}$ are *not* opposite half-lines. Show that $\cos \angle AVC + \cos \angle BVC \neq 0$. [*Hint*: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be the unit vectors in the senses of $\overrightarrow{VA}$, $\overrightarrow{VB}$, and $\overrightarrow{VC}$. Show, first, that $\vec{w} = \vec{u}a + \vec{v}b$, where a and b are both positive or both negative.]
5. Show that, given $\angle AVC$ and $\angle R$, there is one and only one angle congruent to $\angle R$ which is adjacent to $\angle AVC$ and has $\overrightarrow{VC}$ as one side. [*Hint*: Use Theorems 15-8 and 15-7 and Definiton 15-4.]

*

In Exercise 3(a) you probably noticed:

> Theorem 15-9 Vertical angles are congruent.

15.04 Supplementary Angles

Before stating the results obtained in Exercises 3(b) and 4 of the preceding exercises we need:

> Definition 15-6 A first and a second angle are supplementary [and each is a supplement of the other] if and only if they are congruent to adjacent angles whose noncommon sides are opposite half-lines.

In short, angles are supplementary if and only if they are congruent to what we have already called 'adjacent supplementary angles'. By Exercise 3(b) and Theorem 15-8 it follows that the sum of the cosines of supplementary angles is zero. By Exercises 4 and 5 and Theorem 15-8 it follows that the sum of the cosines of nonsupplementary angles is not zero. So, we have:

> Theorem 15-10 $\angle R$ and $\angle C$ are supplementary if and only if $\cos \angle R + \cos \angle C = 0$.

Since any angle is congruent to itself it follows, by Definition 15-6, that adjacent angles whose noncommon sides are opposite half-lines are supplementary. On the other hand, it follows from Exercise 4 and Theorem 15-10 that adjacent angles whose noncommon sides are not opposite half-lines are not supplementary. [Explain.] So, we have:

> Corollary Adjacent angles are supplementary if and only if their noncommon sides are opposite half-lines.

In view of this our use of 'supplementary' in Definition 15-6 is consistent with our use of this word on page 207.

Since, by Theorem 15-8, congruence of angles amounts to equality of their cosines it follows by Definition 15-6 that any two supplements of a given angle are congruent and that any angle which is congruent to some supplement of a given angle is also a supplement of that angle. [Explain.] If an angle is its own supplement, what can you say about its cosine? Are there any such angles?

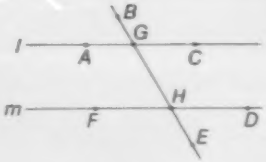
Exercises

Part A

Suppose that the half-lines s and t are the sides of $\angle Q$ and that the half-lines u and v are the sides of $\angle R$.

1. Suppose that $s \parallel u$ and $t \parallel v$. What can you say about $\cos \angle Q$ and $\cos \angle R$? About $\angle Q$ and $\angle R$? [*Hint*: Let $\vec{s}$, $\vec{t}$, $\vec{u}$, and $\vec{v}$ be the unit vectors in the senses of s, t, u, and v. What can you say, for example, about $\vec{s}$ and $\vec{u}$?]
2. Prove:

> Theorem 15-11 If the sides of one angle can be paired with those of another in such a way that paired sides are parallel then the angles are either congruent or supplementary. They are congruent if each two paired sides have the same sense or if each two paired sides have opposite senses. They are supplementary if some paired sides have the same sense and the other two paired sides have opposite senses.



3. Consider the picture at the right, and assume that $l \parallel m$. In each of the following give two angles, one whose vertex is G and one whose vertex is H, which satisfy the stated conditions. Tell whether the angles are congruent or supplementary.
   (a) Sides of the angle at G have same senses as sides of angle at H.
   (b) Each side of the angle at G has the sense opposite that of a side of the angle at H.
   (c) One side of the angle at G has the same sense as a side of the angle at H while the other side of the angle at G has the sense opposite that of the other side of the angle at H.

To see that the sum of the cosines of nonsupplementary angles is not 0, note that, given one of them we can, by Exercise 5, find an angle adjacent to it which is congruent to [and so has the same cosine] as the other. Since the given angles are not supplementary it follows from Definition 15-6 that the noncommon sides of the adjacent angles are not opposite half-lines. So, by Exercise 4, the sum of their cosines is not zero. Hence, the sum of the cosines of the given angles is not zero.

That adjacent angles whose noncommon sides are not opposite half-lines are not supplementary, note that, by Exercise 4 the sum of the cosines of such angles is not zero and, so, by Theorem 15-10, the angles are not supplementary.

Theorem 15-10 could, of course, have been used as a definition of 'supplementary' and Definition 15-6 be deduced as a theorem. Definition 15-6 seems, however, to be better motivated as a definition. In the sequel we will find Theorem 15-10 the more convenient of the two to use.

Suggestions for the exercises of section 15.04:

(i) Part A should be teacher directed to insure proper application of Theorem 15-11.

(ii) Part B may be used as homework since the work has some similarities to Part A.

(iii) Part C may be used either as supervised class exercises or as homework, but should not be omitted.

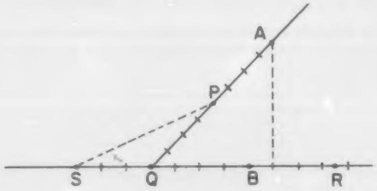
*

Sample Quiz

1. Make a sketch of an angle, $\angle PQR$, whose cosine is 5/8.
2. Locate the point A of $\overrightarrow{QP}$ such that $QA = 8$. What is $d(A, \overleftrightarrow{QR})$?
3. Locate the point B of $\overrightarrow{QR}$ such that $QB = 4$. What is $d(B, \overleftrightarrow{QP})$?
4. Let S be such that $Q \in \overline{SR}$. What is $\cos \angle PQS$?
5. Suppose that $QP = 4$ and $QS = 3$. What is PS?

Key to Sample Quiz

1. Here is an appropriate sketch of an angle, $\angle PQR$, whose cosine is 5/8, together with the points A, B, and S described in 2 - 5.



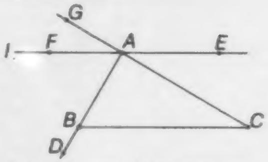
2. $\sqrt{39}$ $[\sqrt{8^2 - 5^2}]$
3. $\sqrt{39}/2$
4. $-5/8$
5. $\sqrt{40}$ [See Exercise 2, page 218. $SP^2 = PQ^2 + QS^2 - 2 \cdot PQ \cdot QS \cdot \cos \angle PQS$.]

An angle is its own supplement if and only if its cosine is its own opposite — that is, if and only if its cosine is 0. Such angles are just those whose sides are perpendicular. [The term 'right angle' is introduced in Definition 15-10.]

Answers for Part A

1. Using the notation introduced in the hint, since $s \parallel u$, $\vec{s} = \vec{u}$, or $\vec{s} = -\vec{u}$; since $t \parallel v$, $\vec{t} = \vec{v}$ or $\vec{t} = -\vec{v}$. It follows that $\vec{s} \cdot \vec{t} = \vec{u} \cdot \vec{v}$ [in two cases] or $\vec{s} \cdot \vec{t} = -(\vec{u} \cdot \vec{v})$ [in the other two cases]. Hence, $\cos \angle Q$ and $\cos \angle R$ are the same or are opposites. [So, $\angle Q$ and $\angle R$ are congruent or are supplementary.]
2. [The proof is given, essentially, in the answer for Exercise 1.]
3. (a) $\angle BGA$ and $\angle GHF$; congruent [$\angle CGH$ and $\angle DHE$, or $\angle BGC$ and $\angle GHD$, or $\angle AGH$ and $\angle FHE$, would do as well.]
   (b) $\angle BGA$ and $\angle DHE$; congruent [As in part (a) there are three other correct choices.]
   (c) $\angle BGA$ and $\angle EHF$; supplementary [This time there are five other correct choices.]

4. Consider the picture in Exercise 3, together with the assumption that $l \parallel m$.
   (a) Give three angles which are congruent to $\angle BGC$.
   (b) Give three angles which are supplements of $\angle BGC$.
   (c) Describe a translation which maps $\angle DHE$ onto $\angle CGH$. What is the image of $\angle EHF$ under this translation?
5. Consider $\triangle ABC$ with $l \parallel \overline{BC}$, as shown in the picture at the right.



   (a) Give an angle which has vertex A and is congruent to $\angle C$.
   (b) Give an angle which has vertex A and is supplementary to $\angle C$.
   (c) Give an angle which has vertex A and is congruent to $\angle CBD$.
   (d) Give an angle which has vertex A and is supplementary to $\angle CBD$.

Part B

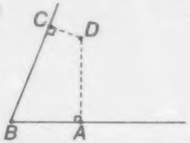
1. Suppose that $\angle Q$ and $\angle R$ are in parallel planes, that s and t are the sides of $\angle Q$, u and v are the sides of $\angle R$, and that $s \perp u$ and $t \perp v$. Draw several pictures to illustrate this situation and make a conjecture concerning $\angle Q$ and $\angle R$. [*Hint*: It will be sufficient to consider the case in which $\angle Q$ and $\angle R$ are in the same plane.]
2. Would your conjecture seem reasonable if the planes of $\angle Q$ and $\angle R$ were not parallel?
3. Prove:

> **Theorem 15–12** Given angles in parallel planes, if the sides of one angle can be paired with those of the other in such a way that paired sides are perpendicular, then the angles are either congruent or supplementary.

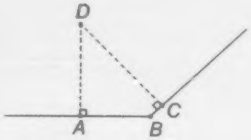
[*Hint*: Let $\overrightarrow{s}, \overrightarrow{t}, \overrightarrow{u}$, and $\overrightarrow{v}$ be the unit vectors in the senses of s, t, u, and v. In the situation described, $(\overrightarrow{s}, \overrightarrow{u})$ and $(\overrightarrow{t}, \overrightarrow{v})$, say, are orthonormal and $[\overrightarrow{s}, \overrightarrow{u}] = [\overrightarrow{t}, \overrightarrow{v}]$. It follows that $\overrightarrow{s} = \overrightarrow{t}(\overrightarrow{s} \cdot \overrightarrow{t}) + \overrightarrow{v}(\overrightarrow{s} \cdot \overrightarrow{v})$ and, since $\|\overrightarrow{s}\| = 1$, that $(\overrightarrow{s} \cdot \overrightarrow{t})^2 + (\overrightarrow{s} \cdot \overrightarrow{v})^2 = 1$. Find an equation similar to the last which will enable you to conclude that $(\overrightarrow{s} \cdot \overrightarrow{t})^2 = (\overrightarrow{u} \cdot \overrightarrow{v})^2$.]

4. In each of the following, you are given a point D in the interior or the exterior of $\angle B$, and that A and C are the feet of the perpendiculars from D to the lines containing the sides of $\angle B$. Decide whether $\angle ADC$ is congruent to, or supplementary to $\angle B$, and be prepared to explain your answer.

(a)



(b)

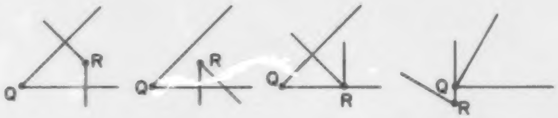


Answers for Part A [cont.]

4. (a) $\angle$GHD, $\angle$FHE, $\angle$AGH
   (b) $\angle$BGA, $\angle$GHF, $\angle$CGH [and $\angle$DHE]
   (c) G - H; $\angle$HGA
5. (a) $\angle$GAF [or: $\angle$EAC] (b) $\angle$GAE [or: $\angle$CAF]
   (c) $\angle$EAB (d) $\angle$FAB

Answers for Part B

1.

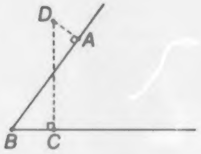


[There are, of course, many suitable pictures.]

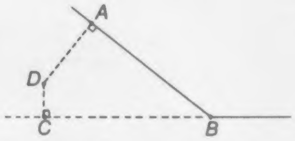
Conjecture: $\angle$Q and $\angle$R are congruent or supplementary.

2. No. [A limiting case may be enlightening. Consider $\angle$ACB in a plane π and $\overrightarrow{CD}$ perpendicular to π. $\overrightarrow{CD}$ might be thought of as a "very small angle" and $\angle$ACB can be of any size. Starting from this it is easy enough to envisage counterexamples to the conjecture in case the angles are not in parallel planes. Another procedure is to consider planes π and σ perpendicular to the sides of a given angle and choose a point in their intersection. Almost any angle with this point as a vertex and whose sides are subsets of π and σ, respectively, will be related to the given angle as in the instructions for this exercise.]
3. Suppose that $\angle$Q has sides s and t, that $\angle$R has sides u and v, and that s $\perp$ u and t $\perp$ v. Let $\vec{s}$, $\vec{t}$, $\vec{u}$, and $\vec{v}$ be unit vectors in the senses of s, t, u, and v, respectively. It follows that $(\vec{s}, \vec{u})$ and $(\vec{t}, \vec{v})$ are orthonormal and, since $\vec{s}$, $\vec{t}$, $\vec{u}$, and $\vec{v}$ all belong to the same bidirection, that both $(\vec{s}, \vec{u})$ and $(\vec{t}, \vec{v})$ are bases for this bidirection. It further follows [see Theorem 11-11] that $\vec{s} = \vec{t}(\vec{s} \cdot \vec{t}) + \vec{v}(\vec{s} \cdot \vec{v})$ and that $\vec{v} = \vec{s}(\vec{v} \cdot \vec{s}) + \vec{u}(\vec{v} \cdot \vec{u})$. Since $\|\vec{s}\| = 1$ and $(\vec{t}, \vec{v})$ is orthonormal, $1 = (\vec{s} \cdot \vec{t})^2 + (\vec{s} \cdot \vec{v})^2$. Since $\|\vec{v}\| = 1$ and $(\vec{s}, \vec{u})$ is orthonormal, $1 = (\vec{v} \cdot \vec{s})^2 + (\vec{v} \cdot \vec{u})^2$. So, $(\vec{s} \cdot \vec{t})^2 = (\vec{u} \cdot \vec{v})^2$ and cos$\angle$Q is the same as, or is the opposite of, cos$\angle$R. Hence, $\angle$Q and $\angle$R are congruent or supplementary.

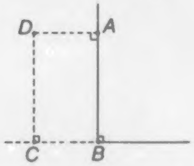
(c)



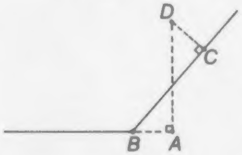
(d)



(e)



(f)



*5. Using the notation of the hint for Exercise 3, show that $\overrightarrow{s} \cdot \overrightarrow{t} = -(\overrightarrow{u} \cdot \overrightarrow{v})$ if $\overrightarrow{u} \cdot \overrightarrow{t}$ and $\overrightarrow{s} \cdot \overrightarrow{v}$ are both positive or both negative, and that $\overrightarrow{s} \cdot \overrightarrow{t} = \overrightarrow{u} \cdot \overrightarrow{v}$ if one of $\overrightarrow{u} \cdot \overrightarrow{t}$ and $\overrightarrow{s} \cdot \overrightarrow{v}$ is positive and the other is negative. [*Hint*: Use the fact that $\overrightarrow{u} \cdot \overrightarrow{s} = 0$.]

Part C

1. Consider an angle, ∠*ABC*, and a point *D* in the interior of ∠*ABC*.
   (a) Draw a picture of an angle coplanar with ∠*ABC* and whose vertex is *D*, whose sides are perpendicular to the sides of ∠*ABC*, and which is supplementary to ∠*ABC*. How many such angles are there?
   (b) Draw a picture of an angle coplanar with ∠*ABC* and whose vertex is *D*, whose sides are perpendicular to the sides of ∠*ABC*, and which is congruent to ∠*ABC*. How many such angles are there?
2. Replace the word 'perpendicular' with 'parallel' in Exercises 1(a) and 1(b) and draw pictures of the angles so described.
3. Consider an angle, ∠*ABC*, and a point *D* in the *exterior* of ∠*ABC*.
   (a) Repeat Exercises 1(a) and 1(b) in this case
   (b) Repeat Exercise 2 in this case.
4. Given ∠*ABC* and a point *D* exterior to ∠*ABC*. In each of the following, draw an angle with vertex *D* which satisfies the given conditions.
   (a) One side of ∠*D* is parallel to a side of ∠*ABC*, ∠*D* is congruent to ∠*ABC*, and ∠*D* and ∠*ABC* have exactly one point in common.
   (b) One side of ∠*D* is parallel to ∠*ABC*, ∠*D* is supplementary to ∠*ABC*, and ∠*D* and ∠*ABC* have exactly one point in common.
   (c) The sides of ∠*D* are perpendicular to the sides of ∠*ABC*, ∠*D* is congruent to ∠*ABC*, and ∠*D* and ∠*ABC* have exactly four points in common.
   (d) The sides of ∠*D* are perpendicular to the sides of ∠*ABC*, ∠*D* is supplementary to ∠*ABC*, and ∠*D* and ∠*ABC* have exactly two points in common.

Answers for Part B [cont.]

4. (a) supplementary [Looking at the picture, cos ∠D < 0 and cos ∠B > 0, so ∠B and ∠D are not congruent and, so, must be supplementary.]
   (b) supplementary [cos ∠D > 0 and cos ∠B < 0]
   (c) congruent [cos ∠B > 0 and cos ∠D > 0]
   (d) congruent [cos ∠B < 0 and cos ∠D < 0]
   (e) congruent and supplementary [cos ∠B = cos ∠D = 0]
   (f) supplementary [cos ∠D > 0 and cos ∠B < 0]

5. Since $\vec{s} = \vec{t}(\vec{s} \cdot \vec{t}) + \vec{v}(\vec{s} \cdot \vec{v})$ and $\vec{u} \cdot \vec{s} = 0$ it follows that $(\vec{u} \cdot \vec{t})(\vec{s} \cdot \vec{t}) + (\vec{u} \cdot \vec{v})(\vec{s} \cdot \vec{v}) = 0$. From this it follows that if $\vec{u} \cdot \vec{t}$ and $\vec{s} \cdot \vec{v}$ are both positive or both negative then one of $\vec{s} \cdot \vec{t}$ and $\vec{u} \cdot \vec{v}$ must be positive and the other negative or both must be 0. Since $\vec{s} \cdot \vec{t}$ and $\vec{u} \cdot \vec{v}$ are either equal or opposite it follows that $\vec{s} \cdot \vec{t} = -(\vec{u} \cdot \vec{v})$. Similarly, if one of $\vec{u} \cdot \vec{t}$ and $\vec{s} \cdot \vec{v}$ is positive and the other is negative then both of $\vec{s} \cdot \vec{t}$ and $\vec{u} \cdot \vec{v}$ must be positive or both must be negative, or both must be 0. So, in this case, $\vec{s} \cdot \vec{t} = \vec{u} \cdot \vec{v}$. [It is interesting to check these results for various figures like those in Exercise 4.]

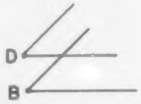
*

Exercise 5, above, gives a way of sorting out the cases in which angles with corresponding sides perpendicular are supplementary from those in which they are congruent. A less definitive, but conceptually simpler method is given in Exercise 3 of Part E on page 259. For the most part we shall expect students to settle this question of congruence vs. supplementarity by inspection of a figure.

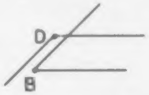
Answers for Part C

1. (a) [There are many figures.] In case ∠ABC is not a right angle there are two "such angles"; in case ∠ABC is a right angle there are four.
   (b) [Remarks given for (a) apply here also.]
2. (a), (b) [See comments for Exercise 1.]
3. (a), (b) [Same comments as for Exercises 1 and 2.]
4. [D must be chosen with some care if ∠ABC and D are to be the same for all four parts.]

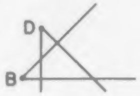
(a)



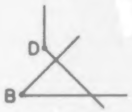
(b)



(c)



(d)



15.05 Parallel Lines and Transversals

A line which crosses two coplanar lines at different points is called *a common transversal* of the given lines. In Fig. 15–11, t is a common transversal of l and m since l and m are coplanar [Why?], $l \cap t = \{A\}$, $m \cap t = \{B\}$, and $A \neq B$. [In the same figure, l is a common transversal of m and t, and m is a common transversal of l and t. Can you draw three coplanar lines such that only one of them is a common transversal of the remaining two?] There are eight angles each of which either has A as its vertex and is contained in $l \cup t$ or has B as its vertex and is contained in $m \cup t$. Each of these angles has one of its sides contained in t and its other side on one or another side of t. The four angles which have $\overrightarrow{AB}$ or $\overrightarrow{BA}$ as a side are called *interior angles*; those which have $-\overrightarrow{BA}$ or $-\overrightarrow{AB}$ as a side are called *exterior angles*. Any two of the eight angles which have different vertices and are both interior angles or both exterior angles are called *consecutive angles* or *alternate angles* according as they have sides on the same side of t or on opposite sides of t. Finally, two angles with different vertices, one of which is an interior angle and the other an exterior

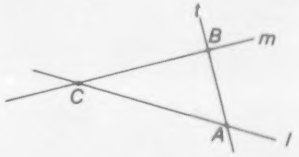


Fig. 15–11

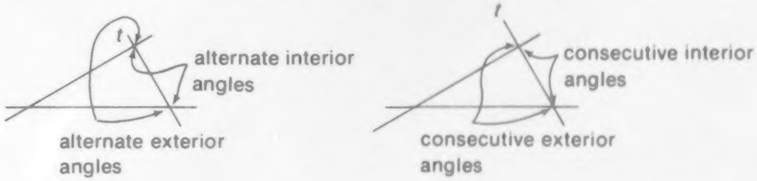

t
alternate interior angles
alternate exterior angles
t
consecutive interior angles
consecutive exterior angles

Fig. 15–12

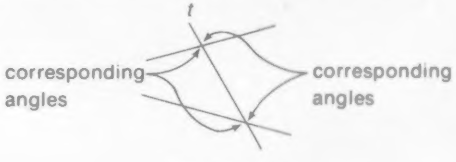


Fig. 15–13

In Figure 15-11, l and m are coplanar because they intersect at C.

Three coplanar lines of which two are parallel, and the third is not parallel to them, furnish an example of three coplanar lines such that one and only one is a common transversal of the other two.

* * *

Two consecutive angles — one at A and one at B — either have as sides $\overrightarrow{AB}$ and $\overrightarrow{BA}$, respectively, or have as sides $-\overrightarrow{AB}$ and $-\overrightarrow{BA}$. [And their other sides are on the same side of t.]

Two alternate angles — one at A and one at B — either have as sides $\overrightarrow{AB}$ and $\overrightarrow{BA}$, respectively, or have as sides $-\overrightarrow{AB}$ and $-\overrightarrow{BA}$. [And their other sides are on opposite sides of t.]

Two corresponding angles — one at A and one at B — either have as sides $\overrightarrow{AB}$ and $-\overrightarrow{BA}$, respectively, or have as sides $-\overrightarrow{AB}$ and $\overrightarrow{BA}$. [And their other sides are on the same side of t.]

The preceding remarks, together with Theorem 15-11, furnish a proof of Theorem 15-13 on page 225. To see how, consider two parallel lines crossed by a transversal t at A and B. Given two consecutive angles we can pair their sides $\overrightarrow{AB}$ and $\overrightarrow{BA}$ or $-\overrightarrow{AB}$ and $-\overrightarrow{BA}$, as the case may be, to obtain corresponding oppositely sensed sides, and pair the other two sides to obtain corresponding sides which, since they are parallel and on the same side of t, have the same sense. [See Exercise 3 of Part A on page 225.] It follows by Theorem 15-11 that consecutive angles are supplementary. Similarly, in the case of alternate angles we obtain two pairs of oppositely sensed sides; and in the case of corresponding angles we obtain two pairs of similarly sensed sides. So, by Theorem 15-11, alternate angles are congruent and corresponding angles are congruent.

Alternatively, we could prove Theorem 15-13 by using the isometry B - A to establish the congruence of corresponding angles, noting that this isometry maps $\overrightarrow{AB}$ onto $-\overrightarrow{BA}$ [and $-\overrightarrow{AB}$ onto $\overrightarrow{BA}$] and maps that one of the parallel lines which contains A onto the one containing B, and, also, that it maps a given side of t onto itself. Then, we could establish the rest of the content of Theorem 15-13 by dealing with vertical angles and adjacent supplementary angles. However, having Theorem 15-11 the former proof of Theorem 15-13 is more natural and much simpler than the one just sketched.

Suggestions for the exercises of section 15.05:

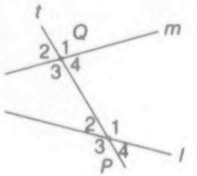
(i) Parts A and B may be used for homework after the terms introduced on pages 225 and 226 have been illustrated.

(ii) Part C could be used as a team assignment or as class discussion material.

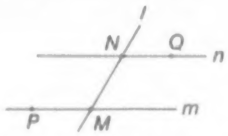
angle, are called *corresponding angles* if they have sides on the same side of t. Note that in the case of two alternate angles or two consecutive angles, the sides of these angles which are contained in t have opposite senses. [Why?] In the case of two corresponding angles, the sides which are contained in t have the same sense. [Why?]

Exercises

Part A

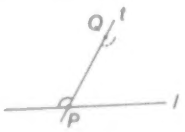


1. In the picture at the right, t is a common transversal of l and m. The numerals can be used in referring to the angles. For example, $\angle P_1$ and $\angle P_3$ are vertical angles, and the adjacent supplements of $\angle Q_2$ are $\angle Q_1$ and $\angle Q_3$. List pairs of angles which are (a) alternate interior angles (b) alternate exterior angles (c) consecutive interior angles (d) consecutive exterior angles (e) corresponding angles.



2. In the picture at the right, $m \parallel n$ and l is a common transversal of m and n. $P \in m$, $Q \in n$, and neither belongs to l. M is the vertex of two opposite half-lines contained in m, and N is the vertex of two opposite half-lines contained in n. Show that P and Q are on opposite sides of l if and only if $P - M$ and $Q - N$ have opposite senses. [*Hint*: Recall Part C on page 211.]
3. Show that, with l, m, and n as in Exercise 2, those sides of alternate angles which are not contained in l have opposite senses and those sides of consecutive or corresponding angles which are not contained in l have the same sense.
4. Prove:

> Theorem 15–13 Of the angles formed by two parallel lines and a common transversal, any two alternate angles are congruent, any two corresponding angles are congruent, and any two consecutive angles are supplementary.



5. Suppose that $l \cap t = \{P\}$, that $Q \in t$, and that $Q \neq P$. How many lines are there through Q, and coplanar with l, such that a given angle at P is congruent to its alternate angle at Q?
6. What theorem [or theorems] from this chapter would you use in justifying your answer for Exercise 5?

Answers for Part A

1. (a) $(\angle P_1, \angle Q_3)$, $(\angle P_2, \angle Q_4)$ (b) $(\angle P_3, \angle Q_1)$, $(\angle P_4, \angle Q_2)$
   (c) $(\angle P_1, \angle Q_4)$, $(\angle P_2, \angle Q_3)$ (d) $(\angle P_3, \angle Q_2)$, $(\angle P_4, \angle Q_1)$
   (e) $(\angle P_1, \angle Q_1)$, $(\angle P_2, \angle Q_2)$, $(\angle P_3, \angle Q_3)$, $(\angle P_4, \angle Q_4)$

2. Since $m \parallel n$, and neither P nor Q belongs to ℓ there is a number — say, t — such that $t \neq 0$ and $P - M = (Q - N)t$. Since $Q - N = (M - N) + (Q - M)$ it follows that $P - M = (N - M) \cdot -t + (Q - M)t$. By Part C on page 212 it follows that P and Q are on opposite sides of ℓ if and only if $t < 0$. But, since $P - M = (Q - N)t$, $P - M$ and $Q - N$ have opposite senses if and only if $t < 0$. Hence, P and Q are on opposite sides of ℓ if and only if $P - M$ and $Q - N$ have opposite senses. [Note that it follows that P and Q are on the same side of ℓ if and only if $P - M$ and $Q - N$ have the same sense.]

3. In the case of alternate angles, the sides not contained in ℓ are contained in opposite sides of t and so, by Exercise 1, have opposite senses. The sides of consecutive or corresponding angles not contained in ℓ are contained in the same side of t and so, by Exercise 1, have the same sense.

4. [A proof, using definitions and Exercise 3 has been given in the preceding commentary.]

5. Just one.

6. Theorems 15-7 and 15-8.

*

If you answered Exercises 5 and 6 correctly, you have in hand most of the proof of the part of the following theorem which deals with alternate angles:

> **Theorem 15–14** If some two alternate or corresponding angles formed by two coplanar lines and a common transversal are congruent, or some two consecutive angles are supplementary, then the lines are parallel.

Here is a proof of the part of the theorem dealing with alternate angles: Suppose that l, t, l', and Q are as in Exercise 5, and let $\angle P$ be one of the four angles contained in $l \cup t$. Let r be the half-line with vertex Q whose sense is opposite to the sense of that side of $\angle P$ which is contained in t. [In Fig. 15–14, r is $\overrightarrow{QP}$ if $\angle P$ is either $\angle P_3$ or $\angle P_4$ and is $-\overrightarrow{QP}$ if $\angle P$ is either $\angle P_1$ or $\angle P_2$.] By Theorems 15–7 and 15–8 there is just one angle which is congruent with $\angle P$, which has r as one of its sides, and whose other side—say, s—is on the opposite side of t from the side of $\angle P$ which is not in t. So, the line which contains the half-line s is the only line through Q and coplanar with l such that $\angle P$ is congruent to its alternate angle at Q. Since, by Theorem 15–13, the parallel to l through Q is such a line it follows that the parallel to l through Q is the only such line.

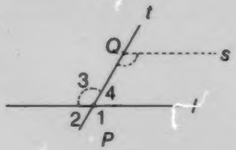


Fig. 15–14

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Part B

1. Prove the part of Theorem 15–14 which deals with corresponding angles.
2. Prove the part of Theorem 15–14 which deals with consecutive angles.
3. Use the two theorems of this section to show that, given two coplanar lines and a common transversal, if some two alternate or corresponding angles are congruent, or some two consecutive

In view of Theorem 15-13 for alternate angles, Theorem 15-14 for alternate angles will be proved once we show that, given an angle at P, there is just one line through Q coplanar with ℓ such that the alternate angle at Q to the given angle at P is congruent to the given angle. For, since, by Theorem 15-13, the parallel to ℓ through Q is one such line it follows that this parallel is the only such line— so, the coplanar lines of Theorem 15-14 must be parallel. To treat the corresponding angle case of Theorem 15-14 it is, similarly, sufficient to show that, given an angle at P, there is just one line through Q coplanar with ℓ such that the angle at Q corresponding to the given angle at P is congruent to the given angle. Similarly, to treat the adjacent angle case it is sufficient to show that, given an angle at P, there is just one line through Q coplanar with ℓ such that the angle at Q adjacent to the given angle at P is supplementary to the given angle.

In all three cases we know by Theorem 15-13 that the parallel to ℓ through Q "works" and, to prove Theorem 15-14, it is just a matter of showing that there can be no other line through Q coplanar with ℓ which "works".

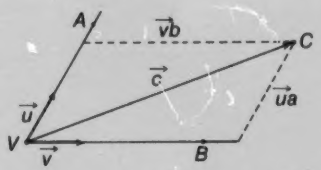
Answers for Part B

1. Using the notation in Exercise 5 of Part A, let $\angle$P be one of the four angles contained in $\ell \cup$ t. Let r be the half-line with vertex Q whose sense is the same as the sense of that side of $\angle$P which is contained in t. By Theorems 15-7 and 15-8 there is just one angle which is congruent with $\angle$P, which has r as one of its sides, and whose other side — say, s — is on the same side of t as the side of $\angle$P not in t. So, the line which contains the half-line s is the only line through Q and coplanar with ℓ such that $\angle$P is congruent with its corresponding angle at Q. Since, by Theorem 15-13, the parallel to ℓ through Q is such a line it follows that the parallel to ℓ through Q is the only such line.
2. Using the notation in Exercise 5 of Part A, let $\angle$P be one of the angles contained in $\ell \cup$ t. Let r be the half-line with vertex Q whose sense is the opposite of the sense of that side of $\angle$P which is contained in t. By Theorems 15-7 and 15-8 there is just one angle which is supplementary to $\angle$P, which has r as one of its sides, and whose other side — say, s — is on the same side of t as the side of $\angle$P not in t. So, the line which contains the half-line s is the only line through Q and coplanar with ℓ such that $\angle$P is supplementary to its consecutive angle at Q. Since, by Theorem 15-13, the parallel to ℓ through Q is such a line it follows that the parallel to ℓ through Q is the only such line.
3. By Theorem 15-14, if some two alternate or corresponding angles are congruent, or some two consecutive angles are supplementary, then the lines are parallel. By 15-13 if the lines are parallel then any two alternate or corresponding angles are congruent, and any two consecutive angles are supplementary.

angles are supplementary, then any two alternate or corresponding angles are congruent and any two consecutive angles are supplementary.

Part C

Given $\angle AVB$, let $\vec{u}$ and $\vec{v}$ be unit vectors in the senses of $A - V$ and $B - V$. Let $\vec{c}$ be the position vector of a point C with respect to V and suppose that C belongs to the plane of $\angle AVB$. More explicitly, suppose that $\vec{c} = \vec{u}a + \vec{v}b$. We wish to investigate the possibility that $\angle AVC$ and $\angle BVC$ are congruent. Show each of the following. [Assume that $C \notin \angle AVB$.]



1. $\angle AVC \cong \angle BVC$ if and only if $\vec{c} \in [\vec{u} + \vec{v}]$ and $\vec{c} \neq \vec{0}$. [*Hint*: Note that $\angle AVC \cong \angle BVC$ if and only if $\vec{c} \cdot \vec{u} = \vec{c} \cdot \vec{v}$. [Why?] What does this tell you about a and b?]
2. For C interior to $\angle AVB$, $\angle AVC \cong \angle BVC$ if and only if $\vec{c} \in [\vec{u} + \vec{v}]^+$.
3. C is equidistant from the lines $\overleftrightarrow{VA}$ and $\overleftrightarrow{VB}$ if and only if $\angle AVC$ and $\angle BVC$ are either congruent or supplementary. [*Hint*: Recall that $d(C, \overleftrightarrow{VA}) = \|\vec{c} - \vec{u}(\vec{c} \cdot \vec{u})\|$. [Why?]]
4. For C interior to $\angle AVB$, $d(C, \overleftrightarrow{VA}) = d(C, \overleftrightarrow{VB})$ if and only if $\angle AVC \cong \angle BVC$.
5. For C interior to $\angle AVB$ and $\angle AVC \cong \angle BVC$,

$$\cos \angle AVC = \sqrt{\frac{1 + \cos \angle AVB}{2}}.$$

6. Given that C is exterior to $\angle AVB$ and $\angle AVC \cong \angle BVC$, express $\cos \angle AVC$ in terms of $\cos \angle AVB$.

15.06 Bisectors of Angles

The exercises of Part C, above, suggest the following:

> Definition 15–7 The bisector of an angle is the half-line interior to it such that the two angles which have this half-line for one side, and one of the sides of the given angle for the other, are congruent.

Note that Exercise 2 of Part C shows that, for any angle, there is one and only one half-line which satisfies the conditions given in Defini-

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Answers for Part C

1. $\angle AVC \cong \angle BVC$ if and only if $\cos\angle AVC = \cos\angle BVC$, and the latter is the case if and only if $\vec{u}\cdot\vec{c}/\|\vec{c}\| = \vec{v}\cdot\vec{c}/\|\vec{c}\|$. So, $\angle AVC \cong \angle BVC$ if and only if $\vec{u}\cdot\vec{c} = \vec{v}\cdot\vec{c}$. Since $\vec{c} = \vec{u}a + \vec{v}b$ it follows that $\angle AVC \cong \angle BVC$ if and only if $a + (\vec{u}\cdot\vec{v})b = (\vec{v}\cdot\vec{u})a + b$ — that is, if and only if $(a - b)(1 - \vec{u}\cdot\vec{v}) = 0$. Since $(\vec{u}, \vec{v})$ is linearly independent [we are "given" $\angle AVB$] it follows that $\vec{u}\cdot\vec{v} \neq 1$ and, so, $\angle AVC \cong \angle BVC$ if and only if $a = b$ — that is, if and only if $\vec{c} = (\vec{u} + \vec{v})a \in [\vec{u} + \vec{v}]$.

2. C is interior to $\angle AVB$, with $\vec{c} = \vec{u}a + \vec{v}b$ if and only if a and b are both positive. Hence, for C interior to $\angle AVB$, $\angle AVC \cong \angle BVC$ if and only if $\vec{c} = (\vec{u} + \vec{v})a$ where $a > 0$ — that is, if and only if $\vec{c} \in [\vec{u} + \vec{v}]^+$.

3. The foot F of the perpendicular from C to $\overleftrightarrow{VA}$ is $V + \vec{u}(\vec{c}\cdot\vec{u})$. So, $d(C, \overleftrightarrow{VA}) = \|C - F\| = \|\vec{c} - \vec{u}(\vec{c}\cdot\vec{u})\| = \sqrt{\|\vec{c}\|^2 - (\vec{c}\cdot\vec{u})^2}$. It follows that $d(C, \overleftrightarrow{VA}) = d(C, \overleftrightarrow{VB})$ if and only if $\|\vec{c}\|^2 - (\vec{c}\cdot\vec{u})^2 = \|\vec{c}\|^2 - (\vec{c}\cdot\vec{v})^2$ — that is, if and only if $(\vec{c}\cdot\vec{u})^2 = (\vec{c}\cdot\vec{v})^2$. This latter is the case if and only if $\vec{c}\cdot\vec{u} = \vec{c}\cdot\vec{v}$ or $\vec{c}\cdot\vec{u} + \vec{c}\cdot\vec{v} = 0$ — that is, if and only if $\angle AVC$ and $\angle BVC$ are congruent or supplementary.

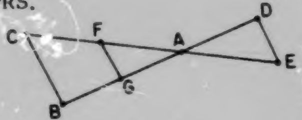
4. By Exercise 3, if $\angle AVC \cong \angle BVC$ then $d(C, \overleftrightarrow{VA}) = d(C, \overleftrightarrow{VB})$. Suppose, then, that C is interior to $\angle AVB$ and that $d(C, \overleftrightarrow{VA}) = d(C, \overleftrightarrow{VB})$. It follows that $\vec{c} = \vec{u}a + \vec{v}b$, with a and b positive, and [by the work for Exercise 3] that $(\vec{c}\cdot\vec{u})^2 = (\vec{c}\cdot\vec{v})^2$. So, $\vec{c}\cdot\vec{u} = a + (\vec{u}\cdot\vec{v})b$, $\vec{c}\cdot\vec{v} = (\vec{u}\cdot\vec{v})a + b$, and $[a + (\vec{u}\cdot\vec{v})b]^2 = [(\vec{u}\cdot\vec{v})a + b]^2$. From the last it is readily shown that $(a^2 - b^2)[1 - (\vec{u}\cdot\vec{v})^2] = 0$. Since $(\vec{u}, \vec{v})$ is linearly independent it follows that $a^2 - b^2 = 0$ and, since a and b are positive, that $a = b$. So, by Exercise 2, $\angle AVC \cong \angle BVC$.

5. Suppose that C is interior to $\angle AVB$ and that $\angle AVC \cong \angle BVC$. It follows by Exercise 2 that $\vec{c} = (\vec{u} + \vec{v})a$, with $a > 0$. So, $\|\vec{c}\|^2 = 2a^2(1 + \vec{u}\cdot\vec{v})$ and $\vec{c}/\|\vec{c}\| = (\vec{u} + \vec{v})/\sqrt{2(1 + \vec{u}\cdot\vec{v})}$. Hence, $\cos\angle AVC = \vec{u}\cdot(\vec{c}/\|\vec{c}\|) = (1 + \vec{u}\cdot\vec{v})/\sqrt{2(1 + \vec{u}\cdot\vec{v})} = \sqrt{1 + \vec{u}\cdot\vec{v}/2}$ $= \sqrt{(1 + \cos\angle AVB)/2}$.

6. In case C is exterior to $\angle AVB$ and $\angle AVC \cong \angle BVC$, the argument in Exercise 5 applies with the exception that $a < 0$ and, so, $\sqrt{a^2} = -a$. So, $\vec{c}/\|\vec{c}\| = -(\vec{u} + \vec{v})/\sqrt{2(1 + \vec{u}\cdot\vec{v})}$, and $\cos\angle AVC$ $= -\sqrt{(1 + \cos\angle AVB)/2}$.

Sample Quiz

1. Suppose that ABCD is a parallelogram.
   (a) Which pairs of angles of ABCD are congruent?
   (b) Which pairs of angles of ABCD are supplementary?
2. Suppose that PQRS is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$.
   (a) Give a supplement of $\angle PQR$.
   (b) Give a supplement of $\angle RSP$.
   (c) Give an angle which is congruent to $\angle PRS$.
   (d) Give an angle congruent to $\angle PQS$.
3. Suppose that $\overline{BC} \parallel \overline{FG} \parallel \overline{DE}$, as shown in the picture at the right.



   (a) Give two angles which are congruent to $\angle CBA$.
   (b) Give two angles which are congruent to $\angle AFG$.
   (c) Give a supplement to $\angle BCF$.
   (d) Give a supplement to $\angle ADE$.
   (e) Give a supplement to $\angle BAC$.

Key to Sample Quiz

1. (a) $\angle DAB$, $\angle DCB$; $\angle ABC$, $\angle ADC$
   (b) $\angle DAB$, $\angle ADC$; $\angle ABC$, $\angle BAC$; $\angle ABC$, $\angle BCD$; $\angle BCD$, $\angle CDA$
2. (a) $\angle QRS$ (b) $\angle SPQ$ (c) $\angle RPQ$ (d) $\angle QSR$
3. (a) $\angle FGA$, $\angle ADE$ (b) $\angle FCB$, $\angle ADE$ (c) $\angle CFG$
   (d) $\angle FGB$ (e) $\angle CAD$

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tion 15–7. So, it is Exercise 2 which justifies our speaking of *the* half-line which satisfies these conditions. Exercise 2 also gives us important information as to what half-line this is:

> **Theorem 15–15** The bisector of $\angle AVB$ is the half-line with vertex V whose sense is that of $\vec{a}/\|\vec{a}\| + \vec{b}/\|\vec{b}\|$, where $\vec{a} = A - V$ and $\vec{b} = B - V$.

Exercise 4 gives us another way of describing the bisector of an angle:

> **Theorem 15–16** The bisector of an angle consists of those points which are interior to the angle and are equidistant from the lines containing the sides of the angle.

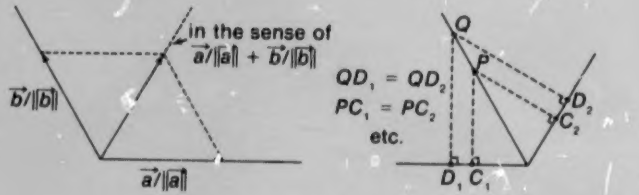
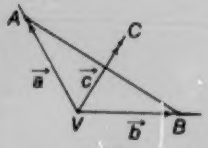


Fig. 15–15

Exercises

Part A

Consider $\angle AVB$ and its bisector $\overrightarrow{VC}$. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be the position vectors of A, B, and C with respect to V.



1. How do you know that $\overrightarrow{VC}$ intersects $\overline{AB}$?
2. In what ratio does the point of intersection of $\overrightarrow{VC}$ and $\overline{AB}$ divide the interval from A to B? [*Hint*: Recall that the point whose position vector is $\vec{a}(1 - r) + \vec{b}r$ divides the interval from A to B in the ratio $r : (1 - r)$, and use Theorem 15–15.]
3. Show that the bisector of $\angle AVB$ bisects the interval $\overline{AB}$ if and only if A and B are equidistant from V.
4. Show that the bisector of $\angle AVB$ is perpendicular to $\overline{AB}$ if and only if A and B are equidistant from V. [*Hint*: Use Theorem 15–15.]

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TC 228

Suggestions for the exercises of section 15.06:

(i) Part A may be used in conjunction with the discussion preceding and following it.

(ii) Parts B and C (except for Exercise 5, Part C) are appropriate for homework.

(iii) Part D should be developed under teacher direction.

(iv) Part F may be used either as a class exercise or as homework.

Answers for Part A

1. C is interior to $\angle AVB$, since C is on its bisector. Therefore, by Theorem 15-2(c), $\overrightarrow{VC}$ intersects $\overline{AB}$.
2. Let D be the point of intersection of $\overrightarrow{VC}$ and $\overline{AB}$. Then $\vec{d} = \vec{a}(1 - r) + \vec{b}r$, where $0 < r < 1$, and $\vec{d} = (\vec{a}/||\vec{a}|| + \vec{b}/||\vec{b}||)c$, where $c > 0$. Solving for 'c', we have $c = ||\vec{b}||\,||\vec{a}||/(||\vec{a}|| + ||\vec{b}||)$. Therefore, $r = ||\vec{a}||/(||\vec{a}|| + ||\vec{b}||)$, and $1 - r = ||\vec{b}||/(||\vec{a}|| + ||\vec{b}||)$. So D divides the interval from A to B in the ratio $||\vec{a}|| : ||\vec{b}||$.
3. By Exercise 2, D is the midpoint of $\overline{AB}$ if and only if $||\vec{a}|| : ||\vec{b}|| = 1$ — that is, if and only if $||\vec{a}|| = ||\vec{b}||$. Since $d(V, A) = ||\vec{a}||$ and $d(V, B) = ||\vec{b}||$, the result follows.
4. By Theorem 15-15, the bisector of $\angle AVB$ is perpendicular to $\overline{AB}$ if and only if $(\vec{a}/||\vec{a}|| + \vec{b}/||\vec{b}||)\cdot(\vec{b} - \vec{a}) = 0$. The latter is the case if and only if $[(\vec{a}\cdot\vec{b})/(||\vec{a}||\,||\vec{b}||) + 1][||\vec{b}|| - ||\vec{a}||] = 0$ and, so, since $\cos\angle AVB \neq -1$, if and only if $||\vec{a}|| = ||\vec{b}||$.

TC 229

Explanations called for in the text: The perpendicular bisector of $\overline{AB}$ is the set of all points equidistant from A and B. It consists of a plane which is perpendicular to $\overline{AB}$ at its midpoint. The bisector of the given angle is a half-line which is clearly a subset of the plane just described.

Placing the point of a pair of compasses at the vertex of an angle and drawing an arc which cuts both sides of the angle, two points of the angle are located each of which is equidistant from the vertex of the angle.

Exercises 3 and 4 suggest a procedure for drawing bisectors of angles. If one marks two points A and B which are on the sides of an angle and are equidistant from its vertex V then, by Exercises 3 and 4, the bisector of the angle is contained in the perpendicular bisector of $\overline{AB}$. [Explain.] More specifically, the bisector of the angle is contained in the line of intersection of the plane of the angle and the perpendicular bisector of $\overline{AB}$. The point V belongs to the line and so does any point of the plane of the angle which is equidistant from A and B. When drawing figures, it is easy to mark such points by using compasses. [Explain.] Draw some angles and practice the procedure just outlined for drawing their bisectors.

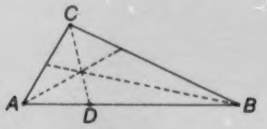


Fig. 15–16

The exercises of Part A suggest a number of theorems concerning triangles. To state these it is convenient to introduce a notion somewhat like the notions of medians and altitudes. Briefly, the angle bisector of $\triangle ABC$ *from* the vertex C, or *to* the side $\overline{AB}$, is the *interval* of the bisector of $\angle BCA$ which has as endpoints the vertex C and the point D in which the bisector of $\angle BCA$ intersects $\overline{AB}$. Note that, like medians of triangles and altitudes of triangles, angle bisectors of triangles are intervals.

When dealing with a triangle it is convenient to describe the three angles each of whose sides contains a side of the triangle as being angles *of* the triangle. For example, $\angle BCA$ is an angle *of* $\triangle ABC$. More specifically, $\angle BCA$ is the angle of $\triangle ABC$ *at* C. Similarly, $\angle CAB$ is the angle of $\triangle ABC$ at A. [Make a similar assertion concerning the third angle of $\triangle ABC$.]

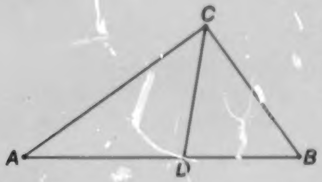
We can now state the following definition:

> Definition 15–8 The angle bisector of a triangle from a given vertex is the interval whose endpoints are the given vertex and the point at which the bisector of the angle of the triangle at this vertex intersects the opposite side.

Note that your answer for Exercise 1 of Part A should be relevant to the question as to whether Definition 15–8 is acceptable as a definition. [Explain.]

The result of Exercise 2 of Part A can be stated as follows:

> Theorem 15–17 In $\triangle ABC$, the endpoint on $\overline{AB}$ of the angle bisector from C divides the interval from A to B in the ratio of CA to CB.



$\overline{CD}$ is the angle bisector from C.

D divides the interval from A to B in the ratio CA:CB.

Fig. 15–17

The results of Exercises 3 and 4 yield the following theorems:

> Theorem 15–18 $\triangle ABC$ is isosceles with base $\overline{AB}$ if and only if its angle bisector from C is its median from C.

> Theorem 15–19 $\triangle ABC$ is isosceles with base $\overline{AB}$ if and only if its angle bisector from C is its altitude from C.

[Of what earlier theorem do these theorems remind you?] There is another theorem about the angle bisectors of a triangle which follows at once from Theorem 15–17 and Ceva's Theorem. [For the latter see page 367 of Volume 1.] From your experience with medians, altitudes, and perpendicular bisectors you may guess that this theorem is:

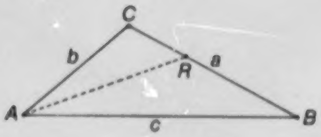
> Theorem 15–20 The angle bisectors of a triangle are concurrent.

This theorem follows easily from Theorem 15–17 and Ceva's Theorem. And, as you will see, there is a slightly longer proof using Theorem 15–16.

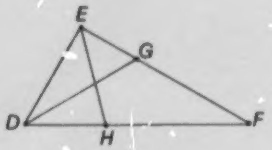
Part B

1. Use your answer for Exercise 2 of Part A to obtain a proof of Theorem 15–17.

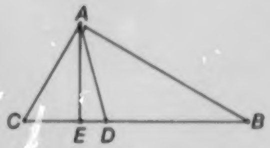
2. Suppose that, in $\triangle ABC$, $\overline{AR}$ is the angle bisector from A. Suppose, as usual, that $BC = a$, $CA = b$, and $AB = c$.
   (a) What is the ratio of BR to RC?
   (b) Suppose that $a = 4$, $b = 3$, and $c = 6$. What is BR? RC? [Check your answers by adding them.]



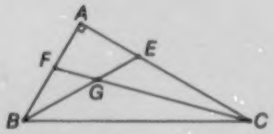
3. Ceva's Theorem says that if R, S, and T are points of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, then $\overline{AR}$, $\overline{BS}$, and $\overline{CT}$ are concurrent if and only if $BR \cdot CS \cdot AT = RC \cdot SA \cdot TB$. [Draw a picture.] Use Ceva's Theorem and Theorem 15–17 to prove Theorem 15–20.
4. In each of the following, you are given a picture of a triangle and some information about it. Do the required computations.
   (a) Given: $DE = 5$, $EF = 6$, and $FD = 8$; $\overline{DG}$ and $\overline{EH}$ are angle bisectors.
   Compute: DH, HF, FG, and GE



   (b) Given: Right triangle, $\triangle ABC$, with hypotenuse $\overline{BC}$, $AB = 8$ and $AC = 6$; $\overline{AE}$ is the altitude from A and $\overline{AD}$ is the angle bisector from A.
   Compute: CE, AE, CD, and AD



   (c) Given: Same information as in (b); $\overline{CF}$ is the angle bisector from C; G is the point of intersection of $\overline{CF}$ and $\overline{AD}$.
   Compute: AF, FB, CF, CG/GF, and AG/GD
   (d) Given: Right triangle, $\triangle ABC$, with hypotenuse $\overline{BC}$; $AB = 5$ and $AC = 12$; angle bisectors $\overline{BE}$ and $\overline{CF}$ intersect in the point G.
   Compute: BE, CF, CG, and GE

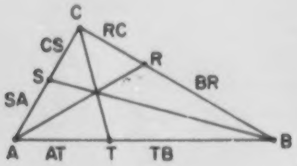


Answers for Part B

1. By Exercise 2 [but using the notation of Theorem 15-17] the end point on $\overline{AB}$ of the angle bisector from the opposite vertex divides the interval from A to B in the ratio $\|\vec{a}\|/\|\vec{b}\|$, where $\vec{a} = A - C$ and $\vec{b} = B - C$. Since $\|A - C\| = CA$ and $\|B - C\| = CB$, the theorem follows.

2. (a) c/b (b) $8/3$; $4/3$

3. $\overline{AR}$, $\overline{BS}$, and $\overline{CT}$ are concurrent if and only if $BR \cdot CS \cdot AT = RC \cdot SA \cdot TB$.



If $\overline{AR}$ is the angle bisector from A of $\triangle ABC$ then, by Theorem 15-17, $BR = ac/(b + c)$ and $RC = ab/(b + c)$. If $\overline{BS}$ is the angle bisector from B then $CS = ba/(c + a)$ and $SA = bc/(c + a)$. If $\overline{CT}$ is the angle bisector from C then $AT = cb/(a + b)$ and $TB = ca/(a + b)$. So, the angle bisectors are concurrent if and only if

$$\frac{ac}{b+c} \cdot \frac{ba}{c+a} \cdot \frac{cb}{a+b} = \frac{ab}{b+c} \cdot \frac{bc}{c+a} \cdot \frac{ca}{a+b}.$$

Since the latter is always the case, the angle bisectors of a triangle are concurrent.

4. (a) $DH = 40/11$; $HF = 48/11$; $FG = 48/13$; $GE = 30/13$

   (b) $CE = 18/5$; $AE = 24/5$; $CD = 30/7$; $AD = 24\sqrt{2}/7$
   $[(AD)^2 = (CD - CE)^2 + (AE)^2]$

   (c) $AF = 3$; $FB = 5$; $CF = 3\sqrt{5}$; $CG/GF = 2$ [Theorem 15-17, $\triangle ACF$]; $AG/GD = 7/5$

   (d) $BE = 5\sqrt{13}/3$ $[(BE)^2 = (AE)^2 + (AB)^2]$; $CF = 12\sqrt{26}/5$; $CG = 2\sqrt{26}$; $GE = 2\sqrt{13}/3$

5. Prove Theorem 15–20 as suggested below. Begin by considering $\triangle ABC$ whose angle bisectors are $\overline{AR}$, $\overline{BS}$, and $\overline{CT}$.
   (a) Show that there is a point—say, P—which belongs to both $\overline{AR}$ and $\overline{BS}$. [*Hint*: For this, all you need know is that $R \in \overline{BC}$ and $S \in \overline{CA}$. What theorem?]
   (b) Show that P is equidistant from $\overleftrightarrow{CA}$ and $\overleftrightarrow{CB}$.
   (c) What else do you need to know in order to infer that P belongs to the bisector of $\angle C$?
   (d) Show that P belongs to the bisector of $\angle C$. [*Hint*: It is relevant that $P \in \overline{AR}$ and that $R \in \overrightarrow{CB}$.]
   (e) Show that $P \in \overline{CT}$.

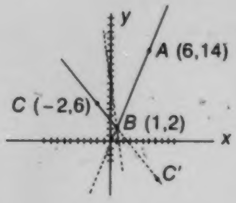
Part C

1. Show that the lines containing the bisectors of adjacent supplementary angles are perpendicular.
2. Given two intersecting lines, describe the set consisting of all points which are in the plane of these lines and are equidistant from these lines.
3. Suppose that π is the plane containing two lines l and m which intersect at a point V. Show that a point P is equidistant from l and m if and only if the foot, F, of the perpendicular from P to π is equidistant from l and m. [*Hint*: Show that the plane perpendicular to l which contains P also contains F. So, obtain a relation between $d(P, l)$, $d(F, l)$, and PF.]
4. Describe the set of all points which are equidistant from two intersecting lines.

*5. Show that the lines containing the bisectors of two coplanar angles which share a side are perpendicular if and only if the angles are adjacent supplementary angles. [*Hint*: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be the unit vectors in the senses of the sides of the angle, $\vec{w}$ being in the sense of the shared side. Since $(\vec{u}, \vec{w})$ is linearly independent and the angles are coplanar, $\vec{v} = \vec{u}a + \vec{w}b$. Show that $\vec{u} + \vec{w} \perp \vec{v} + \vec{w}$ [Why?] if and only if $a + b + 1 = 0$—that is, if and only if $\vec{v} = \vec{u}a - \vec{w}(a + 1)$. Use the fact that $\|\vec{v}\|^2 = 1$ (and the linear independence of $(\vec{u}, \vec{w})$ and $(\vec{v}, \vec{w})$) to show that $\vec{v} = \vec{u}a - \vec{w}(a + 1)$ if and only if $\vec{v} = -\vec{u}$.]

***Part D**

Suppose that A, B, and C are points of a plane σ and that, with respect to the orthonormal coordinate system for σ which is shown in the figure, they have the indicated coordinates. We must find an equation for the line which contains the bisector of $\angle ABC$.



Answers for Part B [cont.]

5. (a) By the corollary to Theorem 15-2, $\overline{AR}$ and $\overline{BS}$ intersect. Let P be the point of intersection.
   (b) Since P belongs to the angle bisector from A, $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{AC})$. Since P belongs to the angle bisector from B, $d(P, \overleftrightarrow{BA}) = d(P, \overleftrightarrow{BC})$. So, P is equidistant from $\overleftrightarrow{CA}$ and $\overleftrightarrow{CB}$.
   (c) In order to use Theorem 15-16 we need to know that D is interior to $\angle C$.
   (d) Since $R \in \overrightarrow{CB}$, $\angle C = \angle RCA$. So, since $P \in \overline{AR}$, P is interior to $\angle C$. [Theorem 15-2(a)] Since, also, P is equidistant from the sides of $\angle C$ it follows that P belongs to the bisector of $\angle C$.
   (e) As in part (a), $\overline{CT}$ and $\overline{AR}$ intersect. Since $P \in \overrightarrow{CT} \cap \overline{AR}$, P must be the point of intersection of $\overline{CT}$ and $\overline{AR}$.

   [By (a) and (e), the angle bisectors of $\triangle ABC$ are concurrent at P.]

Answers for Part C

1. Given two adjacent supplementary angles, let $\vec{u}$ be the unit vector in the sense of their common side. Then the unit vectors in the senses of their other sides are opposites — say, $\vec{v}$ and $-\vec{v}$. The senses of the two angle bisectors are, by Theorem 15-15, that of $\vec{u} + \vec{v}$ and $\vec{u} + -\vec{v}$, respectively. Since $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$ it follows that the angle bisectors are perpendicular.

2. Two intersecting lines contain the sides of four angles, and the points equidistant from the two lines and in their plane consist of the point of intersection and the points in the bisectors of the four angles.

3. The case in which $P \in \pi$ is trivial. So, suppose that $P \notin \pi$ and let F be the foot of the perpendicular from P to π. Let σ be the plane perpendicular to ℓ which contains P. Since $\overleftrightarrow{PF} \perp \pi$ and $\ell \subseteq \pi$ it follows that $\overleftrightarrow{PF} \perp \ell$ and, so, that $\overleftrightarrow{PF} \subseteq \sigma$. Hence, $F \in \sigma$. Now, by the Pythagorean theorem, $d(P, \ell)^2 = d(P, F)^2 + d(F, \ell)^2$ and, similarly, $d(P, m)^2 = d(P, F)^2 + d(F, m)^2$. Hence, $d(P, \ell) = d(P, m)$ if and only if $d(F, \ell) = d(F, m)$.

4. The set of all points equidistant from two intersecting lines is the union of two perpendicular planes each of which is perpendicular to the plane of the given lines and containing the bisectors of the angles contained in the union of the two lines.

5. In Exercise 1 we proved that the angle bisectors of adjacent supplementary angles are perpendicular. So, all that remains is to show that if the angle bisectors of coplanar angles sharing a side are perpendicular then the angles are adjacent and supplementary — that is, that the noncommon sides of these angles are oppositely sensed half-lines. The assumption that the angles are coplanar and share a side is embodied in the notation introduced in the hint — in particular, that $\vec{v} = \vec{u}a + \vec{w}b$. The assumption that the angle bisectors are perpendicular amounts to saying that $\vec{u} + \vec{w}$ and $\vec{v} + \vec{w}$ are orthogonal —that is, that $(\vec{u} + \vec{w}) \cdot [\vec{u}a + \vec{w}(b + 1)] = 0$. This last reduces to '$(a + b + 1)(1 + \vec{u} \cdot \vec{w}) = 0$'. So, since $(\vec{u}, \vec{w})$ is linearly independent, we have that $a + b + 1 = 0$ and, so, that $\vec{v} = \vec{u}a - \vec{w}(a + 1)$. Since $\vec{v}$ is a unit vector it follows [on computing the square of the norm of $\vec{v}$] that $2a(a + 1)(1 - \vec{u} \cdot \vec{w}) = 0$. Since $(\vec{u}, \vec{w})$ is linearly independent it follows that $a(a + 1) = 0$. Since $(\vec{v}, \vec{w})$ is linearly independent, and $\vec{v} = \vec{u}a + \vec{w}b$, it follows that $a \neq 0$. So, $a = -1$ and $\vec{v} = -\vec{u}$.

1. Find equations of the form '$ax + by + c = 0$' for the lines $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$.
2. Show that if P is a point which belongs to σ, has coordinates (x, y), and is on the same side of $\overleftrightarrow{AB}$ as is C then

$$d(P, \overleftrightarrow{AB}) = -\frac{12x - 5y - 2}{13}.$$

[*Hint*: Recall Theorem 14-17 on page 175 and the discussion preceding it.]

3. Show that if P is on the same side of $\overleftrightarrow{BC}$ as is A then

$$d(P, \overleftrightarrow{BC}) = \frac{4x + 3y - 10}{5}.$$

4. Given that $B \in \overline{CC'}$, find an equation for the line which contains the bisector of $\angle ABC$.
5. Find an equation for the line which contains the bisector of $\angle ABC'$,
   (a) by using Theorem 15-16, and
   (b) by using Exercise 1 of Part C and Exercise 4, above.

*6. Find equations for the lines which contain the bisectors of $\angle CAB$ and of $\angle BCA$, respectively. [*Hint*: First find an equation for $\overleftrightarrow{AC}$. Don't expect your answers to be as "neat" as those for Exercises 4 and 5.]

*7. Use the equations obtained in Exercises 4 and 6 to show that the lines containing the bisectors of the angles of $\triangle ABC$ are concurrent. [*Hint*: The easiest way to do this is to show that the left side of an equation for the bisector of $\angle B$ is a linear combination of the left sides of equations for the bisectors of $\angle A$ and $\angle C$. Of course, the equations must be in the form '$ax + by + c = 0$'.]

Part E

As in Part D, all points referred to in the following exercises belong to a plane σ and all equations and coordinates are with respect to an orthonormal coordinate system for σ.

1. Suppose that $\overrightarrow{QR}$ and $\overrightarrow{QP}$ are described by the equations:

$$12x + 5y - 3 = 0, \quad 4x - 3y + 5 = 0$$

Suppose, also, that the point with coordinates $(1, 1)$ is interior to $\angle PQR$. Find an equation for the line containing the bisector of $\angle PQR$.

2. Suppose that U, V, and W have coordinates $(4, 4)$, $(0, 1)$, and $(5, 13)$, respectively.
   (a) Find an equation for the line which contains the bisector of $\angle UVW$.
   (b) Find an equation for the line which contains the bisector of the adjacent supplements of $\angle UVW$.

Answers for Part D

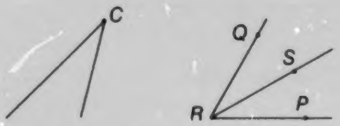
1. $\overleftrightarrow{BA}$: $12x - 5y - 2 = 0$; $\overleftrightarrow{BC}$: $4x + 3y - 10 = 0$
2. We can use Theorem 14-17 by considering the equation of $\overleftrightarrow{BA}$ to be an equation for the plane π containing $\overleftrightarrow{BA}$ and perpendicular to σ. For any $P \in \sigma$, $d(P, \overleftrightarrow{BA}) = d(P, \pi)$. So, by Theorem 14-17, $d(P, \pi) = |12x - 5y - 2|/\sqrt{12^2 + 5^2}$. For P on the same side of $\overleftrightarrow{AB}$ as is C it follows by the second part of Theorem 14-17 that the coordinates (x, y) of P must be such that $(12x - 5y - 2)(12 \cdot -2 - 5 \cdot 6 - 2) > 0$ — that is, such that $12x - 5y - 2 < 0$. So, for such points, $d(P, \overleftrightarrow{AB}) = -(12x - 5y - 2)/13$.
3. By reasoning similar to that in Exercise 2 we see that if P is on the same side of $\overleftrightarrow{BC}$ as is A then $d(P, \overleftrightarrow{BC}) = (4x + 3y - 10)/5$.
4. By Exercises 2 and 3, P is interior to $\angle ABC$ and equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ if and only if, for its coordinates (x, y), $-(12x - 5y - 2)/13$ and $(4x + 3y - 10)/5$ are equal and positive. [If they are equal and negative then P is on the bisector of $\angle ABC$'s vertical angle.] So, the equation '$-(12x - 5y - 2)/13 = (4x + 3y - 10)/5$' is an equation of a line which contains the bisector of $\angle ABC$. A simpler equation is '$8x + y - 10 = 0$'.
5. (a) In this case we are interested in points on the same side of $\overleftrightarrow{BC}$ as is A but on the opposite side of $\overleftrightarrow{AB}$ from C. So, by arguments like those in Exercises 2, 3, and 4 we find the equation '$(12x - 5y - 2)/13 = (4x + 3y - 10)/5$' or, more simply, '$x - 8y + 15 = 0$'.
   (b) By Exercise 1, the line we seek is perpendicular to that of Exercise 4 and, so, has slope $1/8$. Since it also contains B its point-slope equation is '$y - 2 = (x - 1)/8$' which simplifies to '$x - 8y + 15 = 0$'.
6. $(13 + 12\sqrt{2})x - (13 + 5\sqrt{2})y + (104 - 2\sqrt{2}) = 0$;
   $(5 - 4\sqrt{2})x - (5 + 3\sqrt{2})y + (40 + 10\sqrt{2}) = 0$
7. $[(13 + 12\sqrt{2})x - (13 + 5\sqrt{2})y + (104 - 2\sqrt{2})]5 - [(5 - 4\sqrt{2})x - (5 + 3\sqrt{2})y + (40 + 10\sqrt{2})]13 = [8x + y - 10]14\sqrt{2}$. So, any point common to the lines containing the bisectors of $\angle A$ and $\angle C$ also belongs to the line containing the bisector of $\angle B$. There is a point common to the first two lines since they are coplanar and, as their equations show, are not parallel. $[(13 + 12\sqrt{2}) \cdot -(5 + 3\sqrt{2}) - -(13 + 5\sqrt{2})(5 - 4\sqrt{2}) \neq 0]$

Answers for Part E

1. $x + 8y - 10 = 0$
2. (a) $9x - 7y + 7 = 0$ (b) $7x + 9y - 9 = 0$

15.07 Ordering Angles by Size

Whatever method we might use to sort angles according to size, we would probably agree that congruent angles have the same size. Of two noncongruent angles it is customary to say that the "sharper" one is the smaller and the "blunter" one is the larger. It is easy to judge the relative sharpness of angles which are related as are $\angle PRS$ and $\angle PRQ$ in the figure. This suggests that we adopt:



$\angle C$ is smaller than $\angle PRQ$
S is interior to $\angle PRQ$ and $\angle PRS \cong \angle C$

Fig. 15–18

> Definition 15–9 $\angle C$ is smaller than $\angle R$ [and $\angle R$ is larger than $\angle C$] if and only if $\angle C$ is congruent to an angle which shares a side with $\angle R$ and whose other side is interior to $\angle R$.

Recall that, given $\angle C$ and $\angle PRQ$, there is one and only one angle – say, $\angle PRS$ – which has $\overrightarrow{RP}$ for a side, which is congruent with $\angle C$, and whose other side is contained in the same side of $\overleftrightarrow{RP}$ as is Q. [What theorems?] Moreover, the side $\overrightarrow{RS}$ is interior to $\angle PRQ$ if and only if $\cos \angle PRS > \cos \angle PRQ$. [Why?] So, by Definition 15–9 and Theorem 15–8, we have:

> Theorem 15–21 A first angle is smaller than a second if and only if the cosine of the first angle is greater than the cosine of the second.

It is also worth recording here a theorem which is almost an immediate consequence of Definition 15–9:

> Theorem 15–22 If C is interior to $\angle AVB$ then $\angle AVC$ is smaller than $\angle AVB$.

[Explain. Why 'almost'?]

For Theorem 15-22 we need, besides Definition 15-9, the fact that an angle [$\angle AVC$] is congruent to itself and the fact that if C is interior to $\angle AVB$ then so is each point of $\overrightarrow{VC}$.

The properties of isometries which underlie properties (i) - (iii) of congruence are, for (i), that $\overrightarrow{0}$ is an isometry, for (ii), that the inverse of an isometry is an isometry, and for (iii), that a resultant of isometries is an isometry.

Suggestions for the exercises of section 15.07:
(i) Part A and the discussion preceding it should be treated in class.
(ii) Part B and Exercises 1-6 of Part C can be used for homework.
(iii) Exercise 6 of Part C and Part D should be teacher directed.

We have seen in the preceding chapter that the relation of being congruent to has some of the properties of the relation of being identical to. For example,

(i) $\angle A \cong \angle A$ $\qquad$ $\angle A = \angle A$

(ii) $\angle A \cong \angle B \longrightarrow \angle B \cong \angle A$ $\qquad$ $\angle A = \angle B \longrightarrow \angle B = \angle A$

(iii) $(\angle A \cong \angle B \text{ and } \angle B \cong \angle C) \longrightarrow \angle A \cong \angle C$ $\qquad$ $(\angle A = \angle B \text{ and } \angle B = \angle C) \longrightarrow \angle A = \angle C$

In other words, both congruence and identity are (i) reflexive, (ii) symmetric, and (iii) transitive. That identity has these properties follows from our rules of logic, alone; that congruence has these properties follows [by our rules of logic] from properties of isometries. [What properties?]

The same properties of congruence [at least, for angles] also follow from the corresponding properties of identity and Theorem 15–8. This suggests that, by using Theorem 15–21, we can obtain properties of the relation of being smaller than [for angles] from properties of the greater-than relation for real numbers.

Exercises

Part A

Use Theorems 15–8 and 15–21 [and real number theorems] to establish each of the following. In each case, give the corresponding postulate or other theorem concerning greater-than.

1. Of two noncongruent angles, one is smaller than the other.
2. No angle is smaller than itself.
3. If a first angle is smaller than a second, and the second is smaller than a third, then the first angle is smaller than the third.
4. If $\angle A \cong \angle B$, and $\angle B$ is smaller than $\angle C$, then $\angle A$ is smaller than $\angle C$.

Part B

1. Prove:

> Theorem 15–23 If, in $\angle ABC$ and $\angle PQR$, $QP = BA$ and $QR = BC$ then $\angle PQR$ is smaller than $\angle ABC$ if and only if $PR < AC$.

2. Suppose that $\triangle ABC$ is an isosceles triangle with base $\overline{BC}$, that $AB = 5 = AC$, and that $BC = 6$.
   (a) Make use of Theorem 15–23 to show that $\angle ABC$ is not smaller than $\angle ACB$.

Answers for Part A

1. Since, of two real numbers, one is greater than the other it follows, by Theorem 15-21 that, of two angles with different cosines, one is smaller than the other. And, by Theorem 15-8, noncongruent angles have different cosines.

2. Since no real number is greater than itself, the cosine of no angle is greater than itself. So, by Theorem 15-21, no angle is smaller than itself.

3. Suppose that a first angle is smaller than a second and the second is smaller than a third. It follows, by Theorem 15-21, that the cosine of the first angle is greater than the cosine of the second angle and the cosine of the second angle is greater than the cosine of the third. Since if a first real number is greater than a second, and the second is greater than a third, then the first real number is greater than the third it follows that the cosine of the first angle is greater than the cosine of the third angle. So, by Theorem 15-21, the first angle is smaller than the third angle. Hence, if a first angle is smaller than a second, and the second is smaller than a third, then the first angle is smaller than the third.

4. Suppose that $\angle A \cong \angle B$ and $\angle B$ is smaller than $\angle C$. It follows, by Theorem 15-8, that $\cos \angle A = \cos \angle B$ and, by Theorem 15-21, that $\cos \angle B > \cos \angle C$. Hence, $\cos \angle A > \cos \angle C$ and, so, by Theorem 15-21, $\angle A$ is smaller than $\angle C$. Hence, if $\angle A \cong \angle B$, and $\angle B$ is smaller than $\angle C$, then $\angle A$ is smaller than $\angle C$.

Answers for Part B

1. By Exercise 2 of Part D on page 218, $\cos \angle Q > \cos \angle B$ if and only if $PR < AC$. So, by Theorem 15-21, $\angle Q$ is smaller than $\angle B$ if and only if $PR < AC$.

2. (a) Consider $\angle ABC$ and $\angle ACB$. Since $AB = AC$ and $BC = CB$, the conditions of Theorem 15-23 hold. But $AC \not< AB$, therefore, $\angle ABC$ is not smaller than $\angle ACB$.

(b) Now, show that $\angle ACB$ is not smaller than $\angle ABC$.

(c) What do the results in (a) and (b) suggest? Is this a familiar result?

(d) Let D be the point of $\overline{BC}$ such that $BD = 5$. Show that $AD < 6$, and use this fact to show that $\angle B$ is smaller than $\angle BAC$.

(e) Is the angle opposite the base of an isosceles triangle always larger than the other angles of the triangle? Explain.

3. Suppose that $\triangle ABC$ is such that $AB = 5$, $BC = 6$, and $AC = 8$. Let D be the point on $\overline{AC}$ such that $AD = 6$. [Draw a picture.]

   (a) Show that $BD < 8$.

   (b) Show that $\angle A$ is smaller than $\angle ABC$.

4. Using the procedure suggested in Exercise 3, give an argument to support the following:

   > If a first side of a triangle is longer than a second side of the triangle then the angle opposite the first side is larger than the angle opposite the second side.

5. In each of the following, you are given the lengths of the three sides of a triangle. In each case, you are to order the angles of the triangle according to size from largest to smallest.

   (a) $AB = 3, AC = 6, BC = 8$ (b) $AB = 13, BC = 12, AC = 5$

   (c) $AB = 10, BC = 10, AC = 8$ (d) $AB = 10, BC = 10, AC = 12$

*

> **Definition 15-10**
>
> (a) A right angle is an angle whose sides are perpendicular.
>
> (b) An acute angle is an angle which is smaller than a right angle.
>
> (c) An obtuse angle is an angle which is larger than a right angle.

Part C

1. Show that an angle is a right angle if and only if its cosine is 0, is an acute angle if and only if its cosine is positive, and is an obtuse angle if and only if its cosine is negative.

2. (a) Suppose that an angle is obtuse. What kind of an angle is a supplement of the given angle? Explain.

   (b) What kind of an angle is a supplement of an acute angle? Explain.

   (c) Is there an angle which is its own supplement? Justify your answer.

3. Show that any two right angles are congruent, any angle congruent to an acute angle is acute, and any angle congruent to an obtuse angle is obtuse.

Answers for Part B [cont.]

2. (b) Since $AB \not< AC$, $\angle ACB$ is not smaller than $\angle ABC$.

   (c) It follows from Exercise 1 of Part A and parts (a) and (b) that $\angle ACB \cong \angle ABC$. Yes, it is a familiar result that the base angles of an isosceles triangle are congruent.

   (d) Since $AD < DC + AC = 1 + 5$ it follows that $AD < 6$. Now, consider $\triangle DBA$ and $\triangle BAC$. $BD = AB$, $BA = AC$, and $DA < BC$. So, by Theorem 15-23, $\angle DBA$ is smaller than $\angle BAC$.

   (e) No. The angle opposite the base of an isosceles triangle is larger than the other angles if and only if the base is longer than the other sides.

3. (a) Since $BD < CD + BC = 2 + 6$ it follows that $BD < 8$.

   (b) Consider $\triangle BAD$ and $\triangle ABC$. $DA = CB$, $AB = BA$, and $BD < CA$. So, by Theorem 15-23, $\angle A$ is smaller than $\angle ABC$.

4. In $\triangle ABC$, suppose that $AC > BC$. Let D be the point of $\overline{AC}$ such that $AD = BC$. Then $BD < DC + BC = (AC - BC) + BC = AC$. So, $BD < AC$. Comparing $\angle DAB$ and $\angle ABC$, we have $AD = BC$ and $AB = AB$. Since $BD < AC$, it follows that $\angle DAB$ is smaller than $\angle ABC$.

5. (a) $\angle A, \angle B, \angle C$ (b) $\angle C, \angle A, \angle B$ (c) $\angle C \cong \angle A, \angle B$ (d) $\angle B, \angle C \cong \angle A$

In connection with Definition 15-10(a) you might point out that we may now say that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$ if and only if $\angle C$ is a right angle. [Compare with Definition 14-[illegible].]

*

Answers for Part C

1. If $\vec{u}$ and $\vec{v}$ are unit vectors in the senses of the sides of an angle then the cosine of the angle is $\vec{u} \cdot \vec{v}$ and is 0 if and only if $\vec{u} \perp \vec{v}$ — that is, if and only if the sides are perpendicular. By definition, the last is the case if and only if the angle is a right angle. It follows, using Theorem 15-21 and Definition 15-10, that an angle is acute if and only if its cosine is greater than 0, and is obtuse if and only if its cosine is less than 0. So, an angle is acute if and only if its cosine is positive, and is obtuse if and only if its cosine is negative.

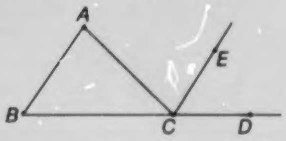
2. (a) A supplement of an obtuse angle is an acute angle. For $\angle B$ is a supplement of $\angle A$ if and only if $\cos\angle B + \cos\angle A = 0$ and if the latter holds for $\cos\angle A < 0$ then $\cos\angle B > 0$.

   (b) A supplement of an acute angle is an obtuse angle. For if $\cos\angle B + \cos\angle A = 0$, where $\cos\angle A > 0$, then $\cos\angle B < 0$.

   (c) An angle is its own supplement if and only if it is a right angle. For $\cos\angle A + \cos\angle A = 0$ if and only if $\cos\angle A = 0$, and $\cos\angle A = 0$ if and only if $\angle A$ is a right angle.

3. Any two right angles have the same cosine, to wit, 0. So, by Theorem 15-8, any two right angles are congruent. Since congruent angles have the same cosine it follows that an angle congruent to an angle whose cosine is positive has a positive cosine and an angle congruent to an angle whose cosine is negative has a negative cosine. So, by Exercise 1, an angle congruent to an acute angle is acute and an angle congruent to an obtuse angle is obtuse.

4. Consider $\triangle ABC$, with D any point such that $C \in \overline{BD}$, and with $\overrightarrow{CE}$ in the sense of $\overrightarrow{BA}$, as shown in the picture at the right.

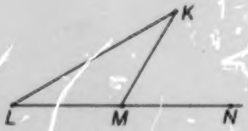


   (a) Show that $\angle ACE \cong \angle BAC$ and $\angle ECD \cong \angle ABC$.

   (b) Which is larger, $\angle ACE$ or $\angle ACD$? Explain.

   (c) Which is the larger of $\angle ACD$ and $\angle BAC$? Of $\angle ACD$ and $\angle ABC$? Explain.

5. Consider $\triangle KLM$, with obtuse angle $\angle LMK$, and with N any point such that $M \in \overline{LN}$, as shown in the picture at the right.



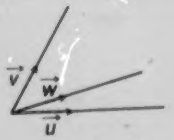
   (a) How do you show that $\angle KMN$ is larger than either $\angle L$ or $\angle K$?

   (b) Can a triangle have more than one of its angles obtuse? Explain.

   (c) Can a triangle have more than one of its angles right? Explain.

   (d) Given that a triangle has a right angle, what can you say about the other two angles of the triangle?

6. Show that the common side of adjacent acute angles is interior to the angle whose sides are the noncommon sides of the acute angles. [*Hint*: Suppose that $\vec{w}$ is the unit vector in the sense of the common side and that $\vec{u}$ and $\vec{v}$ are unit vectors in the senses of the other sides. Use Exercise 1 of Part B on page 208 and Exercise 1 of Part C on page 212.]



*7. Show that the common side of adjacent angles is interior to the angle whose sides are the noncommon sides of the adjacent angles if and only if the sum of the cosines of these angles is positive.

Part D

1. Suppose that $(\vec{u}, \vec{v})$ is orthonormal and that $\vec{w}$ is a unit vector in $[\vec{u}, \vec{v}]$. Show that $(\vec{w} \cdot \vec{u})^2 + (\vec{w} \cdot \vec{v})^2 = 1$.

2. Suppose that $(\vec{u}, \vec{v})$ is orthonormal and that $\vec{w}$ is a unit vector such that $(\vec{w} \cdot \vec{u})^2 + (\vec{w} \cdot \vec{v})^2 = 1$. Show that $\vec{w} \in [\vec{u}, \vec{v}]$. [*Hint*: Let $\vec{k}$ be a unit vector orthogonal to both $\vec{u}$ and $\vec{v}$.]

Answers for Part C [cont.]

4. (a) $\angle ACE \cong \angle BAC$ because these angles are alternate interior angles formed by the transversal $\overleftrightarrow{AC}$ of the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$. $\angle ECD \cong \angle ABC$ because these angles are corresponding angles formed by the transversal $\overleftrightarrow{BC}$ of the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$.

   (b) $\angle ACD$ is larger than $\angle ACE$. This will follow from Theorem 15-22 once we have shown that E is interior to $\angle ACD$. To see this, begin by noting that [as in Exercise 2 of Part A on page 225], since $\overrightarrow{BA}$ and $\overrightarrow{CE}$ have the same sense, A and E are on the same side of $\overleftrightarrow{CD}$. Since, similarly, B and E are on opposite sides of $\overleftrightarrow{AC}$, D and E are on the same side of $\overleftrightarrow{AC}$. Hence, E is interior to $\angle ACD$. [See Exercise 4 of Part C on page 212.]

   (c) Since $\angle BAC \cong \angle ACE$ and $\angle ACD$ is larger than $\angle ACE$ it follows that $\angle ACD$ is larger than $\angle BAC$. Since $\angle ABC \cong \angle ECD$ and $\angle ACD$ is larger than $\angle ECD$ [by the same argument as given for part (b)] it follows that $\angle ACD$ is larger than $\angle ABC$.

5. (a) The arguments given in answer to Exercise 4 apply to any triangle — in particular to $\triangle KLM$.

   (b) No. Referring to the notation of the figure, if $\angle KML$ is obtuse then, by Exercise 2(a), $\angle KMN$ is acute and, by Exercise 4, $\angle K$ and $\angle L$ are smaller than $\angle KMN$. And, an angle smaller than an acute angle cannot be obtuse [since a negative number cannot be larger than a positive number].

   (c) No, by the same reasoning as in (b) [and since a negative number cannot be greater than 0].

   (d) If a triangle has a right angle [or an obtuse angle] then its other two angles are acute.

6. Using the notation of the hint it follows, since the angles are adjacent, that $\vec{v} = \vec{w}a + \vec{u}b$, where $b < 0$. [See Definition 15-4 and Exercise 1 of Part C on page 212.] Since the angles are acute we know that $\vec{w} \cdot \vec{v} > 0$ and $\vec{w} \cdot \vec{u} > 0$. Since $\vec{w} \cdot \vec{v} = a + (\vec{w} \cdot \vec{u})b > 0$, and since $\vec{w} \cdot \vec{u} > 0$ and $b < 0$, it follows that $a > 0$. So, $\vec{w} = \vec{u}(-b/a) + \vec{v}/a$ where $-b/a > 0$ and $/a > 0$. Hence, by Exercise 1 of Part B on page 208, the common side [whose sense is that of $\vec{w}$] is interior to the angle formed by the noncommon sides [whose senses are those of $\vec{u}$ and $\vec{v}$].

7. As in Exercise 6, $\vec{v} = \vec{w}a + \vec{u}b$, where $b < 0$. We wish to show that $a > 0$ if and only if $\vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v} > 0$. [For, then, $\vec{w}$ will be $\vec{u}(-b/a) + \vec{v}/a$, with $-b/a > 0$ and $/a > 0$ if and only if $\vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v} > 0$.] Now, $1 = \vec{v} \cdot \vec{v} = (\vec{w} \cdot \vec{v})a + (\vec{u} \cdot \vec{v})b$ and $\vec{u} \cdot \vec{v} = (\vec{w} \cdot \vec{u})a + b$. Combining these results we see that $1 + \vec{u} \cdot \vec{v} = (\vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v})a + (1 + \vec{u} \cdot \vec{v})b$ and, so, that $(1 + \vec{u} \cdot \vec{v})(1 - b) = (\vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v})a$. Since $1 + \vec{u} \cdot \vec{v} > 0$ and $b < 0$ [so that $1 - b > 0$] it follows that $a > 0$ if and only if $\vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v} > 0$.

TC 237 (2)

Answers for Part D

1. Since $(\vec{u}, \vec{v})$ is orthonormal and $\vec{w} \in [\vec{u}, \vec{v}]$ it follows that $\vec{w} = \vec{u}a + \vec{v}b$ where, since $\vec{w}\cdot\vec{u} = (\vec{u}\cdot\vec{u})a + (\vec{u}\cdot\vec{v})b$ and $\vec{w}\cdot\vec{v} = (\vec{u}\cdot\vec{v})a + (\vec{v}\cdot\vec{v})b$, $a = \vec{w}\cdot\vec{u}$ and $b = \vec{w}\cdot\vec{v}$. Since $\vec{w}\cdot\vec{w} = 1$, $(\vec{u}\cdot\vec{u})a^2 + (\vec{v}\cdot\vec{v})b^2 = 1$. Hence, $(\vec{w}\cdot\vec{u})^2 + (\vec{w}\cdot\vec{v})^2 = 1$. [Instead of going through the algebra, we might merely have referred to Theorems 11-11 and 11-12.]

2. Suppose that $\vec{k}$ is a unit vector orthogonal to $\vec{u}$ and $\vec{v}$. Then $(\vec{u}, \vec{v}, \vec{k})$ is orthonormal and $\vec{w} = \vec{u}(\vec{w}\cdot\vec{u}) + v(\vec{w}\cdot\vec{v}) + \vec{k}(\vec{w}\cdot\vec{k})$ and $1 = \vec{w}\cdot\vec{w}$ $= (\vec{w}\cdot\vec{u})^2 + (\vec{w}\cdot\vec{v})^2 + (\vec{w}\cdot\vec{k})^2$. So, $\vec{w}\cdot\vec{k} = 0$ if and only if $(\vec{w}\cdot\vec{u})^2 + (\vec{w}\cdot\vec{v})^2 = 1$ — that is, $\vec{w} \in [\vec{u}, \vec{v}]$ if and only if $(\vec{w}\cdot\vec{u})^2 + (\vec{w}\cdot\vec{v})^2 = 1$. [Note that the only if-part is another solution for Exercise 1.]

TC 238

3. Following the hint it is clear that the desired angle is $\angle(A + \vec{w})A(A + \vec{v})$ if it is the case that $(\vec{u}\cdot\vec{w})^2 + (\vec{v}\cdot\vec{w})^2 = 1$ and $\vec{v}\cdot\vec{w} > 0$. Since $||\vec{w} - \vec{u}(\vec{w}\cdot\vec{u})||^2 = 1 - (\vec{w}\cdot\vec{u})^2$ it follows that $\vec{v} = [\vec{w} - \vec{u}(\vec{w}\cdot\vec{u})]/\sqrt{1 - (\vec{w}\cdot\vec{u})^2}$. So, $\vec{v}\cdot\vec{w} = \sqrt{1 - (\vec{w}\cdot\vec{u})^2} > 0$ and, so, $(\vec{u}\cdot\vec{w})^2 + (\vec{v}\cdot\vec{w})^2 = 1$.

3. Suppose that $\angle A$ and $\angle B$ are acute angles such that $(\cos \angle A)^2 + (\cos \angle B)^2 = 1$. Show that there is an angle which is congruent with $\angle B$, adjacent to $\angle A$, shares a given side with $\angle A$, and whose other side is perpendicular to the other side of $\angle A$. [*Hint*: Let $\vec{w}$ be the unit vector in the sense of the given side of $\angle A$, and let $\vec{u}$ be the unit vector in the sense of the other side of $\angle A$. Consider the half-line $\overrightarrow{A(A + \vec{v})}$ where $\vec{v}$ is the unit vector in the sense of $\vec{w} - \vec{u}(\vec{w}\cdot\vec{u})$.]

15.08 Complementary Angles

We have defined supplementary angles as being angles which are congruent to adjacent angles whose noncommon sides are opposite half-lines. In a similar vein, we adopt:

> **Definition 15–11** A first and a second angle are complementary [and each is a complement of the other] if and only if they are congruent to adjacent acute angles whose noncommon sides are perpendicular half-lines.

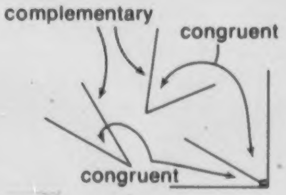


Fig. 15–19

As is the case with supplementary angles, whether or not angles are complementary depends on their cosines:

> **Theorem 15–24** $\angle R$ and $\angle C$ are complementary if and only if $\cos \angle R > 0$, $\cos \angle C > 0$, and $(\cos \angle R)^2 + (\cos \angle C)^2 = 1$.

We also have:

> **Corollary** Adjacent angles are complementary if and only if they are acute and their noncommon sides are perpendicular half-lines.

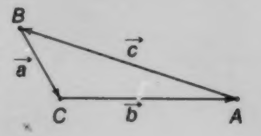
Exercises

Part A

1. What is the cosine of an angle which is its own complement? Check your answer by using the formula in Exercise 5 of Part C, page 227.
2. Suppose that $\triangle ABC$ is equilateral, and F is the foot of its altitude from A. Show that $\angle CAB$ and $\angle FAB$ are complementary.
3. Show that any angle which is congruent to some complement of a given angle is also a complement of that angle.
4. Show that any two complements of a given angle are congruent.
5. Prove Theorem 15–24. [*Hint*: Refer to earlier exercises.]
6. Prove the corollary. [*Hint*: The if-part is trivial. For the only if-part, recall Exercise 5 of Part E on page 220, and use Exercise 3 of Part D on page 238.]

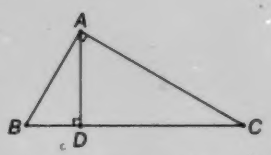
Part B

Consider $\triangle ABC$ with, as usual, $\vec{a} = C - B$, $\vec{b} = A - C$, and $\vec{c} = B - A$. Show that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$ if and only if $\angle A$ and $\angle B$ are complementary. [*Hint*: Let $D = A - \vec{a}$ and show that the corollary to Theorem 15–24 can be applied to $\angle CAB$ and $\angle BAD$.]



Part C

Suppose that $\triangle ABC$ is a right triangle with hypotenuse $\overline{BC}$ and that $\overline{AD}$ is the altitude of $\triangle ABC$ from A, as shown at the right. Make use of the results in Parts A and B to do the following.



1. Show that each of the following pairs of angles are complementary.
   (a) $\angle B, \angle C$ (b) $\angle B, \angle BAD$ (c) $\angle C, \angle CAD$ (d) $\angle BAD, \angle CAD$
2. What can you say about $\angle B$ and $\angle CAD$? About $\angle C$ and $\angle BAD$? Justify your answers.
3. Assume that $\angle B$ is its own complement.
   (a) Show that both $\triangle ABD$ and $\triangle ACD$ are isosceles triangles.
   (b) Is $\triangle ABC$ isosceles? Explain.
   (c) Show that $\overrightarrow{AD}$ is the angle bisector of $\angle BAC$.
   (d) Describe an isometry which maps $\triangle ABD$ onto $\triangle ACD$. What does this tell you about $\triangle ABD$ and $\triangle ACD$?
4. Let l be the line through A and parallel to $\overleftrightarrow{BC}$. Given that E and F are points of l such that $A \in \overline{EF}$ and B is in the interior of $\angle EAC$, show that $\angle EAB$ and $\angle FAC$ are complementary.

If you use Part A of the exercises to illustrate the ideas introduced in this section, then Parts B and C may be used as homework.

Answers for Part A

1. By Theorem 15-24, an angle is its own complement if and only if the square of its cosine is $1/2$ — that is, if and only if its cosine is $\sqrt{2}/2$.

   By Definition 15-11 and Exercise 5 of Part C on page 303, an angle is its own complement if and only if its cosine is $\sqrt{(1 + 0)/2}$ — that is, if and only if its cosine is $\sqrt{2}/2$.

2. By Theorem 15-19, $\overline{AF}$ is the angle bisector from A of $\triangle ABC$. We have seen earlier that $\cos\angle CAB = 1/2$ [Exercise 5(e) of Part C on page 218]. It follows [by Exercise 5 of Part C on page 227] that $\cos\angle FAB = \sqrt{(1 + 1/2)/2} = \sqrt{3}/2$. Since $1/2$ and $\sqrt{3}/2$ are both positive and the sum of their squares is 1 it follows by Theorem 15-24, that $\angle CAB$ and $\angle FAB$ are complementary.

3. Suppose that $\angle B$ is a complement of $\angle A$ and that $\angle C \cong \angle B$. It follows by Theorem 15-24 that $\cos\angle A$ and $\cos\angle B$ are positive and that $(\cos\angle A)^2 + (\cos\angle B)^2 = 1$; and it follows by Theorem 15-8 that $\cos\angle C = \cos\angle B$. So, $\cos\angle A$ and $\cos\angle C$ are positive and $(\cos\angle A)^2 + (\cos\angle C)^2 = 1$ whence, by Theorem 15-24, $\angle C$ is a complement of $\angle A$.

4. Suppose that $\angle A$ and $\angle B$ are complements of the same angle. It follows from Theorem 15-24 that $\cos\angle A > 0$, $\cos\angle B > 0$, and $(\cos\angle A)^2 = (\cos\angle B)^2$. So, it follows that $\cos\angle A = \cos\angle B$ and, by Theorem 15-8, that $\angle A \cong \angle B$.

5. The if-part has been established in Exercise 3 of Part D on page 238. Suppose, then, that $\angle R$ and $\angle C$ are complementary and let $\angle AVB$ and $\angle BVD$ be adjacent acute angles congruent to $\angle R$ and $\angle C$, respectively, which are such that $\overleftrightarrow{VA} \perp \overleftrightarrow{VD}$. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be unit vectors in the senses of $\overrightarrow{VA}$, $\overrightarrow{VB}$, and $\overrightarrow{VD}$, respectively. It follows that $(\vec{u}, \vec{v})$ is orthonormal and that $\vec{w} \in [\vec{u}, \vec{v}]$. So, by Exercise 1 of Part D, $(\vec{u}\cdot\vec{w})^2 + (\vec{v}\cdot\vec{w})^2 = 1$ — that is, $(\cos\angle AVB)^2 + (\cos\angle BVD)^2 = 1$. Since $\angle AVB$ and $\angle BVD$ are acute angles, $\cos\angle AVB$ and $\cos\angle BVD$ are positive. Since, by Theorem 15-8, $\cos\angle R = \cos\angle AVB$ and $\cos\angle C = \cos\angle BVD$ it follows that $\cos\angle R > 0$, $\cos\angle C > 0$, and $(\cos\angle R)^2 + (\cos\angle C)^2 = 1$.

6. The if-part is a trivial consequence of Definition 15-11 because each angle is congruent to itself. Suppose, then, that $\angle ABC$ and $\angle CBD$ are complementary adjacent angles. It follows from the theorem that $\cos\angle ABC > 0$, $\cos\angle CBD > 0$, and $(\cos\angle ABC)^2 + (\cos\angle CBD)^2 = 1$. By Exercise 5 of Part E on page 220, $\angle CBD$ is the only angle with side $\overrightarrow{BC}$ which is adjacent to $\angle ABC$ and whose cosine is $\cos\angle CBD$. By Exercise 3 of Part D on page 238, there is such an angle which, in addition has its side other than $\overrightarrow{BC}$ perpendicular to $\overrightarrow{BA}$. It follows that $\overrightarrow{BD} \perp \overrightarrow{BA}$. So, adjacent complementary angles have their noncommon sides perpendicular.

Answer for Part B

Let D = A - $\vec{a}$. Since C = B + $\vec{a}$ it follows that C and D are on opposite sides of $\overleftrightarrow{AB}$ [Exercise 2 of Part A on page 225] and, so, that $\angle CAB$ and $\angle BAD$ are adjacent angles. By the corollary, these angles are complementary if and only if they are acute and $\overrightarrow{AC} \perp \overrightarrow{AD}$. Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, $\overrightarrow{AC} \perp \overrightarrow{AD}$ if and only if $\angle C$ is a right angle. Also, $\angle BAD \cong \angle B$. [Computation shows that they have the same cosine.] So, $\angle CAB$ and $\angle B$ are complementary if and only if they are acute and $\angle C$ is a right angle. Suppose, now, that $\angle C$ is a right angle. It follows by Exercise 5(d) of Part C on page 237 that $\angle CAB$ and $\angle B$ are acute. So, if $\angle C$ is a right angle then $\angle CAB$ and $\angle B$ are complementary. On the other hand, we have shown that if $\angle CAB$ and $\angle B$ are complementary then $\angle C$ is a right angle. Hence, in $\triangle ABC$, $\angle A$ and $\angle B$ are complementary if and only if $\angle C$ is a right angle — that is, if and only if $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$.

Answers for Part C

1. (a) $\angle B$ and $\angle C$ are complementary because they are acute angles of right $\triangle ABC$.
   (b) $\angle B$ and $\angle BAD$ are complementary because they are acute angles of right $\triangle BAD$.
   (c) $\angle C$ and $\angle CAD$ are complementary because they are acute angles of right $\triangle CAD$.
   (d) $\angle BAD$ and $\angle CAD$ are complementary because they are acute adjacent angles whose noncommon sides are perpendicular. [They are complementary, also, because, by parts (a), (b), and (c), they are complements of complementary angles; and it is easily proved that complements of complementary angles are complementary.]
2. $\angle B \cong \angle CAD$ because their sides can be matched so that corresponding sides are perpendicular and, since both are acute, they cannot be supplementary. For the same reason, $\angle C \cong \angle BAD$. [Another reason for the congruence of $\angle B$ and $\angle CAD$ is that both are complements of $\angle BAD$, by parts (b) and (d) of Exercise 1. Similarly, $\angle C$ and $\angle BAD$ are complements of $\angle CAD$ and, so, are congruent.]

Answers for Part C [cont.]

3. (a) Since $\angle B$ is its own complement it is congruent to its complements, $\angle BAD$ and $\angle C$. Also, $\angle B$ is congruent to $\angle CAD$. It follows that $\triangle BAD$ and $\triangle CAD$ have congruent angles at B and A and at C and A, respectively. So, $\triangle BAD$ and $\triangle CAD$ are isosceles, the respective bases being $\overline{BA}$ and $\overline{CA}$.
   (b) As in part (a), $\triangle ABC$ has congruent angles at B and C. So, it is isosceles with base $\overline{BC}$.
   (c) It follows from Theorem 15-19 that the altitude $\overline{AD}$ to the base of isosceles $\triangle ABC$ is the angle bisector of the triangle from A. So, $\overline{AD}$ is the bisector of $\angle BAD$.
   (d) The reflection in the plane containing $\overleftrightarrow{AD}$ and perpendicular to $\overline{ABC}$ maps each of A and D on itself and maps B on C. [Recall that, by Theorem 15-18, the altitude from A is the median from A.] So, this reflection is an isometry which, by Theorem 14-27, maps $\triangle ABD$ onto $\triangle ACD$. It follows that $\triangle ABD \cong \triangle ACD$.
4. Since E and C are on opposite sides of $\overleftrightarrow{AB}$ it follows that E - A and C - B have opposite senses and, so, F - A and B - C have opposite senses. So, F and B are on opposite sides of $\overleftrightarrow{AC}$. Hence, $\angle EAB$ and $\angle B$, as well as $\angle FAC$ and $\angle C$ are congruent [as alternate interior angles]. Since $\angle B$ and $\angle C$ are complementary, so are $\angle EAB$ and $\angle FAC$.

*

Sample Quiz

Given right $\triangle ABC$ with hypotenuse $\overline{BC}$, assume that AB = 5, AC = 12, and that $\overline{AD}$ is the angle bisector from A.

1. Compute BD, DC, and $\cos \angle ABC$.
2. Compute AD. [Express answer in simplest radical form.]
3. Order the angles of $\triangle ABD$ according to size from largest to smallest.

Key to Sample Quiz

1. 65/17; 156/17; 5/13 [See Exercise 2, page 218.]
2. $60\sqrt{2}/17$ [$AD^2 = 5^2 + (\frac{65}{17})^2 - 2\cdot 5\cdot \frac{65}{17}\cdot\frac{5}{13} = \frac{5^2}{17^2}(17^2 + 13^2 - 170)$
   $= \frac{5^2\cdot 12^2\cdot 2}{17^2}$.]
3. $\angle ADB$, $\angle ABD$, $\angle BAD$ [This follows from the fact that AB > AD > BD.]

[Note: The result of Exercise 2 of the Sample Quiz suggests the following theorem:

> Given right $\triangle ABC$ with hypotenuse $\overline{BC}$, AB = c, and AC = b, the angle bisector from A has measure $bc\sqrt{2}/(b + c)$.

This is an appropriate problem to assign as extra-credit work or as an "original" on an examination. The algebra in the proof, based on the cosine law, is not difficult.]

15.09 Sines of Angles

When dealing with vectors $\vec{a}$ and $\vec{b}$ we have frequently made use of the vector

$$(*) \qquad \vec{a} - \vec{b}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right).$$

This last vector is orthogonal to $\vec{b}$ and, in fact, is in the direction which is orthogonal to $[\vec{b}]$ and is contained in $[\vec{a}, \vec{b}]$. More specifically, the

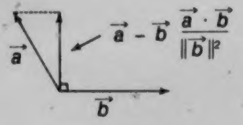

$\vec{a} - \vec{b}\,\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}$
$\vec{a}$
$\vec{b}$

Fig. 15–20

vector (*) is the projection of $\vec{a}$ in the bidirection $[\vec{b}]^{\perp}$. Since (*) belongs to $[\vec{a}, \vec{b}]$ and is orthogonal to $\vec{b}$, the square of the norm of (*) is the dot product of $\vec{a}$ with (*):

$$\begin{aligned}\left\|\vec{a} - \vec{b}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\right\|^2 &= \vec{a} \cdot \left(\vec{a} - \vec{b}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\right)\\ &= \|\vec{a}\|^2 - \frac{(\vec{a} \cdot \vec{b})^2}{\|\vec{b}\|^2}\\ &= \frac{\|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2}{\|\vec{b}\|^2}\end{aligned}$$

The numerator of this last fraction is also familiar. As you should recall, its value is nonnegative, and this value is zero if and only if $(\vec{a}, \vec{b})$ is linearly dependent. Moreover, for $\vec{a}$ and $\vec{b}$ not $\vec{0}$,

$$\|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2 = \|\vec{a}\|^2\|\vec{b}\|^2\left(1 - \frac{(\vec{a} \cdot \vec{b})^2}{\|\vec{a}\|^2\|\vec{b}\|^2}\right).$$

Hence, in case the senses of $\vec{a}$ and $\vec{b}$ are those of the sides of $\angle C$, the norm of (*) is

$$\|\vec{a}\| \sqrt{1 - (\cos \angle C)^2}.$$

Explanation of the range of sin: Suppose given a number a such that $0 < a \leq 1$. It follows that $0 \leq \sqrt{1 - a^2} < 1$ and, so, that there is an angle — say, $\angle C$ — such that $\cos \angle C = \sqrt{1 - a^2}$. So, $\sqrt{1 - (\cos \angle C)^2} = |a|$ and, since $a > 0$, it follows by Definition 15-12 that $\sin \angle C = a$. Hence, for any positive number not greater than 1, there is an angle whose sine is that number. [If, in addition, $a \neq 1$ then there are noncongruent angles which have a as sine. For, we can choose $\angle C$ so that $\cos \angle C = -\sqrt{1 - a^2}$ and, as above, show that $\sin \angle C = a$.]

An angle whose sine is 1 is an angle whose cosine is 0 — that is, is a right angle.

Supplementary angles have the same sine.

The sum of the squares of the sines of complementary angles is 1. [Apply Definition 15-12 to each of a pair of complementary angles.]

* * *

Suggestions for the exercises of section 15.09:
(i) Part A and the discussion preceding should be developed in class. Be sure to illustrate applications of Definition 15-12.
(ii) Parts B and C may be used as homework.
(iii) Part D may be developed as a class project.

So, $\sqrt{1 - (\cos \angle C)^2}$ is the ratio in which an interval contained in one side of $\angle C$ is foreshortened when it is projected into a plane perpendicular to the other side of $\angle C$. Finally, $\sqrt{1 - (\cos \angle C)^2}$ is referred to implicitly in Theorem 15-24. In fact, this theorem can be restated as:

> $\angle R$ and $\angle C$ are complementary if and only if $\angle C$ is acute and $\cos \angle R = \sqrt{1 - (\cos \angle C)^2}$.

The preceding suggests that we introduce an abbreviation for '$\sqrt{1 - (\cos \_)^2}$'. The usual one is 'the sine of _' or, merely, 'sin _'. We do so in:

Definition 15-12 $\sin \angle C = \sqrt{1 - (\cos \angle C)^2}$

Notice that the sine of any angle is a positive number which is not greater than 1 and that, given any such number, there is an angle which has this number as its sine. [Explain.] What can you say of an angle whose sine is 1? What can you say about the sines of supplementary angles? About the sines of complementary angles?

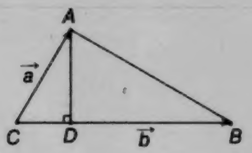
Exercises

Part A

Earlier we learned that, for any linearly independent vectors $\vec{a}$ and $\vec{b}$,

$$(^{**})\qquad \cos \angle ACB = \frac{\|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{b} - \vec{a}\|^2}{2\|\vec{a}\|\,\|\vec{b}\|}$$

where $\vec{a} = A - C$ and $\vec{b} = B - C$. Thus, knowing the norms of $\vec{a}$, $\vec{b}$, and $\vec{b} - \vec{a}$, we can make use of (**) and compute the cosine of a given angle, $\angle ACB$. And, making use of Definition 15-12, we can compute the sine of that angle. In each of the following, you are given some information about $\vec{a}$, $\vec{b}$, and $\vec{b} - \vec{a}$. You are to do the required computations.

1. 

Given: $\|\vec{a}\| = 15$, $\|\vec{b}\| = 14$, and $\|\vec{b} - \vec{a}\| = 13$

Compute: $\cos \angle C$, $\sin \angle C$, and the length AD of the perpendicular from A to $\overleftrightarrow{BC}$

2. Given the same information as in Exercise 1, let E be such that $B \in \overline{CE}$. Compute $\cos \angle B$, $\sin \angle B$, $\cos \angle ABE$, and $\sin \angle ABE$.

Answers for Part A

1. $\cos\angle C = 3/5$; $\sin\angle C = 4/5$; $AD = 12$
2. $\cos\angle B = 5/13$; $\sin\angle B = 12/13$; $\cos\angle ABE = -5/13$; $\sin\angle ABE = 12/13$

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3. $AB = 13$; $\cos\angle A = 5/13$; $\sin\angle A = 12/13$; $\cos\angle B = 12/13$; $\sin\angle B = 5/13$; $AD = 25/13$; $CD = 60/13$
4. $\cos\angle ACD = 12/13$; $\sin\angle ACD = 5/13$; $\cos\angle DCB = 5/13$; $\sin\angle DCB = 12/13$
5. $\cos\angle A = 13/20$; $\cos\angle B = 37/40$; $\cos\angle C = -5/16$; $\sin\angle A = \sqrt{231}/20$; $\sin\angle B = \sqrt{231}/40$; $\sin\angle C = \sqrt{231}/16$ [If students are struck by the ubiquity of '$\sqrt{231}$', you may be able to nudge them into discovering that, with $d = \sqrt{231}/80$, $\sin\angle A = da$, $\sin\angle B = db$, and $\sin\angle C = dc$. If so, they will have discovered an instance of the Sine Law — Theorem 16-4 on page 259.]
6. $CD = \sqrt{231}/5$ [Recalling the discussion preceding these exercises, $CD = 8 \sin\angle B = 4 \sin\angle A$]; $AD = 13/5$ [$= 4 \cos\angle A$]; $\cos\angle ACD = \sqrt{231}/20$; $\sin\angle ACD = 13/20$; $\cos\angle BCD = \sqrt{231}/40$; $\sin\angle BCD = 37/40$
7. $AE = 26/5$; $\cos\angle ACE = 31/200$; $\sin\angle ACE = 13\sqrt{231}/200$

Answers for Part B

1. [See (**) on page 241 and Definition 15-12.]

2. Congruent angles have the same sine; supplementary angles have the same sine. For congruent angles have the same cosine and the cosines of supplementary angles are opposites and opposites have the same square. [See Definition 15-12.]

3. Angles which have the same sine are either congruent or supplementary. For, if $\sin\angle B = \sin\angle C$ then, by Definition 15-12, $(\cos\angle B)^2 = (\cos\angle C)^2$ and, so, $\cos\angle B = \cos\angle C$ or $\cos\angle B + \cos\angle C = 0$.

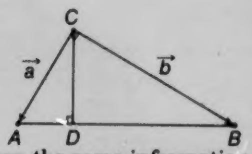
4. (a) $\cos\angle B = \sin\angle A$; $\sin\angle B = \cos\angle A$; If $\angle A$ and $\angle B$ are complementary then both are acute and $(\cos\angle A)^2 + (\cos\angle B)^2 = 1$. It follows that $\cos\angle B = \sqrt{1 - (\cos\angle A)^2} = \sin\angle A$ and $\sin\angle B = \sqrt{1 - (\cos\angle B)^2} = \cos\angle A$. [Note that it is essential for this argument that $\cos\angle B > 0$ and $\cos\angle A > 0$.]

   (b) Given that $\angle A$ is acute and $\sin\angle A = \cos\angle B$ it follows that $\angle A$ and $\angle B$ are complementary. For, since $\cos\angle B = \sin\angle A > 0$ it follows that $\angle B$ is also acute and since $\cos\angle B = \sin\angle A = \sqrt{1 - (\cos\angle A)^2}$ it follows that $(\cos\angle A)^2 + (\cos\angle B)^2 = 1$.

5. (a) If $\sin\angle BAC = 1$ then $\cos\angle BAC = 0$ and, so, $\angle BAC$ is a right angle. By Theorem 15-7 there is just one right angle which has $\overrightarrow{AB}$ as one side and its other side on a given side of l.

   (b) Since $\sin\angle BAC < 1$, $\cos\angle BAC \neq 0$ and, so, $-\cos\angle BAC \neq \cos\angle BAC$. There is, by Theorem 15-7 just one angle — say, $\angle BAD$ — whose cosine is $-\cos\angle BAC$, which has $\overrightarrow{BA}$ as one side and whose other side is on the same side of l as is $\overrightarrow{AC}$. Since this angle is supplementary to $\angle BAC$, it has the same sine as does $\angle BAC$.

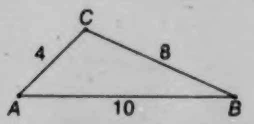
   (c) There are two such angles. They are the angles with $\overrightarrow{AB}$ as side and with other side on the same side of l as is $\overrightarrow{AC}$ and whose cosines are $\sqrt{1 - a^2}$ and $-\sqrt{1 - a^2}$. By Theorem 15-7 there is just one angle of each kind and, by Definition 15-12, any angle whose sine is a has one of the specified cosines.

3. 

Given: $\|\vec{a}\| = 5$, $\|\vec{b}\| = 12$, and $\vec{a} \perp \vec{b}$

Compute: AB, $\cos\angle A$, $\sin\angle A$, $\cos\angle B$, $\sin\angle B$, AD, and CD

4. Given the same information as in Exercise 3. Compute $\cos\angle ACD$, $\sin\angle ACD$, $\cos\angle DCB$, and $\sin\angle DCB$.

5. 

Given: $\triangle ABC$, with side-measures as indicated.

Compute: $\cos\angle A$, $\cos\angle B$, $\cos\angle C$, $\sin\angle A$, $\sin\angle B$, and $\sin\angle C$.

6. Given $\triangle ABC$ as described in Exercise 5, let $\overline{CD}$ be its altitude from C. Compute CD, AD, $\cos\angle ACD$, $\sin\angle ACD$, $\cos\angle BCD$, and $\sin\angle BCD$.

7. Given $\triangle ABC$ as described in Exercise 5, let E be the point on line $\overleftrightarrow{AB}$ such that $\triangle ACE$ is isosceles with base $\overline{AE}$. Compute AE, $\cos\angle ACE$, and $\sin\angle ACE$.

Part B

1. Consider $\angle ACB$, where $\vec{a} = A - C$ and $\vec{b} = B - C$. Show that

$$\sin\angle ACB = \frac{\sqrt{\|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2}}{\|\vec{a}\|\,\|\vec{b}\|}.$$

2. What can you say about the sines of two congruent angles? About the sines of two supplementary angles? Give arguments in support of your answers.

3. Suppose that two angles have the same sine. What can you say about the angles? Give an argument to support your answer.

4. Consider an acute angle, $\angle A$, and another angle, $\angle B$.
   (a) Given that $\angle A$ and $\angle B$ are complementary, what can you say about $\cos\angle B$? About $\sin\angle B$? Explain your answers.
   (b) Assume that $\sin\angle A = \cos\angle B$. What can you say about $\angle A$ and $\angle B$? Explain your answer.

5. Suppose that A and B are two points of a line l, and that $C \notin l$.
   (a) Given that $\sin\angle BAC = 1$, what can you say about $\angle BAC$? How many angles have the same sine as $\angle BAC$, have $\overrightarrow{AB}$ as one side, and have the other side on the same side of l as is $\overrightarrow{AC}$? Explain your answers.
   (b) Assume that $\sin\angle BAC < 1$. Show that there is an angle – say, $\angle BAD$ – which is different from $\angle BAC$, whose side $\overrightarrow{AD}$ is on the same side of l as is $\overrightarrow{AC}$, and whose sine is that of $\angle BAC$.
   (c) Given any number a such that $0 < a < 1$, how many angles are there whose sine is a, which have $\overrightarrow{AB}$ as one side, and whose other sides are on the side of l which contains $\overrightarrow{AC}$? Justify your answer.

*

There are a number of theorems concerning sines which are analogous to some of our earlier theorems concerning cosines. Some which are suggested by the exercises just completed are:

Theorem 15-25 $\sin \angle ACB = \dfrac{\sqrt{\|\vec{a}\|^2\,\|\vec{b}\|^2 - (\vec{a}\cdot\vec{b})^2}}{\|\vec{a}\|\,\|\vec{b}\|}$, where $\vec{a} = Z - C$ and $\vec{b} = B - C$.

Theorem 15-26 Angles are congruent or supplementary if and only if they have the same sine.

Theorem 15-27 Given a number a such that $0 < a < 1$, and given a half-line r, there are two and only two angles whose sine is a, which have r as one side, and whose other sides are on a given side of the line which contains r.

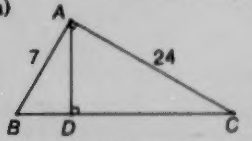
Theorem 15-28 Two angles are complementary if and only if one of them is acute and its sine is the cosine of the other.

Notice that the *co*sine of an acute angle is the sine of any *co*mplement of that angle.

Part C

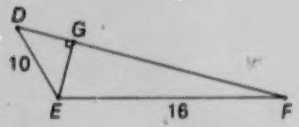
1. Show that, for any $\angle A$, $(\cos \angle A)^2 + (\sin \angle A)^2 = 1$.
2. Complete each of the following.
   (a) Given that $\angle A$ is obtuse and $\sin \angle A = \frac{1}{2}$, $\cos \angle A =$ __.
   (b) Given that $\angle A$ is acute and $\sin \angle A = \frac{3}{4}$, $\cos \angle A =$ __.
   (c) Given that $\sin \angle B = \frac{12}{13}$ and $\angle C$ is a complement of $\angle B$, $\cos \angle C =$ __ and $\sin \angle C =$ __.
   (d) Given that $\angle A$ and $\angle C$ are supplements and $\cos \angle C = \frac{1}{2}$, $\sin \angle A =$ __.
   (e) $\triangle ABC$ is such that $AB = 13$, $BC = 14$, and $AC = 15$. $\sin \angle A =$ __ and $\sin \angle B =$ __.
3. In each of the following, you are given a picture and some information about it. Do the indicated computations.

(a)



Compute: BC, BD, AD, $\sin \angle B$, and $\sin \angle CAD$

(b)



Given: $\cos \angle D = \frac{3}{5}$
Compute: EG, GF, $\cos \angle DEG$, $\sin \angle F$, and $\sin \angle GEF$

Answers for Part C

1. [This comes directly from Definition 15-12.]
2. (a) $-\sqrt{3}/2$ (b) $\sqrt{7}/4$
   (c) $12/13$; $5/13$ (d) $\sqrt{3}/2$
   (e) $56/65$; $12/13$
3. (a) $BC = 25$; $BD = 49/25$; $AD = 168/25$; $\sin\angle B = 24/25$; $\sin\angle CAD = 24/25$
   (b) $[DG = 6]$ $EG = 8$; $GF = 8\sqrt{3}$; $\cos\angle DEG = 4/5$; $\sin\angle F = 1/2$; $\sin\angle GEF = \sqrt{3}/2$

Answers for Part C [cont.]

4. (a) $(4 - 3\sqrt{3})/10$ [Use (**) of Part A on page 241.]
   (b) $\frac{4}{5} \cdot \frac{1}{2} - \frac{3}{5} \cdot \frac{\sqrt{3}}{2} = \frac{4 - 3\sqrt{3}}{10}$
   (c) They are equivalent.

Answers for Part D

1. Since $\angle P_1$ and $\angle P_2$ are adjacent it follows by Definition 15-4 that Q and R are on opposite sides of $\overleftrightarrow{PS}$.
2. Since Q and R are on opposite sides of $\overleftrightarrow{PS}$ it follows by Exercise 3 of Part D on page 212 that $\vec{v} - \vec{w}(\vec{v} \cdot \vec{w})$ and $\vec{u} - \vec{w}(\vec{u} \cdot \vec{w})$ have opposite senses.
3. $\sqrt{1 - (\vec{v} \cdot \vec{w})^2}$; $\sqrt{1 - (\vec{u} \cdot \vec{w})^2}$
4. $[\vec{u} \cdot \vec{v} - (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{w})]/\sqrt{1 - (\vec{v} \cdot \vec{w})^2}$; $\sqrt{1 - (\vec{u} \cdot \vec{w})^2}$
5. The angles are supplementary; so, the sum of their cosines is 0.
6. By Exercises 4 and 5, $[\vec{u} \cdot \vec{v} - (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{w})]/\sqrt{1 - (\vec{v} \cdot \vec{w})^2}$ $= -\sqrt{1 - (\vec{u} \cdot \vec{w})^2}$. So, $\vec{u} \cdot \vec{v} = (\vec{u} \cdot \vec{w})(\vec{v} \cdot \vec{w}) - \sqrt{1 - (\vec{u} \cdot \vec{w})^2}\sqrt{1 - (\vec{v} \cdot \vec{w})^2}$ — that is,

$$\cos \angle QPR = \cos \angle P_1 \cos \angle P_2 - \sin \angle P_1 \sin \angle P_2.$$

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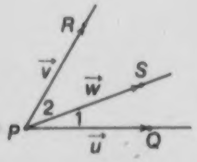
4. (a) Given the information in Exercise 3(b), compute $\cos \angle DEF$.
   (b) Evaluate the expression:

$$\cos \angle DEG \cos \angle FEG - \sin \angle DEG \sin \angle FEG$$

   (c) Compare your results in (a) and (b).

Part D

Suppose that $\angle P_1$ and $\angle P_2$ are adjacent angles which are *not* supplementary, as shown in the picture at the right. Let $\vec{u}$, $\vec{w}$, and $\vec{v}$ be the unit vectors $Q - P$, $S - P$, and $R - P$, as shown.



1. Q and R are on opposite sides of $\overleftrightarrow{PS}$. Why?
2. Show that the vectors $\vec{v} - \vec{w}(\vec{v} \cdot \vec{w})$ and $\vec{u} - \vec{w}(\vec{u} \cdot \vec{w})$ have opposite senses.
3. What is the norm of $\vec{v} - \vec{w}(\vec{v} \cdot \vec{w})$? Of $\vec{u} - \vec{w}(\vec{u} \cdot \vec{w})$?
4. Compute the dot products with $\vec{u}$ of the unit vectors in the senses of $\vec{v} - \vec{w}(\vec{v} \cdot \vec{w})$ and $\vec{u} - \vec{w}(\vec{u} \cdot \vec{w})$.
5. What is the relationship between angles whose cosines are the numbers you computed in Exercise 4? What does this tell you about these cosines?
6. Make use of your results so far to show that

$$\cos \angle QPR = \cos \angle P_1 \cos \angle P_2 - \sin \angle P_1 \sin \angle P_2.$$

15.10 Dihedral Angles

We have found the notion of angle to be a quite useful one. Basically, angles are two-dimensional objects, for any two noncollinear rays with the same vertex are contained in exactly one plane and planes are two-dimensional objects. An analogue in 3-dimensional space to an angle is *a dihedral angle*, which is the union of two noncoplanar half-planes with a common edge. The half-planes which make up a given dihedral angle are sometimes called *the faces* of the dihedral angle.

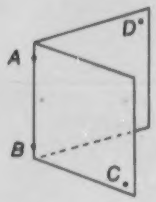


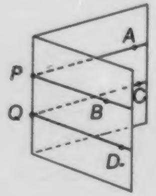
Fig. 15–21

In referring to a dihedral angle, it is customary to give a "four point" name by naming a point in one face, two points on the edge, and a point in the other face. For example, the dihedral angle picture in Fig. 15–21 is $\angle D\text{-}AB\text{-}C$. [Read '$\angle D\text{-}AB\text{-}C$' as 'dihedral angle D, A, B, C'.] Alternate ways of referring to this same dihedral angle are '$\angle D\text{-}BA\text{-}C$', '$\angle C\text{-}AB\text{-}D$', and '$\angle C\text{-}BA\text{-}D$'. [This corresponds to "three point" names for angles, such as '$\angle ABC$'.]

Exercises

Part A

1. Given two intersecting planes, how many dihedral angles are contained in their union?
2. Suppose that $\angle P\text{-}QR\text{-}S$ is a dihedral angle.
   (a) Give the two planes whose union contains $\angle P\text{-}QR\text{-}S$.
   (b) What is the edge of the given dihedral angle?
   (c) Give at least two other ways of referring to $\angle P\text{-}QR\text{-}S$.
3. Fold a piece of cardboard so as to form a model of a dihedral angle. Mark a point A on the edge of the dihedral angle and draw a ray $\overrightarrow{AB}$ in one of its faces.
   (a) Draw a ray $\overrightarrow{AC}$ in the other face so that $\angle BAC$ is acute.
   (b) Draw a ray $\overrightarrow{AD}$ in the other face so that $\angle BAD$ is obtuse.
   (c) Now, try to locate a ray – say, $\overrightarrow{AE}$ – in the other face so that $\angle BAE$ is right.
4. Of which of the following might the intersection of a plane and a dihedral angle consist? Explain your answers.
   (a) A line. (b) A point.
   (c) Two parallel lines. (d) Two intersecting lines.
   (e) An angle. (f) Two disjoint rays.
5. Suppose that $\overleftrightarrow{PQ}$ is the edge of a dihedral angle and the points A, B, C, and D are chosen in the faces of the dihedral angle so that the rays $\overrightarrow{PA}$, $\overrightarrow{PB}$, $\overrightarrow{QC}$, and $\overrightarrow{QD}$ are perpendicular to $\overleftrightarrow{PQ}$, as shown in the picture at the right.



   (a) What can you say about $\angle APB$ and $\angle CQD$? Explain your answer.
   (b) Suppose that $\angle APB$ is a right angle. What can you say about the faces of $\angle A\text{-}PQ\text{-}D$? About $\angle CQD$? Explain your answers.

Answers for Part A

1. Four.
2. (a) $\overline{PQR}$ and $\overline{SQR}$
   (b) $\overleftrightarrow{QR}$
   (c) The most obvious answers are the following:
   $\angle S\text{-}QR\text{-}P$, $\angle S\text{-}RQ\text{-}P$, $\angle P\text{-}RQ\text{-}S$
3. [It is probably best to have this done as seatwork, discussing the results when all the students have had a chance to see "what is going on" with their own models. For making the models, index cards should serve quite well.]
4. (a) Yes. Any plane which contains the edge of a given dihedral angle and does not contain either of the faces of the dihedral angle is such that its intersection with the dihedral angle consists of a line.
   (b) No. Any plane which contains a point of a given dihedral angle intersects the plane of one of the faces in a line. If this line is parallel to the edge of the dihedral angle, it either is the edge or is contained in one of the faces of the dihedral angle, so that the intersection of the plane and dihedral angle consists of a line. If this line is not parallel to the edge then it intersects the edge, so that the intersection of the plane and dihedral angle consists of an angle. In any case, the intersection cannot consist of a point.
   (c) Yes. Any plane which is parallel to the edge of a dihedral angle and intersecting both faces of the dihedral angle is such that its intersection with the dihedral angle consists of two parallel lines.
   (d) No. By the argument in (b), we can obtain two rays with the same vertex as the faces of a dihedral angle are half-planes.
   (e) Yes. See the argument in (b).
   (f) No. A plane cannot contain exactly two points of the edge of a dihedral angle and, for the intersection of a plane and a dihedral angle to consist of two disjoint rays, that would have to be the case.
5. (a) They are congruent, for their corresponding sides are in the same sense.
   (b) The faces of $\angle A\text{-}PQ\text{-}D$ are perpendicular half-planes, for they are subsets of perpendicular planes. $\angle CQD$ is a right angle for it is congruent to right $\angle APB$.

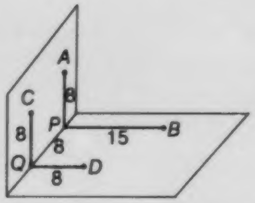
6. Consider the angle, $\angle APB$, described in Exercise 5. $\angle APB$ is a subset of $\angle A\text{-}PQ\text{-}D$. Describe another dihedral angle which has $\angle APB$ as one of its subsets. Are the sides of $\angle APB$ perpendicular to the edge of any one of these other dihedral angles?

*

The angles, $\angle APB$ and $\angle CQD$, described in Exercise 5, above, are sometimes called *plane angles* of the dihedral angle, $\angle A\text{-}PB\text{-}D$. By definition, a plane angle of a dihedral angle is an angle which is a subset of the dihedral angle and whose sides are perpendicular to the edge of the dihedral angle.

Part B

1. How many plane angles does a given dihedral angle have?
2. What can you say about the plane angles of a dihedral angle?
3. Can the intersection of a plane and a dihedral angle be a plane angle of the dihedral angle? Explain.
4. Suppose that $\angle APB$ and $\angle COD$ are plane angles of $\angle A\text{-}PQ\text{-}B$ and $\angle C\text{-}OR\text{-}D$, respectively, and that $\angle APB \cong \angle COD$. Do you think that $\angle A\text{-}PQ\text{-}B$ and $\angle C\text{-}OR\text{-}D$ are congruent? [Remember, to answer 'Yes.' means that you believe there is an isometry which maps one of the dihedral angles onto the other.] Justify your answer.
5. Suppose that we define the cosine of a dihedral angle to be the cosine of any of its plane angles. How would you define the following.
   - (a) acute dihedral angle
   - (b) right dihedral angle
   - (c) obtuse dihedral angle
   - (d) supplementary dihedral angles
6. Suppose that plane angles, $\angle APB$ and $\angle CQD$, of $\angle A\text{-}PQ\text{-}D$ are right angles [so that $\angle A\text{-}PQ\text{-}D$ is a right dihedral angle] and that $AP = PQ = CQ = QD = 8$ and $PB = 15$, as shown in the picture at the right.
   - (a) Find AB, AQ, PD, AD, CD, QB, and CB.
   - (b) Find the cosine values of $\angle ABP$, $\angle ADP$, $\angle CDQ$, $\angle CPQ$, and $\angle CBQ$.
   - (c) Order the angles given in (b) according to size from smallest to largest.



Answers for Part A [cont.]

6. Choose any line which contains P and is not a subset of either plane which contains a face of ∠A-PQ-D. The union of the two half-planes with this line as edge and containing A and B is a dihedral angle one of whose subsets is ∠APB. Since $\overleftrightarrow{PQ}$ is the only line which contains P and is perpendicular to $\overleftrightarrow{APB}$, we have that the sides of ∠APB are not perpendicular to the edge of the newly constructed dihedral angle. Thus, ∠APB is not a plane angle of that dihedral angle.

Answers for Part B

1. Infinitely many. [One for each point of the edge of the dihedral angle.]
2. They are all congruent.
3. Yes. Any plane which is perpendicular to the edge of a dihedral angle is such that its intersection with the dihedral angle is an angle whose sides are perpendicular to that edge and whose vertex is on that edge. And, such an angle is a plane angle of the dihedral angle.
4. Yes. Let f be any isometry which maps ∠APB onto ∠COD. We lose no generality by assuming that PA = OC and PB = OD. Then, f(A) = C, f(P) = O, and f(B) = D. Also, f maps the line through P and perpendicular to $\overleftrightarrow{APB}$ onto the line through O and perpendicular to $\overleftrightarrow{COD}$. Thus, f maps $\overleftrightarrow{PQ}$ onto $\overleftrightarrow{OR}$. Hence, f maps ∠A-PQ-B onto ∠C-OR-D.
5. (a) One whose cosine is positive.
   - (b) One whose cosine is 0.
   - (c) One whose cosine is negative.
   - (d) A pair of dihedral angles the sum of whose cosines is 0.
6. (a) $AB = 17$; $AQ = 8\sqrt{2}$; $PD = 8\sqrt{2}$; $AD = 8\sqrt{3}$; $CD = 8\sqrt{2}$; $QB = 17$; $CB = \sqrt{353}$
   - (b) $15/17$; $\sqrt{2/3}$; $\sqrt{2}/2$; $\sqrt{2}/2$; $17/\sqrt{353}$
   - (c) ∠CBQ, ∠ABP, ∠ADP, ∠CDQ ≅ ∠CPQ

15.11 Chapter Summary

Vocabulary Summary

- angle
  - vertex of
  - side of
  - interior of
  - exterior of
  - plane of
  - cosine of
  - sine of
  - bisector of
- vertical angles
- adjacent angles
- right angle
- acute angle
- obtuse angle
- dihedral angle
  - faces of
  - edge of
  - plane angles of
- supplementary angles
- complementary angles
- opposite sides of l
- same side of l
- angle bisectors of a triangle
- transversal
- alternate interior angles
- corresponding angles
- consecutive interior angles

Definitions

15-1. An angle is a set of points which is the union of two noncollinear rays which have the same vertex.

15-2. (a) C is interior to $\angle AVB$ if and only if there exist points X and Y on $\overrightarrow{VA}$ and $\overrightarrow{VB}$, respectively, such that $C - V = (X - V) + (Y - V)$.

(b) C is exterior to $\angle AVB$ if and only if $C \in \overline{AVB}$ but belongs neither to $\angle AVB$ nor to the interior of $\angle AVB$.

15-3. (a) P and Q are on opposite sides of l if and only if neither P nor Q belongs to l but $\overline{PQ} \cap l \neq \varnothing$.

(b) P and Q are on the same side of l if and only if P and Q are [together] coplanar with l but $\overleftrightarrow{PQ} \cap l = \varnothing$.

15-4. Two angles are adjacent if and only if they have a common side and their other sides are on opposite sides of the line containing their common side.

15-5. The cosine of an angle is the dot product of the unit vectors in the senses of the sides of the angle.

15-6. A first and a second angle are supplementary [and each is a supplement of the other] if and only if they are congruent to adjacent angles whose noncommon sides are opposite half-lines.

15-7. The bisector of an angle is the half-line interior to it such that the two angles which have this half-line for one side, and one of the sides of the given angle for the other, are congruent.

15-8. The angle bisector of a triangle from a given vertex is the interval whose endpoints are the given vertex and the point at which the bisector of the angle of the triangle at this vertex intersects the opposite side.

15-9. $\angle C$ is smaller than $\angle R$ [and $\angle R$ is larger than $\angle C$] if and only if $\angle C$ is congruent to an angle which shares a side with $\angle R$ and whose other side is interior to $\angle R$.

15-10. (a) A right angle is an angle whose sides are perpendicular.

(b) An acute angle is an angle which is smaller than a right angle.

(c) An obtuse angle is an angle which is larger than a right angle.

15-11. A first and a second angle are complementary [and each is a complement of the other] if and only if they are congruent to adjacent acute angles whose noncommon sides are perpendicular half-lines.

15-12. $\sin \angle C = \sqrt{1 - (\cos \angle C)^2}$

Other Theorems

15-1. Each segment whose endpoints are interior to an angle is a subset of the interior of that angle.

15-2. (a) Each point of $\overline{AB}$ is interior to $\angle AVB$.

(b) If C is interior to $\angle AVB$ then so is each point of $\overrightarrow{VC}$.

(c) C is interior to $\angle AVB$ if and only if $\overline{AB}$ intersects $\overrightarrow{VC}$.

Corollary. If, in $\triangle ABC$, $D \in \overline{BC}$ and $E \in \overline{CA}$ then $\overline{AD}$ and $\overline{BE}$ intersect.

15-3. If $R \in l$, $P \notin l$, and $Q \in \overleftrightarrow{RP}$ then P and Q are on the same side of l if and only if $Q \in \overrightarrow{RP}$, and are on opposite sides of l if and only if Q belongs to the opposite of $\overrightarrow{RP}$.

15-4. $\angle AVC$ and $\angle BVC$ are adjacent if and only if $\overline{AB}$ intersects $\overleftrightarrow{VC}$.

15-5. If B and C are on the same side of $\overleftrightarrow{VA}$ then either C is interior to $\angle AVB$, or $C \in \overrightarrow{VB}$, or B is interior to $\angle AVC$.

15-6. $\cos \angle ACB = (\vec{a} \cdot \vec{b})/(\|\vec{a}\| \, \|\vec{b}\|)$, where $\vec{a} = A - C$ and $\vec{b} = B - C$.

15-7. Given a number k such that $|k| < 1$, and given a half-line r, there is one and only one angle whose cosine is k, which has r as one of its sides, and whose side is contained in a given side of the line containing r.

15-8. Angles are congruent if and only if they have the same cosine.

15-9. Vertical angles are congruent.

15-10. $\angle R$ and $\angle C$ are supplementary if and only if $\cos \angle R + \text{cox} \angle C = 0$.

Corollary. Adjacent angles are supplementary if and only if their noncommon sides are opposite half-lines.

15-11. If the sides of one angle can be paired with those of another in such a way that paired sides are parallel then the angles are either congruent or supplementary. They are congruent if each two paired sides have the same sense or if each two paired sides have opposite senses. They are supplementary if some two paired sides have the same sense and the other two paired sides have opposite senses.

15-12. Given angles in parallel planes, if the sides of one angle can be paired with those of the other in such a way that paired sides are perpendicular, then the angles are either congruent or supplementary.

15-13. Of the angles formed by two parallel lines and a common transversal, any two alternate angles are congruent, any two corresponding angles are congruent, and any two consecutive angles are supplementary.

15-14. If some two alternate or corresponding angles formed by two coplanar lines and a common transversal are congruent, or some two consecutive angles are supplementary, then the lines are parallel.

15-15. The bisector of $\angle AVB$ is the half-line with vertex V whose sense is that of $\vec{a}/\|\vec{a}\| + \vec{b}/\|\vec{b}\|$, where $\vec{a} = A - V$ and $\vec{b} = B - V$.

15-16. The bisector of an angle consists of those points which are interior to the angle and are equidistant from the lines containing the sides of the angle.

15-17. In $\triangle ABC$, the endpoint of $\overline{AB}$ of the angle bisector from C divides the interval from A to B in the ratio of CA to CB.

15-18. $\triangle ABC$ is isosceles with base $\overline{AB}$ if and only if its angle bisector from C is its median from C.

15-19. $\triangle ABC$ is isosceles with base $\overline{AB}$ if and only if its angle bisector from C is its altitude from C.

15-20. The angle bisectors of a triangle are concurrent.

15-21. A first angle is smaller than a second if and only if the cosine of the first angle is greater than the cosine of the second.

15-22. If C is interior to $\angle AVB$ then $\angle AVC$ is smaller than $\angle AVB$.

15-23. If, in $\angle ABC$ and $\angle PQR$, $QP = BA$ and $QR = BC$ then $\angle PQR$ is smaller than $\angle ABC$ if and only if $PR < AC$.

15-24. $\angle R$ and $\angle C$ are complementary if and only if $\cos \angle R > 0$, $\cos \angle C > 0$, and $(\cos \angle R)^2 + (\cos \angle C)^2 = 1$.

Corollary. Adjacent angles are complementary if and only if they are acute and their noncommon sides are perpendicular half-lines.

15-25. $\sin \angle ACB = \dfrac{\sqrt{\|\vec{a}\|^2\,\|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2}}{\|\vec{a}\|\,\|\vec{b}\|}$,

where $\vec{a} = A - C$ and $\vec{b} = B - C$.

15-26. Angles are congruent or supplementary if and only if they have the same sine.

15-27. Given a number a such that $0 < a < 1$, and given a half-line r, there are two and only two angles whose sine is a, which have r as one side, and whose other sides are on a given side of the line which contains r.

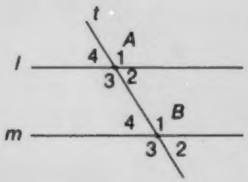
15-28. Two angles are complementary if and only if one of them is acute and its sine is the cosine of the other.

Chapter Test

1. Suppose that $\angle C$ is an angle whose cosine is $\sqrt{2}/2$. Compute each of the following.
   - (a) $\sin \angle C$
   - (b) $\cos \angle P$, where $\angle C$ and $\angle P$ are complementary
   - (c) $\cos \angle Q$, where $\angle C$ and $\angle Q$ are supplementary
   - (d) $\sin \angle Q$, where $\angle C$ and $\angle Q$ are supplementary
2. In $\triangle ABC$, $AB = 6$, $BC = 15$, and $CA = 12$. Suppose that $\overline{AD}$ and $\overline{BE}$ are the angle bisectors from A and B, respectively.
   - (a) Compute BD, DC, AE, and EC.
   - (b) Let P be the point of intersection of $\overline{AD}$ and $\overline{BE}$. In what ratio does P divide the segment from A to D? The segment from B to E?
   - (c) Compute $\cos \angle A$, $\cos \angle B$, and $\cos \angle C$.
3. Assume that $\cos \angle A = \frac{3}{5}$, $\cos \angle B = \frac{5}{13}$, $\cos \angle C = -\frac{3}{5}$, and $\cos \angle D = \frac{12}{13}$.
   - (a) Which of the angles are acute and which are obtuse?
   - (b) Order the angles according to size from smallest to largest.
   - (c) Which, if any, of the angles are complementary?
   - (d) Which, if any, of the angles are supplementary?
4. Suppose that $\angle ABC$ is a right angle, that $\vec{a} = A - B$ and $\vec{c} = C - B$, and that $\vec{a}$ and $\vec{c}$ are unit vectors. Let D, E, F, and G be points such that

$$D = B + \vec{a} + \vec{c}, \qquad E = B + \vec{a}\tfrac{1}{2} + \vec{c}\tfrac{1}{6},$$
$$F = B + \vec{a} \cdot -\tfrac{1}{3} + \vec{c}\tfrac{5}{6}, \text{ and } \qquad G = B + \vec{a}\tfrac{1}{4} + \vec{c} \cdot -\tfrac{3}{4}.$$

   - (a) Which of $\angle ABD$ and $\angle ABE$ is larger? Explain your answer.
   - (b) Which is the larger of $\angle CBG$ and $\angle ABF$? Explain your answer.
   - (c) Show that E and F are on the same side of $\overleftrightarrow{BD}$.
   - (d) Show that E and G are on opposite sides of $\overleftrightarrow{BD}$.
5. Consider the parallel lines l and m and transversal t shown in the picture at the right. In each of the following, give a pair of angles of the specified type and tell whether they are congruent or supplementary.
   - (a) alternate exterior angles
   - (b) corresponding angles
   - (c) consecutive angles
   - (d) alternate interior angles

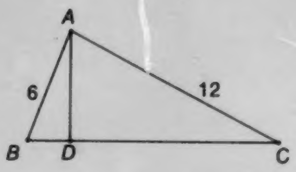


6. Suppose that C is a point such that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are the angle bisectors of $\angle A_2$ and $\angle B_1$, respectively, where $\angle A_2$ and $\angle B_1$ are as described in Exercise 5. Prove the following.
   - (a) $\angle CAB$ and $\angle CBA$ are complementary.
   - (b) $\triangle ACB$ is a right triangle.

Answers for Chapter Test

1. (a) $\sqrt{2}/2$ (b) $\sqrt{2}/2$ (c) $-\sqrt{2}/2$ (d) $\sqrt{2}/2$
2. (a) $BD = 5$; $DC = 10$; $AE = 24/7$; $EC = 60/7$
   - (b) $6/5$; $7/4$
   - (c) $-5/16$; $13/20$; $37/40$
3. (a) $\angle A$, $\angle B$, and $\angle D$ are acute; $\angle C$ is obtuse.
   - (b) $\angle D$, $\angle A$, $\angle B$, $\angle C$
   - (c) $\angle B$ and $\angle D$ are complementary.
   - (d) $\angle A$ and $\angle C$ are supplementary.
4. (a) $\angle ABE$. For, $\cos \angle ABD = 1/\sqrt{2}$, $\cos \angle ABE = 3/\sqrt{25}$, and $3/\sqrt{25} < 1/\sqrt{2}$.
   - (b) $\angle CBG$. For, $\cos \angle CBG = -2/\sqrt{5}$, $\cos \angle ABF = -2/\sqrt{13}$, and $-2/\sqrt{5} < -2/\sqrt{13}$.
   - (c) $E - B = (D - B)\frac{1}{2} + \vec{c}\frac{1}{6}$ and $F - B = (D - B) \cdot -\frac{1}{3} + \vec{c}\frac{5}{6}$. So, E and F are on the same side of $\overleftrightarrow{BD}$ as is C.
   - (d) $E - B = (D - B)\frac{1}{2} + \vec{c}\frac{1}{6}$ and $G - B = (D - B) \cdot \frac{1}{4} + \vec{c} \cdot -\frac{3}{4}$. So, E and G are on opposite sides of $\overleftrightarrow{BD}$.
5. (a) $(\angle A_4, \angle B_2)$ [or: $(\angle A_1, \angle B_3)$]; congruent
   - (b) $(\angle A_1, \angle B_1)$ [or: $([A_2, \angle B_2)$, $(\angle A_3, \angle B_3)$, $(\angle A_4, \angle B_4)$]; congruent
   - (c) $(\angle A_1, \angle B_2)$ [or: $(\angle A_2, \angle B_1)$, $(\angle A_3, \angle B_4)$, $(\angle A_4, \angle B_3)$]; supplementary
   - (d) $(\angle A_2, \angle B_4)$ [or: $(\angle A_3, \angle B_1)$]; congruent
6. (a) $\cos \angle CAB = \sqrt{(1 + \cos \angle A_2)/2}$ and $\cos \angle CBA = \sqrt{(1 + \cos \angle B_1)/2}$. So, $\angle CAB$ and $\angle CBA$ are both acute [since their cosines are positive] and $(\cos \angle CAB)^2 + (\cos \angle CBA)^2 = (1 + \cos \angle A_2)/2 + (1 + \cos \angle B_1)/2 = 1$ since $\cos \angle A_2 + \cos \angle B_1 = 0$ [since $\angle A_2$ and $\angle B_1$ are supplementary]. Hence, $\angle CAB$ and $\angle CBA$ are complementary.
   - (b) It was proved in Part B on page 239 that a triangle two of whose angles are complementary is a right triangle.
7. (a) $6/\sqrt{5}$ (b) $12/\sqrt{5}$ (c) $24/\sqrt{5}$
   (d) $1/\sqrt{5}$ (e) $2/\sqrt{5}$ (f) $2/\sqrt{5}$

7. Assume that $\triangle ABC$ is a right triangle with hypotenuse $\overline{BC}$, that $\overline{AD}$ is the altitude from A, and that $AB = 6$ and $AC = 12$, as shown in the picture at the right. Compute each of the following.



(a) BD (b) AD (c) DC
(d) $\cos \angle B$ (e) $\cos \angle C$ (f) $\sin \angle DAC$

Background Topic

In the exercises at the end of Chapter 14 [pages 199–205] we learned how to use three postulates:

(Nn_1) $0 \in Nn$
(Nn_2) $a \in Nn \longrightarrow a + 1 \in Nn$
(Nn_3) $F0$ and $\forall_x\, (x \in Nn$ and $Fx) \longrightarrow F(x + 1)$
$\longrightarrow [a \in Nn \longrightarrow Fa]$

to prove theorems concerning nonnegative integers. [Recall that '*Nn*' was adopted as a name for the set of nonnegative integers.] Among the theorems proved were:

(1) $a \in Nn \longrightarrow a \geq 0$ [(3) on page 204]
(2) $a \in Nn \longrightarrow [a > 0 \longrightarrow a \geq 1]$ [(4) on page 204]

In the following pages we shall see how to prove theorems about all integers – that is, about the numbers

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots.$$

[We shall use '*I*' as a name for the set of all integers.] To prove such theorems we need a postulate telling us what numbers are integers. Since the integers are just the numbers which either are nonnegative integers or have nonnegative integers for their opposites, we shall adopt:

(I) $a \in I \longleftrightarrow (a \in Nn \text{ or } -a \in Nn)$

For example, Postulate (I) tells us that, since, by (Nn_1) and (Nn_2), $1 \in Nn$ and since $--1 = 1$, $-1 \in I$. As another example of the use of (I) we shall prove:

(3) $a \in I \longrightarrow [a < 0 \longrightarrow a \leq -1]$

[As you may guess, (3) is not very far in meaning from (2), and is not, in itself very startling. Nevertheless, we need it as a lemma to more important theorems.]

In these exercises we continue the study of the integers which was begun in the Background Topic at the end of Chapter 14. As in those exercises, we are not here attempting to teach students to write proofs by mathematical induction. Students will have an opportunity to practice this in a later chapter. At present, we are aiming at only the ability to recognize and understand inductive proofs. In addition we wish to call attention to the important result (8) on page 254. This result — which we shall use later — is intuitively obvious; but the difficulty of the proof may give students a respect for its importance!

Proof of (3): By Postulate (I) it is sufficient to show that $a \epsilon Nn \longrightarrow [a < 0 \longrightarrow a \leq -1]$ and $-a \epsilon Nn \longrightarrow [a < 0 \longrightarrow a \leq -1]$. The first is trivial since, assuming that $a \epsilon Nn$ and that $a \nleq -1$, it follows from the former and (1) that $a \nless 0$, and so, that $a \nleq -1 \longrightarrow a \nless 0$. Thus, $a < 0 \longrightarrow a \leq -1$ so that $a \epsilon Nn \longrightarrow [a < 0 \longrightarrow a \leq -1]$. To establish the second it is sufficient to note that, by (2), $-a \epsilon Nn \longrightarrow [-a > 0 \longrightarrow -a \geq 1]$ and to note that $-a > 0$ if and only if $a < 0$ and that $-a \geq 1$ if and only if $a \leq -1$.

As a third example of the use of (I) we shall prove a theorem which, like (3), is a lemma to a more important theorem:

$$(4) \qquad a \epsilon I \longrightarrow a - 1 \epsilon I$$

Proof of (4): As in the proof of (3) it is sufficient to show that $a \epsilon Nn \longrightarrow a - 1 \epsilon I$ and that $-a \epsilon Nn \longrightarrow a - 1 \epsilon I$. [We shall prove the first by induction.] As we have seen it follows by (Nn_1), (Nn_2), and (I) that $-1 \epsilon I$. Since $0 - 1 = -1$ it follows that $0 - 1 \epsilon I$. Suppose, now, that $b \epsilon Nn$ and that $b - 1 \epsilon I$. Since $(b + 1) - 1 = b$ it follows that $(b + 1) - 1 \epsilon Nn$ and so, by (I), that $(b + 1) - 1 \epsilon I$. [It doesn't matter that we did not need to use the assumption that $b - 1 \epsilon I$.] Hence, for each x, if $x \epsilon Nn$ and $x - 1 \epsilon I$ then $(x + 1) - 1 \epsilon I$. Since, also, $0 - 1 \epsilon I$ it follows that if $a \epsilon Nn$ then $a - 1 \epsilon I$.

To complete the proof of (4), suppose that $-a \epsilon Nn$. It follows by (Nn_2) that $-a + 1 \epsilon Nn$. So, since $-(a - 1) = -a + 1$ it follows that $-(a - 1) \epsilon Nn$ and, so, by (I) that $a - 1 \epsilon I$. Hence, if $-a \epsilon Nn$ then $a - 1 \epsilon I$.

Part A

1. Show that $-a \epsilon I \longleftrightarrow a \epsilon I$. [*Hint*: Begin with 'By (I), $-a \epsilon I$ if and only if'.]
2. Show that $a \epsilon I \longrightarrow a + 1 \epsilon I$. [*Hint*: Use Exercise 1 and (4).]

*

For ease of reference we shall list the results you obtained in Part A:

$$(5) \qquad -a \epsilon I \longleftrightarrow a \epsilon I$$
$$(6) \qquad a \epsilon I \longrightarrow a + 1 \epsilon I$$

Before proceeding further we need to modify (Nn_3), our principle of mathematical induction, to justify a method similar to mathematical induction for establishing theorems about all integers. It is not difficult to see intuitively what is needed. Suppose given a sentence Fa which we wish to show holds for all integral values of 'a'. It should be sufficient to establish $F0$, to show that if Fa holds for $a = b \epsilon Nn$ then it holds for $a = b + 1$, and to show that if it holds for $a = b$ where $-b \epsilon Nn$ then it holds for $a = b - 1$. In other words, we need two "inductions"—one up through the nonnegative integers,

TC 252

Proving (4) is a first step toward proving that the difference of any two integers is an integer — that is, that I is closed with respect to subtraction. The latter theorem together with the closure of I with respect to oppositing [see Exercise 1 of Part A] implies that I is closed with respect to addition.

Answers for Part A

1. By (I), $-a \in I$ if and only if $(-a \in Nn$ or $--a \in Nn)$. Since $--a = a$ it follows that $(-a \in Nn$ or $--a \in Nn)$ if and only if $(a \in Nn$ or $-a \in Nn)$ — that is, if and only if $a \in I$.
2. Suppose that $a \in I$. It follows by Exercise 1 that $-a \in I$ and, so, by (4), that $-a - 1 \in I$. Since $-a - 1 = -(a + 1)$ it follows that $-(a + 1) \in I$ and, so, by Exercise 1, that $a + 1 \in I$. Hence, if $a \in I$ then $a + 1 \in I$.

TC 253

The "Theorem" — which might be thought of as the principle of mathematical induction for I [rather than merely for Nn] — is important and should be made intuitively obvious to all students. The same applies to its corollary on page 253. The proof of the theorem is very simple but, possibly, subtle.

The proof of (7) is complicated by the fact that the sentence Fa of the corollary is a universal generalization. We have done our best to point out what is to be proved at each stage and to choose variables and indices which may be of some mnemonic aid. Nevertheless, the proof is a difficult one, and you need not expect all students to follow it. All students should, however, understand the theorem which is being proved.

TC 254

Theorem (8) is important and, in the text, we have formulated what it says in several ways. Typical consequences of the theorem are that an integer which is less than 3 is less than or equal to 2, and that an integer which is less than -1 is less than or equal to -2. Contrast this with the fact that if a real number is less than 3 [or, less than -1], that's all you know about it.

Answers for Part B

1. By (7) if a and b are integers then so is $b - a$ and, hence, by (3), if $b - a < 0$ then $b - a \leq -1$. But, $b - a < 0$ if and only if $b < a$ and $b - a \leq -1$ if and only if $b \leq a - 1$. Hence, if a and b are integers then $b < a$ only if $b \leq a - 1$.
2. By (8), if $0 \in I$ and $a \in I$ then $a < 0$ only if $a \leq 0 - 1$. But, $0 - 1 = -1$, and it is a theorem that $0 \in I$. Hence, (3).
3. Since, by a previous theorem, $a - 1 < (a - 1) + 1 = a$ it follows that if $b \leq a - 1$ then $b < a$.
4. $(a \in I \text{ and } b \in I) \Longrightarrow [b < a \Longleftrightarrow b \leq a - 1]$
5. Suppose that $a \in I$ and $b \in I$. It follows by (5) that $-a \in I$ and, so, by (7) that $b - -a \in I$. But, $b - -a = b + a$. Hence, if $a \in I$ and $b \in I$ then $a + b \in I$.

and one down through the nonpositive integers. This suggests the following:

Theorem

$$F0 \text{ and } \forall_x\,[(x \in Nn \text{ and } Fx) \longrightarrow F(x+1)] \text{ and } \forall_x\,[(-x \in Nn \text{ and } Fx) \longrightarrow F(x-1)] \longrightarrow [a \in I \longrightarrow Fa]$$

In view of (Nn_3) and (I) all that remains to a proof of this theorem is to show that

$$(F0 \text{ and } \forall_x\,[(-x \in Nn \text{ and } Fx) \longrightarrow F(x-1)]) \longrightarrow [-a \in Nn \longrightarrow Fa].$$

or, equivalently [since $--a = a$] that

$$(F0 \text{ and } \forall_x\,[(x \in Nn \text{ and } F[-x]) \longrightarrow F(-x-1)]) \longrightarrow [a \in Nn \longrightarrow F[-a]].$$

But, this is merely (Nn_3) for the sentence Ga obtained from Fa by substituting '$-a$' for 'a'.

In place of the theorem we can often use the simpler:

Corollary

$$(F0 \text{ and } \forall_x\,[(x \in I \text{ and } Fx) \longrightarrow (F(x+1) \text{ and } F(x-1))]) \longrightarrow [a \in I \longrightarrow Fa]$$

This follows from the theorem and (I) since, by the latter, if $a \in Nn$ then $a \in I$ and if $-a \in Nn$ then $a \in I$.

As an example, we shall use the corollary and Theorems (4) and (6) to prove that I is closed with respect to subtraction. Specifically, we shall prove:

$$(7) \qquad a \in I \longrightarrow \forall_y\,[y \in I \longrightarrow y - a \in I]$$

which, for the purpose of using the corollary, is a convenient restatement of '$(a \in I \text{ and } b \in I) \longrightarrow b - a \in I$'. In the proof our sentence Fa will be '$\forall_y\,[y \in I \longrightarrow y - a \in I]$'. We shall need to establish $F0$. Then, assuming that $b \in I$ and Fb, we shall need to derive $F(b+1)$ and $F(b-1)$.

Proof of (7): Since $b - 0 = b$ it follows that if $b \in I$ then $b - 0 \in I$. Hence, $\forall_y\,[y \in I \longrightarrow y - 0 \in I]$.

Suppose, now, that $b \in I$ and $\forall_y\,[y \in I \longrightarrow y - b \in I]$. We wish to show that

$$\forall_y\,[y \in I \longrightarrow y - (b+1) \in I] \text{ and } \forall_y\,[y \in I \longrightarrow y - (b-1) \in I].$$

For the first, we note that, for any $c \in I$, $c - (b+1) = (c-1) - b$ and that, by (4), $c - 1 \in I$. It follows from our assumption that $(c-1) - b \in I$ and, so, that $c - (b+1) \in I$. Hence, $\forall_y\,[y \in I \longrightarrow y - (b+1) \in I]$.

For the second, we note that, for any $c \in I$, $c - (b-1) = (c+1) - b$ and that, by (6), $c + 1 \in I$. It follows from our assumption that $(c+1) - b \in I$ and, so, that $c - (b-1) \in I$. Hence, $\forall_y\,[y \in I \longrightarrow y - (b-1) \in I]$.

Since we have established that $\forall_y\,[y \in I \longrightarrow y - 0 \in I]$ and that

$$\forall_x\,[(x \in I \text{ and } \forall_y\,[y \in I \longrightarrow y - x \in I]) \longrightarrow (\forall_y\,[y \in I \longrightarrow y - (x+1) \in I] \text{ and } \forall_y\,[y \in I \longrightarrow y - (x-1) \in I])]$$

we may conclude from the corollary that

$$a \in I \longrightarrow \forall_y\,[y \in I \longrightarrow y - a \in I].$$

[This is a much more complex inductive proof than any you will be asked to give in this course.]

It is now easy to use (7) and (3) to prove an important generalization of (3):

$$(8) \qquad (a \in I \text{ and } b \in I) \longrightarrow [b < a \longrightarrow b \leq a - 1]$$

This theorem expresses the "discreteness" of the set of integers—any integer less than one of them is at most as large as the preceding one. Another way of putting it is to say that, for any $a \in I$, $a - 1$ is the greatest integer which precedes a—in particular, each integer has, among the integers, an immediate predecessor. This is an essential difference between the set I of all integers and the set $\mathscr{R}$ of all real numbers. Between any two real numbers there is another.

Part B

1. Prove (8). [*Hint*: In proving (7) we have shown that if $a \in I$ and $b \in I$ then $b - a \in I$. Consider the instance of (3) obtained by substituting '$b - a$' for 'a'.]
2. Show that (3) follows from an instance of (8).
3. Show that, for any real numbers a and b, if $b \leq a - 1$ then $b < a$. [*Hint*: Recall that on page 205 you have proved that, for any real number c, $c < c + 1$.]
4. Use (8) and (3) to obtain a stronger result which includes (8).
5. Show that I is closed with respect to addition. [*Hint*: Use (7) and (5).]

Chapter Sixteen

Triangles and Quadrilaterals

16.01 The Cosines of the Angles of a Triangle

We shall adopt our usual notation as given in Fig. 16-1. According to this, in $\triangle ABC$, $\vec{a} = C - B$, $\vec{b} = A - C$, $\vec{c} = B - A$, $a = \|\vec{a}\|$, $b = \|\vec{b}\|$, and $c = \|\vec{c}\|$. Note that each of $(\vec{a}, \vec{b})$, $(\vec{b}, \vec{c})$, and $(\vec{c}, \vec{a})$ is linearly independent and that

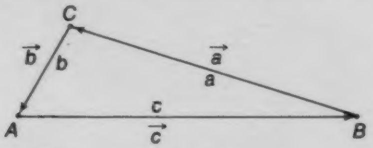


Fig. 16-1

(1) $$\vec{a} + \vec{b} + \vec{c} = \vec{0}.$$

In an earlier exercise we have seen that

(2) $$\cos\angle A = -\frac{\vec{b}\cdot\vec{c}}{bc},\ \cos\angle B = -\frac{\vec{c}\cdot a}{ca},\ \text{and } \cos\angle C = -\frac{\vec{a}\cdot\vec{b}}{ab}.$$

[Explain.] The adjacent supplements of the angles of a triangle are called *exterior angles* of the triangle. So, for example, $\triangle ABC$ has two exterior angles with vertex A [or: *at* A]. These exterior angles are congruent [Why?] and their cosine is $(\vec{b}\cdot\vec{c})/(bc)$. [Explain.] The angles of $\triangle ABC$ at B and C are said to be *opposite* each of the exterior angles at A.

Using (1) and (2) it is easy to establish two useful relations among the measures of the sides of a triangle and the cosines of its angles:

Since $-\vec{b}$ and $\vec{c}$ are vectors in the senses of the sides of $\angle A$,

$$\cos\angle A = \frac{-\vec{b}\cdot\vec{c}}{bc} = -\frac{\vec{b}\cdot\vec{c}}{bc}.$$

Since supplementary angles have opposite cosines, the cosine of an exterior angle of $\triangle ABC$ at A is $(\vec{b}\cdot\vec{c})/(bc)$.

The Projection Theorem and The Cosine Law are very powerful theorems. They are roughly equivalent in that each can be derived from the other and various consequences of the other. This is, however, not an easy task and it is more sensible to prove each independently of the other.

Students should be asked to give two similar formulas for each of those in the two parts of Theorem 16-1 and Theorem 16-2.

$$b\cos\angle C + c\cos\angle B = a \qquad c\cos\angle A + a\cos\angle C = b$$

$$\cos\angle B = \frac{a - b\cos\angle C}{c} \qquad \cos\angle C = \frac{b - c\cos\angle A}{a}$$

$$a^2 = b^2 + c^2 - 2bc\cos\angle A \qquad b^2 = c^2 + a^2 - 2ca\cos\angle B$$

$$\cos\angle A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos\angle B = \frac{c^2 + a^2 - b^2}{2ca}$$

In addition, there are three formulas which can be obtained by transforming the three of Theorem 16-1(b):

$$\cos\angle B = \frac{c - b\cos\angle A}{a},\quad \cos\angle C = \frac{a - c\cos\angle B}{b},$$

$$\cos\angle A = \frac{b - a\cos\angle C}{c}$$

Students should be assured that they are not required to memorize all these formulas. They should remember parts (a) of both theorems in some form which enables them to apply the appropriate alphabetic variants to any given triangle; and they should be aware of the possibility of transforming these expressions as is done in establishing parts (b).

* * *

Suggestions for the exercises of section 16.01:

- (i) Do Exercises 1 and 2 of Part A under teacher direction. Use Exercise 3, Part A to illustrate applications.
- (ii) Exercises 1-4, Part B should be teacher directed. Use Exercise 5, Part B to illustrate applications.
- (iii) Part C may be used for homework, but no student needs to do all seven exercises. Be sure to illustrate each of the seven results in class.
- (iv) Part D should be done by all students. These exercises are important applications.
- (v) Part E can be used as either class discussion material or as homework. Be sure students understand applications of Theorems 16-3 and 16-4.
- (vi) Part F should be directed by the teacher. Be sure to illustrate applications of Theorems 16-5 and 16-6.
- (vii) Parts G and H may be used for homework.
- (viii) Part I makes a nice seat activity in class. The teacher can offer individual help during this activity.

Theorem 16–1 [The Projection Theorem] In $\triangle ABC$,

(a) $a \cos \angle B + b \cos \angle A = c$

and (b) $\cos \angle A = \dfrac{c - a \cos \angle B}{b}$.

Theorem 16–2 [The Cosine Law] In $\triangle ABC$,

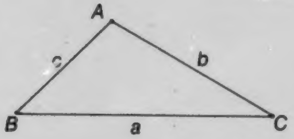
(a) $c^2 = a^2 + b^2 - 2ab \cos \angle C$

and (b) $\cos \angle C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Exercises

Part A

1. Prove Theorem 16–1.
2. Prove Theorem 16–2.
3. In each of the following parts, you are given certain information about a triangle, $\triangle ABC$. Make use of the results in Theorems 16–1 and 16–2 to do the indicated computations.



   (a) $a = 5$, $b = 9$, and $c = 8$. Compute $\cos \angle A$ and $\cos \angle B$.
   (b) $b = 10$, $c = 6$, and $\cos \angle A = \frac{3}{5}$. Compute a, $\cos \angle C$, and $\cos \angle B$.
   (c) $a = 10$, $b = 6$, and $\cos \angle C = -\frac{3}{5}$. Compute c and $\cos \angle A$.
   (d) $a = 9$, $c = 8$, and $\cos \angle B = \frac{1}{2}$. Compute $\cos \angle C$ and b.

Part B

Given $\triangle ABC$ where, as usual, $AB = c$, $BC = a$, and $CA = b$.

1. Show that
   (a) $\cos \angle A + \cos \angle B = (a + b)(1 - \cos \angle C)/c$, and
   (b) $\cos \angle A - \cos \angle B = (b - a)(1 + \cos \angle C)/c$.
   [*Hint*: Use two instances of part (b) of the Projection Theorem.]
2. Show that
   (a) $1 + \cos \angle C = \dfrac{(a + b)^2 - c^2}{2ab}$ and
   (b) $1 - \cos \angle C = \dfrac{c^2 - (a - b)^2}{2ab}$.
   [*Hint*: Use part (b) of the Cosine Law.]
   (c) Make use of the results in (a) and (b) to show that

$$1 + \cos \angle C = \frac{(a + b + c)(a + b - c)}{2ab} \text{ and}$$

$$1 - \cos \angle C = \frac{(c + a - b)(c + b - a)}{2ab}.$$

Answers for Part A

1. (a) $a \cos \angle B + b \cos \angle A = a[-\vec{c} \cdot \vec{a}/(ca)] + b[-\vec{b} \cdot \vec{c}/(bc)]$
$= (-\vec{c} \cdot \vec{a} + -\vec{b} \cdot \vec{c})/c = -(\vec{a} + \vec{b}) \cdot \vec{c}/c$ [Since $a \neq 0 \neq b$]
$= \vec{c} \cdot \vec{c}/c = c$ [Since $c \neq 0$]
   (b) This follows at once from part (a) since $b \neq 0$.
2. (a) $a^2 + b^2 - 2ab \cos \angle C = a^2 + b^2 + 2ab\,[\vec{a} \cdot \vec{b}/(ab)]$
$= a^2 + b^2 + 2(\vec{a} \cdot \vec{b})$ [Since $ab \neq 0$]
$= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$
$= \|-\vec{c}\|^2 = c^2$
   (b) This follows at once from part (a) since $ab \neq 0$.
3. (a) $\cos \angle A = 5/6$, $\cos \angle B = 1/10$
   (b) $a = 8$, $\cos \angle C = 4/5$, and $\cos \angle B = 0$
   (c) $c = 4\sqrt{13}$, $\cos \angle A = 3/\sqrt{13}$
   (d) $b = \sqrt{73}$, $\cos \angle C = 5/\sqrt{73}$

Answers for Part B

1. (a) $\cos \angle A + \cos \angle B = (b - a \cos \angle C)/c + (a - b \cos \angle C)/c$
$= (a + b)(1 - \cos \angle C)/c$
   (b) $\cos \angle A - \cos \angle B = (b - a \cos \angle C)/c - (a - b \cos \angle C)/c$
$= (b - a)(1 + \cos \angle C)/c$
2. (a) $1 + \cos \angle C = 1 + (a^2 + b^2 - c^2)/(2ab) = (2ab + a^2 + b^2 - c^2)/(2ab)$
$= [(a + b)^2 - c^2]/(2ab)$
   (b) $1 - \cos \angle C = (2ab - a^2 - b^2 + c^2)/(2ab) = [c^2 - (a - b)^2]/(2ab)$
   (c) $1 + \cos \angle C = [(a + b)^2 - c^2]/(2ab) = (a + b + c)(a + b - c)/(2ab)$
$1 - \cos \angle C = (c^2 - (a - b)^2)/2ab = (c + (a - b))(c - (a - b))/(2ab)$

[For each of parts (a), (b), and (c), students should be asked to give two formulas similar to the given one.]

*

At this point it would be worthwhile to ask students what theorems concerning triangles they can deduce from Exercises 1 and 2 of Part B. [Seven such theorems are given them in Part C to prove, but the more they think of for themselves, the better.]

3. Show that

$$\sin \angle C = \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \Big/ (2ab).$$

[*Hint*: Use Exercise 2.]

4. Let $s = (a+b+c)/2$. [s is sometimes called *the semiperimeter* of $\triangle ABC$.] Show that
   (a) $s - a = (-a+b+c)/2$, and [so] that
   (b) $\sin \angle C = 2\sqrt{s(s-a)(s-b)(s-c} \big/ (ab)$.
5. In each of the following, you are given measures for three sides of $\triangle ABC$. Do the indicated computations.
   (a) $a = 5$, $b = 9$, $c = 7$. Compute $\sin \angle A$ by Exercise 3 and $\sin \angle B$ by Exercise 4.
   (b) $a = 8$, $b = 8$, $c = 12$. Compute $\sin \angle A$, $\sin \angle B$, and $\sin \angle C$.
   (c) $a = 12$, $b = 13$, $c = 5$. Compute $\sin \angle B$ and $\sin \angle C$.
   (d) $a = 24$, $b = 7$, $c = 25$. Compute $\sin \angle A$ and $\sin \angle B$.

Part C

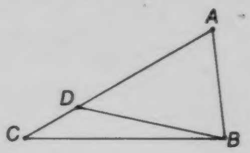
Prove the following corollaries of the Projection Theorem.

1. No two angles of a triangle are supplementary. [*Hint*: Use Exercise 1(a) of Part B. Recall that $|\cos \angle C| < 1$.]
2. Each triangle has at least two acute angles. [*Hint*: Show that at least one of any two angles of a triangle must be acute.]
3. Any exterior angle of a triangle is larger than each of the angles of the triangle opposite it.
4. A triangle is isosceles with a given side as base if and only if the angles of the triangle at the endpoints of the given side are congruent.
5. A triangle is equilateral if and only if it is equiangular. [By definition, a triangle is equiangular if and only if each two of its angles are congruent.]
6. If two sides of a triangle are not congruent then the angle opposite the longer side is larger than the angle opposite the shorter side.
7. If two angles of a triangle are not congruent then the side opposite the larger angle is longer than the side opposite the smaller angle.

Part D

In each of the following, you are given a figure and some information about it. Give arguments to support the stated conclusions.

1. Given: $BD = AD$
   Conclude: **(a)** $AC > BC$
   (b) $\angle A$ is smaller than $\angle ABC$
   (c) $\angle ADB$ is larger than $\angle C$



Answers for Part B [cont.]

3. By definition, $(\sin\angle C)^2 = 1 - (\cos\angle C)^2 = (1+\cos\angle C)(1-\cos\angle C)$

$$= \frac{(a+b+c)(a+b-c)}{2ab} \cdot \frac{(c+a-b)(c+b-a)}{2ab} \quad \text{[By Exercise 2(c)].}$$

So, $\sin\angle C$ can be computed from the given formula. [In taking square roots on both sides, note that, by definition, $\sin\angle C > 0$ and that $2ab > 0$.]

4. (a) $s - a = \dfrac{a+b+c}{2} - a = \dfrac{a+b+c-2a}{2} = \dfrac{-a+b+c}{2}$

   [Similarly, $s - b = (a-b+c)/2$ and $s - c = (a+b-c)/2$.]

   (b) [By substitution from Exercise 4(a) into the formula of Exercise 3.]

5. (a) $\sin\angle A = \sqrt{11}/6$, $\sin\angle B = 3\sqrt{11}/10$

   (b) $\sin\angle A = \sqrt{7}/4 = \sin\angle B$, $\sin\angle C = 3\sqrt{7}/8$

   (c) $\sin\angle B = 1$, $\sin\angle C = 5/13$

   (d) $\sin\angle A = 24/25$, $\sin\angle B = 7/25$

   [Lead students to notice a partial check: $(s-a) + (s-b) + (s-c) = s$]

Answers for Part C

1. By Exercise 1(a) of Part B, for any two angles, $\angle A$ and $\angle B$, of a triangle, $\cos\angle A + \cos\angle B \neq 0$. [For, $a + b \neq 0$, $\cos\angle C \neq 1$, and $c \neq 0$.] Hence, no two angles of a triangle are supplementary.

2. By Exercise 1(a) of Part B, for any two angles, $\angle A$ and $\angle B$, of a triangle, $\cos\angle A + \cos\angle B > 0$. [For, $a + b > 0$, $\cos\angle C < 1$, and $c > 0$.] Hence, at least one of any two angles of a triangle must be acute. So, each triangle has at least two acute angles.

3. Suppose that $\angle A_1$ is an exterior angle at A of $\triangle ABC$. It follows that $\cos\angle A + \cos\angle A_1 = 0$. But, as in Exercise 2, $\cos\angle A + \cos\angle B > 0$. Hence $\cos\angle B > \cos\angle A_1$ and, so, $\angle A_1$ is larger than $\angle B$. Similarly, $\angle A_1$ is larger than $\angle C$.

4. From Exercise 1(b) of Part B it follows, since $1 + \cos\angle C \neq 0$ and $c \neq 0$, that $a = b$ if and only if $\cos\angle A = \cos\angle B$ — that is, if and only if $\angle A \cong \angle B$. Hence, $\triangle ABC$ is isosceles with base $\overline{AB}$ if and only if $\angle A \cong \angle B$.

5. [This follows directly from Exercise 4.]

6. Suppose that, in $\triangle ABC$, $b > a$. Since $1 + \cos\angle C > 0$ and $c > 0$ it follows from Exercise 1(b) of Part B that $\cos\angle A > \cos\angle B$ — that is, that $\angle B$ is larger than $\angle A$.

7. Suppose that, in $\triangle ABC$, $\angle B$ is larger than $\angle A$. It follows that $\cos\angle A > \cos\angle B$ and, since $1 + \cos\angle C > 0$ and $c > 0$, it follows from Exercise 1(b) of Part B that $b > a$.

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Answers for Part D

1. (a) $AC = AD + DC = BD + DC > BC$, by Theorem 14-8(a).

   (b) Since $BD = AD$ it follows by Exercise 4 of Part C that $\angle A \cong \angle ABD$. Since $D \in \overline{AC}$, D is interior to $\angle ABC$ and, so, $\angle ABD$ is smaller than $\angle ABC$ [Theorem 15-22]. So, $\angle A$ is smaller than $\angle ABC$.

   (c) $\angle ADB$ is an exterior angle of $\triangle BDC$ and $\angle C$ is one of the angles opposite it. So [by Exercise 3 of Part C], $\angle ADB$ is larger than $\angle C$.

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2. By two applications of Theorem 14-8(a), $RS < RQ + QS < RQ + QP + PS$.

3. (a) By two applications of the exterior angle inequality, $\angle DCF$ is larger than $\angle CBF = \angle CBE$, and $\angle CBE$ is larger than $\angle AEB$. Hence, $\angle DCF$ is larger than $\angle AEB$.

   (b) $\angle FEA$ is larger than $\angle ABE = \angle ABF$, and $\angle ABF$ is larger than each of $\angle F$ and $\angle BCF$.

   [Note how the "Given" ensures equality of $\angle CBF$ and $\angle CBE$ and of $\angle ABE$ and $\angle ABF$, as well as justifying assertions about which angles are exterior angles, etc.]

4. Let T be the point in which $\overrightarrow{PS}$ intersects $\overline{QR}$. Then $PT + TQ = PS + ST + TQ > PS + SQ$. Similarly, $PR + RQ = PR + RT + TQ > PT + TQ$. So, $PR + RQ > PS + SQ$.

5. (a) By the triangle inequality, $SP + SQ > PQ$, $SQ + SR > QR$, and $SR + SP > RP$. Adding, we see that $(SP + SQ + SR)2 > PQ + QR + RP$.

   (b) By Exercise 4, $SP + SQ < QR + RP$, $SQ + SR < RP + PQ$, and $SR + SP < PQ + QR$. Adding, we see that $(SP + SQ + SR)2 < (PQ + QR + RP)2$.

6. (a) By hypothesis, $AC = AB = DB$. By the triangle inequality, $AD > DC - AC = DC - DB = BC$.

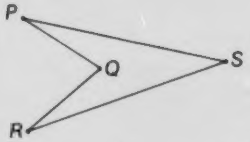
   (b) By hypothesis and Exercise 4 of Part C, $\angle C \cong \angle ABC$. By the exterior angle inequality, $\angle ABC$ is larger than $\angle D$. Hence, $\angle C$ is larger than $\angle D$.

   (c) Since $\angle C$ is larger than $\angle D$, $AD > AC$. And, since $AB = AC$, $AD > AB$. On the other hand $AD < AB + DB = AB + AB = 2AB$. So, $AB < AD < 2AB$.

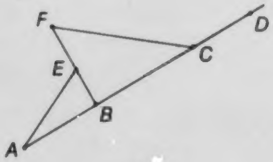
Answers for Part E

1. Suppose that, in $\triangle ABC$, $BC > CA$ and let D be the point of $\overline{BC}$ such that $CD = CA$. Use the notation in the figure. Since D is interior to $\angle CAB = \angle A$, $\angle A$ is larger than $\angle A_2$. By Exercise 4 of Part C, since $CD = CA$, $\angle A_2 \cong \angle D_1$. By the exterior angle inequality $\angle D_1$ is larger than $\angle B$. Hence, $\angle A$ is larger than $\angle B$.

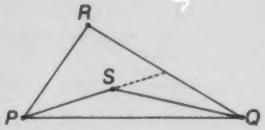
2. Given: *PQRS* is a quadrilateral.
Conclude: $RS < RQ + QP + PS$



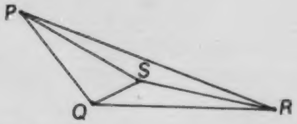
3. Given: $B \in \overline{AC}$, $C \in \overline{BD}$, and $E \in \overline{BF}$.
Conclude: **(a)** $\angle DCF$ is larger than $\angle AEB$
(b) $\angle FEA$ is larger than each of $\angle F$ and $\angle BCF$



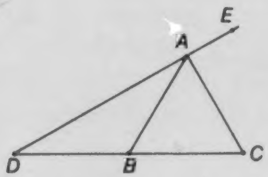
4. Given: *S* is a point in the interior of $\triangle PQR$
Conclude: $PS + SQ < PR + RQ$



5. Given: *S* is a point in the interior of $\triangle PQR$.
Conclude: **(a)** $SP + SQ + SR$ is greater than $\frac{1}{2}(PQ + QR + RP)$
(b) $SP + SQ + SR$ is less than $PQ + QR + RP$ [*Hint*: Use Exercise 4.]

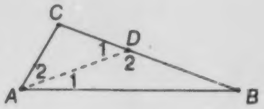


6. Given: Isosceles triangle, $\triangle ABC$, with base $\overline{BC}$, $B \in \overline{DC}$, $AB = BD$, and $A \in \overline{DE}$.
Conclude: **(a)** $AD > BC$
(b) $\angle C$ is larger than $\angle D$
(c) $AB < AD < 2AB$

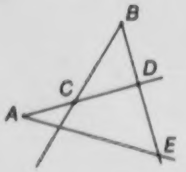


Part E

1. In proving the theorems in Exercises 6 and 7 of Part C you probably used Exercise 1(b) of Part B. The accompanying picture suggests another way of solving Exercise 6 by using Exercises 3 and 4 of Part B and two theorems from Chapter 15. Give this proof. [*Hint*: It is assumed, as in the figure, that $BC > CA$ and that $CD = CA$.]



2. Use the theorems of Exercises 4 and 6 of Part C to prove the theorem of Exercise 7.
3. In applying Theorem 15–26 it is useful to have ways of being certain that two angles are *not* supplementary. Show that if each side of one angle intersects both sides of another angle, then the angles are not supplementary. [*Hint*: Assume that the angles are $\angle A$ and $\angle B$. Label three of the points of intersection so that, as in the picture, they are in the order A, C, D on a side of $\angle A$ and in the order B, D, E on a side of $\angle B$. Then $\angle BDA = \angle BDC$ and $\angle A$ is opposite the exterior $\angle BDA$ of $\triangle ADE$. Now use Exercise 1(a) of Part B and Exercise 3 of Part C.]



4. In Exercise 3 of Part B, above, we saw that

$$\sin \angle C = \sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)} \Big/ (2ab).$$

Make use of this result to show that, in $\triangle ABC$,

$$\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c}.$$

5. According to Theorem 14–8, in $\triangle ABC$, $|a - b| < c < a + b$. Derive this result by making use of Cosine Law. [*Hint*: See Exercise 2 of Part B, above.]
6. Assume that a, b, and c are positive numbers such that $|a - b| < c < a + b$.
   (a) Show that $-1 < \dfrac{a^2 + b^2 - c^2}{2ab} < 1$.
   (b) Make use of part (a) to show that there is a triangle whose side measures are a, b, and c. [*Hint*: Consult Theorem 15–7 and the Cosine Law.]

*

In the exercises just completed, you proved the following theorems:

Theorem 16–3 If a, b, and c are positive numbers such that

$$|a - b| < c < a + b$$

then there is a triangle whose side measures are a, b, and c.

Theorem 16–4 [The Sine Law] In $\triangle ABC$,

$$\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c}.$$

Answers for Part E [cont.]

2. Suppose that, in $\triangle ABC$, $\angle A$ is larger than $\angle B$. Then either $BC < CA$, or $BC = CA$, or $BC > CA$. By Exercise 6 of Part C, if $BC < CA$ then $\angle A$ is smaller than $\angle B$; and, by Exercise 4, if $BC = CA$ then $\angle A \cong \angle B$. Since neither of these is the case it follows that $BC > CA$.

3. Following the hint, we note that by the exterior angle inequality, $\angle BDA$ is larger than $\angle A$ and, so, $\cos\angle A > \cos\angle BDA$. Also, by Exercise 1(a) of Part B, $\cos\angle BDC + \cos\angle B > 0$. Hence, since $\angle BDC = \angle BDA$ it follows that $\cos\angle BDA + \cos\angle B > 0$. So, since $\cos\angle A > \cos\angle BDA$ it follows that $\cos\angle A + \cos\angle B > 0$ and, so, that $\angle A$ and $\angle B$ are not supplementary.

4. Since $c \neq 0$,

$$\frac{\sin\angle C}{c} = \frac{\sqrt{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}}{2abc}.$$

Making use of the cyclic symmetry of our notation it is easy to see that the right sides of similar formulas for '$\sin\angle A/a$' and '$\sin\angle B/b$' are equivalent fractions. Hence,

$$\frac{\sin\angle A}{a} = \frac{\sin\angle B}{b} = \frac{\sin\angle C}{c}.$$

5. Since $1 + \cos\angle C > 0$ and $1 - \cos\angle C > 0$ it follows by Exercise 2 of Part B [which follows from the cosine law] that $(a + b)^2 - c^2 > 0$ and $c^2 - (a - b)^2 > 0$. Hence, $|a - b|^2 < c^2 < (a + b)^2$ and, since $|a - b|$, c, and $a + b$ are all nonnegative, $|a - b| < c < a + b$.

6. (a) Since $|a - b|$, c, and $a + b$ are all nonnegative, $|a - b| < c < a + b$, and $|a - b|^2 = (a - b)^2$, it follows that

$$a^2 + b^2 - 2ab < c^2 < a^2 + b^2 + 2ab,$$

$$a^2 + b^2 - c^2 - 2ab < 0 < a^2 + b^2 - c^2 + 2ab,$$

$$\frac{a^2 + b^2 - c^2}{2ab} - 1 < 0 < \frac{a^2 + b^2 - c^2}{2ab} + 1, \quad [\text{since } 2ab > 0]$$

and, so, that

$$-1 < \frac{a^2 + b^2 - c^2}{2ab} < 1.$$

(b) By Theorem 15-7 and part (a) there is, at any chosen point C, an angle — say, $\angle C$ — such that $\cos\angle C = (a^2 + b^2 - c^2)/(2ab)$. On one side of this angle let A be the point at a distance b from C and, on the other side, let B be the point at a distance a from C. Then $\triangle ABC$ is a triangle which has two sides of measure a and b, respectively. By the cosine law, the third side — that opposite $\angle C$ — has measure c [because, in part, of the choice made for $\cos\angle C$].

*

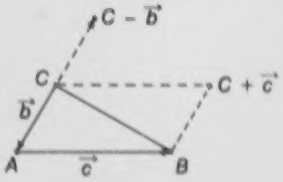
Theorem 16-3 will be used later to show that spheres [and coplanar circles] intersect "as they should."

Part F

1. Making use of the Projection Theorem and the Sine Law show that in a right triangle,
   (a) the cosine of an acute angle is the quotient of the adjacent leg [that is, the leg contained in the angle] by the hypotenuse, and
   (b) the sine of an acute angle is the quotient of the opposite leg by the hypotenuse.
2. Given a triangle, $\triangle ABC$, we know that $c = b \cos \angle A + a \cos \angle B$.
   (a) Show that $c \sin \angle C = b \sin \angle C \cos \angle A + a \sin \angle C \cos \angle B$.
   (b) Make use of the result in (a) and the Sine Law to show that

$$(*) \qquad \sin \angle C = \sin \angle A \cos \angle B + \cos \angle A \sin \angle B.$$

3. Consider the picture of $\triangle ABC$ shown at the right. Note the exterior angle at C. The picture, together with an exercise from Chapter 15 suggests a proof for the following:



(**) The cosine of an exterior angle at C is $\cos \angle A \cos \angle B - \sin \angle A \sin \angle B$.

Prove (**).

4. (a) In $\triangle ABC$, what can be said about $\angle C$ and an exterior angle of $\triangle ABC$ at C? What can be said of the cosines of these angles?
   (b) Making use of (**) and your answers in (a), express $\cos \angle C$ in terms of the sines and cosines of $\angle A$ and $\angle B$.

*

Some of the results of Part F are summarized in the next two theorems:

> **Theorem 16–5** In a right triangle, the cosine of an acute angle is the quotient of the adjacent leg by the hypotenuse, and the sine of an acute angle is the quotient of the opposite leg by the hypotenuse.

> **Theorem 16–6** In $\triangle ABC$,
> $\cos \angle C = -(\cos \angle A \cos \angle B - \sin \angle A \sin \angle B)$, and
> $\sin \angle C = \sin \angle A \cos \angle B + \cos \angle A \sin \angle B$.

Part G

In each of the following, you are given a figure and some information about it. Make use of the results in Theorems 16–3 through 16–6 to do the indicated computations.

Answers for Part F

1. (a) Suppose that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AC}$. Then, $\cos \angle B = 0$ and, by Theorem 16-1(b), $\cos \angle A = c/b$. Similarly, $\cos \angle C = a/b$. [Alternative solution: Suppose that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$. Then, $\sin \angle C = 1$ and, by the sine law, $\sin \angle A = a/c$ and $\sin \angle B = b/c$. The fact that $\cos \angle A = b/c$ and $\cos \angle B = a/c$ follows from the Pythagorean theorem and the fact (see Part B on page 239) that $\cos \angle A > 0$ and $\cos \angle B > 0$.]

   (b) $\sin \angle A = \sqrt{1 - (\cos \angle A)^2} = \sqrt{1 - c^2/b^2} = \sqrt{b^2 - c^2}/b = a/b$ [since, in the notation of part (a), b is the measure of the hypotenuse]. Similarly, $\sin \angle C = c/b$.

   [These results are of sufficient importance that it would be well for you to stress the alternate solution for part (a) in class — with, of course, an appropriate picture. Do not be surprised, however, if students prefer to use the sine law directly, rather than these classical formulas for the sines and cosines of the acute angles of a right triangle. On the other hand, the cosine formulas have been foreshadowed in Part A on page 215 and the text immediately following it; and the sine formulas have been foreshadowed on page 240.]

2. (a) This results immediately from Theorem 16-1(a), as quoted in the preamble.

   (b) By part (a),

$$\begin{aligned} \sin \angle C &= b \cos \angle A \cdot \frac{\sin \angle C}{c} + a \cos \angle B \cdot \frac{\sin \angle C}{c} \\ &= b \cos \angle A \cdot \frac{\sin \angle B}{b} + a \cos \angle B \cdot \frac{\sin \angle A}{a} \\ &= \cos \angle A \sin \angle B + \cos \angle B \sin \angle A. \end{aligned}$$

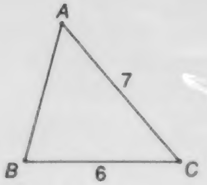
   [The result (*) tells us that once we know the cosines — and, hence, the sines — of two angles of a triangle then we know (at least) one sine of the third angle. The result of Exercise 4(b) shows us that we also know the cosine of the third angle and, so, know its size.]

3. Since $B = (C + \vec{c}) + \vec{b}$, B and $C - \vec{b}$ are on opposite sides of $\overleftrightarrow{C(C + \vec{c})}$ and, so, $\angle BC(C + \vec{c})$ and $\angle(C + \vec{c})C(C - \vec{b})$ are adjacent angles. They are not supplementary since $(-\vec{b}, \vec{c})$ is linearly independent. It follows that Exercise 6 of Part D on page 244 applies. Direct computation of their cosines shows that $\angle BC(C + \vec{c})$ and $\angle CBA$ have the same cosine, as do $\angle(C + \vec{c})C(C - \vec{b})$ and $\angle A$. Hence, (**).

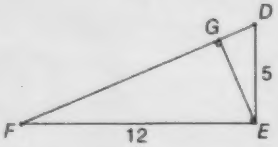
4. (a) $\angle C$ and an exterior angle of $\triangle ABC$ at C are supplementary and, so, have opposites for cosines.

   (b) $\cos \angle C = \sin \angle A \sin \angle B - \cos \angle A \cos \angle B$.

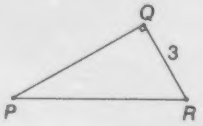
1. Given: In $\triangle ABC$, $AC = 7$, $BC = 6$, $\sin \angle B = \frac{1}{3}$, and $\angle B$ is acute.
   Compute: $\sin \angle A$, $\sin \angle C$, and AB



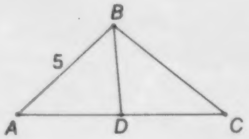
2. In Exercise 1, replace 'acute' by 'obtuse' and do the required computing.
3. Given: Right triangle, $\triangle DEF$, with hypotenuse $\overline{DF}$, altitude $\overline{EG}$, $DE = 5$, and $EF = 12$.
   Compute: $\cos \angle F$, $\cos \angle D$, DG, $\sin \angle GEF$, and GE



4. Given: Right triangle, $\triangle PQR$, with hypotenuse $\overline{PR}$, $QR = 3$, and $\sin \angle P = \frac{1}{2}$.
   Compute: PR, PQ, $\sin \angle R$, and $\cos \angle R$



5. Given: $\triangle ABC$ with median $\overline{BD}$, $AB = 5$, $AC = 8$, and $\cos \angle A = \frac{1}{2}$.
   Compute: BC, $\sin \angle C$, BD, $\sin \angle DBC$, and $\sin \angle ABC$



Part H

1. A ship sails due north for 15 miles from a point P to a point Q and then sails due west for 20 miles to a point R.
   (a) How far is R from P?
   (b) A second ship starts at a point S which is 5 miles due east from P and sails directly towards R. How far is S from R?
   (c) Compute $\sin \angle RSP$ and $\sin \angle SQR$.
2. A ladder leaning against a wall is said to be in a "safe" position when its foot is 1 foot from the base of the wall for each 4 feet of ladder.
   (a) How far from the base of a wall should the foot of an 18-foot ladder be placed?
   (b) How far up a wall would the top of a 14-foot ladder in safe position reach?
   (c) What is the cosine of the "safe angle" of a ladder which is 16 feet long?
3. Assume that ports A and C are 28 miles apart. A ship leaves A sailing toward a point B and a ship leaves C sailing toward the same point B.
   (a) Given that $\cos \angle BAC = \frac{3}{5}$ and $\cos \angle BCA = \frac{5}{13}$, tell how far B is from A and from C.

Answers for Part G

1. $\sin\angle A = 2/7$, $\sin\angle C = (4\sqrt{2} + 3\sqrt{5})/21$, $AB = 4\sqrt{2} + 3\sqrt{5}$ [Use the sine law and the given data to find $\sin\angle A$. Since $\angle B$ is acute, $\angle A$ which is opposite a shorter side is also acute. So, since $\sin\angle A = 2/7$ and $\sin\angle B = 1/3$, $\cos\angle A = 3\sqrt{5}/7$ and $\cos\angle B = 2\sqrt{2}/3$. Find $\sin\angle C$ by Theorem 16-6 and AB by the sine law.]
2. $\sin\angle A = 2/7$, $\sin\angle C' = (3\sqrt{5} - 4\sqrt{2})/21$, $AB = 3\sqrt{5} - 4\sqrt{2}$ [Computations similar to those for Exercise 1 except that, since $\angle B$ is obtuse, $\cos\angle B = -2\sqrt{2}/3$. Note that since $\angle B$ is obtuse, $\angle A$ must, as in Exercise 1, be acute. (See Exercise 2 of Part C on page 257.)]
3. $\cos\angle F = 12/13$, $\cos\angle D = 5/13$, $DG = 25/13$, $\sin\angle GEF = 12/13$, $GE = 60/13$
4. $PR = 6$, $PQ = 3\sqrt{3}$, $\sin\angle R = \sqrt{3}/2$, $\cos\angle R = 1/2$
5. $BC = 7$, $\sin\angle C = 5\sqrt{3}/14$, $BD = \sqrt{21}$, $\sin\angle DBC = 10/(7\sqrt{7})$, $\sin\angle ABC = 4\sqrt{3}/7$

Answers for Part H

1. (a) 25 miles (b) $5\sqrt{34}$ miles [$\cos\angle SPR = -4/5$]
   (c) $\sin\angle RSP = 3/\sqrt{34}$, $\sin\angle SQR = 3/\sqrt{10}$
2. (a) 9/2 feet (b) $7\sqrt{15}/2$ feet (c) 1/4
3. (a) B is 30 miles from A and 26 miles from C. [Use Exercise 2 of Part F to find that $\sin\angle B = 56/65$.]

Answers for Part H [cont.]

3. (b) $25\sqrt{65}/4$ miles [Use the cosine law, computing $\cos\angle B$ from part (a) as $33/65$.]
   (c) $10\sqrt{397/13}$ miles [$\doteq$ 55.3 miles] [Rough check: If the ship from A travelled at 26 mi/hr, the distance between the ships after 2 hours would be twice the distance between A and C.]
   (d) $20/\sqrt{397/13}$ hours [$\doteq$ 3 hours and 37 minutes]
4. (a) $10\sqrt{442}$ pounds (b) $103/(5\sqrt{442})$
5. (a) 80 pounds (b) $-1/8$ (c) $9/16$

Answers for Part I

1. (b) No.
2. (a) [Students may wish to refer to page 216 for a method of constructing $\angle A$.]
   (b) No.
3. (b) No.
4. (b) Yes. There are two triangles which satisfy the given conditions.
5. (b) No.
6. (b) No.

*

Exercise 1 of Part I foreshadows the side-side-side congruence theorem; Exercise 2 foreshadows s.a.s.; Exercise 6 foreshadows a.s.a.; Exercises 3, 4, and 5 explore some of the possibilities of the "ambiguous" case. These are included in Theorem 16-7 on page 266 and in its first corollary on page 269.

(b) Assume that the ships leave B at the same time, move at the same rate, and continue to travel along their original courses. How far must each ship travel before they are 50 miles apart?

(c) Assume, again, that the ships leave B at the same time traveling along their original courses. However, assume that the ship from A travels at 25 miles per hour, while the ship from C travels at 30 miles per hour. How far apart are the ships after 2 hours?

(d) Given the conditions in part (c), how long is it before the ships are 100 miles apart?

4. Forces of 70 pounds and 150 pounds act on the same point at an angle whose cosine is $\frac{4}{5}$.
(a) What is the magnitude of the resultant of these forces?
(b) What is the cosine of the angle between the resultant force and the force of 150 pounds?

5. Two forces, one of f pounds and one of 100 pounds, act on the same point. The resultant of these forces is a force of 120 pounds acting at an angle whose cosine is $\frac{3}{4}$ with the 100-pound force.
(a) Compute the magnitude of the force f.
(b) What is the cosine of the angle between the force of f pounds and the force of 100 pounds?
(c) What is the cosine of the angle between the force of f pounds and the [resultant] force of 120 pounds?

Part I

[For these exercises, use compass and ruler.]

1. (a) Draw a triangle whose side lengths are 2, 3, and 4 inches.
(b) Can you carry out the instructions in part (a) so as to obtain a triangle which does not have the same size and shape as the one you have drawn?

2. (a) Draw $\triangle ABC$ such that $\cos\angle A = \frac{1}{4}$ and the lengths of $\overline{AB}$ and $\overline{AC}$ are 2 inches and 3 inches, respectively.
(b) Repeat Exercise 1(b) for $\triangle ABC$.

3. (a) Draw $\triangle PQR$ such that $\cos\angle P = -\frac{3}{4}$ and the lengths of $\overline{PQ}$ and $\overline{QR}$ are 2 inches and 3 inches, respectively.
(b) Repeat Exercise 1(b) for $\triangle PQR$.

4. (a) Draw $\triangle HJK$ such that $\cos\angle H = \frac{3}{4}$ and the lengths of $\overline{HJ}$ and $\overline{JK}$ are 4 inches and 3 inches, respectively.
(b) Repeat Exercise 1(b) for $\triangle HJK$.

5. (a) Draw $\triangle LMN$ such that $\cos\angle L = \frac{3}{4}$ and the lengths of $\overline{LM}$ and $\overline{MN}$ are 3 inches and 4 inches, respectively.
(b) Repeat Exercise 1(b) for $\triangle LMN$.

6. (a) Draw $\triangle STU$ such that $\cos\angle S = \frac{1}{4}$, $\cos\angle T = \frac{3}{4}$, and the length of $\overline{ST}$ is 3 inches.
(b) Repeat Exercise 1(b) for $\triangle STU$.

16.02 Congruence Theorems

Recall that two sets are congruent if and only if there is an isometry – that is, a distance-preserving mapping of $\mathscr{E}$ onto itself – which maps one set onto the other. We know that, given points A, B, P, and Q, there is an isometry which maps A on P and B on Q if and only if $AB = PQ$, and we know that any such isometry maps $\overline{AB}$ onto $\overline{PQ}$. Also, given A, B, C, P, Q, and R, there is an isometry which maps A on P, B on Q, and C on R if and only if $AB = PQ$, $BC = QR$, and $CA = RP$. Moreover, if $\{A, B, C\}$ is noncollinear then there are just two such isometries, and both map $\overleftrightarrow{ABC}$ onto $\overleftrightarrow{PQR}$ in the same manner. [Each of these two isometries is the resultant of the other followed by the reflection in $\overleftrightarrow{PQR}$.]

It follows that an isometry which maps the vertices of $\triangle ABC$ on those of $\triangle PQR$ also maps $\triangle ABC$ onto $\triangle PQR$. [Explain.] Conversely, an isometry f which maps $\triangle ABC$ onto $\triangle PQR$ must map $\{A, B, C\}$ onto $\{P, Q, R\}$. [To see that this is so, suppose that f maps A on a point of $\overline{PQ}$. Then f^{-1} must be an isometry which maps $\overline{PQ}$ onto an interval contained in $\triangle ABC$ which contains A. Since there is no such interval, f does not map A on a point of $\overline{PQ}$.]

From the preceding paragraph we see that a first triangle is congruent with a second if and only if there is an isometry which maps the vertices of the first triangle on those of the second. So, as far as points in the planes of the triangles are concerned, such an isometry is determined by a matching of vertices of one triangle with those of the other.

We shall indicate a matching of the vertices of, say, $\triangle ABC$ and $\triangle PQR$ by expressions like:

$$ABC \longleftrightarrow PQR, \quad CBA \longleftrightarrow QRP, \text{ etc.}$$

The first of these indicates that A is matched with P, B with Q, and C with R. The second indicates that A is matched with P, B with R, and C with Q. [In how many ways can the vertices of one triangle be matched with those of another?] With respect to the first of these two matchings, A and P are *corresponding vertices*, $\angle B$ and $\angle Q$ are *corresponding angles*, and $\overline{CA}$ and $\overline{RP}$ are *corresponding sides*. [Give some vertices, some angles, and some sides which are corresponding vertices, angles, or sides with respect to the second matching.]

We shall say that a matching of the vertices of a first triangle with those of a second *is a congruence* if and only if there is an isometry which maps each vertex of the first triangle on the corresponding ver-

The statements made in the first paragraph can be justified by references to Theorems 14-27, 14-29, and 14-30.

An isometry which maps the vertices of one triangle on those of another must, by Theorem 14-27, map the sides of the first triangle onto those of the second and, so, must map the first triangle, itself, onto the second triangle.

The vertices of one triangle can be matched with those of another triangle [or, with its own vertices] in exactly six ways. [Note that each of these matchings can be described by any one of six "⟷ -sentences."]

Here is a list of pairs of corresponding vertices, corresponding angles, and corresponding sides, with respect to the matching CBA ⟷ QRP:

$$(A, P),\ (B, R),\ (C, Q),\ (\angle A, \angle P),\ (\angle B, \angle R),$$
$$(\angle C, \angle Q),\ (\overline{BC}, \overline{RQ}),\ (\overline{CA}, \overline{QP}),\ (\overline{AB}, \overline{PR})$$

Pictures with curved arrows indicating corresponding "parts" of two triangles may be a help at first; but students should learn early to list corresponding vertices, angles, and sides, under a given correspondence, without the help of figures.

* * *

Suggestions for the exercises of section 16.02:

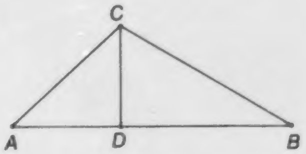
(i) Part A may be used to illustrate the discussion preceding it.
(ii) Part B should be teacher directed, followed by examples of the application of the various triangle congruences.
(iii) Part C and D may be used for homework.
(iv) Part E may be used as a class exercise or as homework.

tex of the second triangle. Clearly, if a matching of vertices of triangles is a congruence then corresponding "parts" [that is, corresponding sides or corresponding angles] of the triangles are congruent. As we shall see, the converse is also a theorem – a matching is a congruence if each side and each angle of the first triangle is congruent to the corresponding part of the second triangle.

Exercises

Part A

1. (a) Draw two noncongruent triangles, $\triangle ABC$ and $\triangle PQR$, and list all pairs of parts which correspond with respect to the matching $ABC \longleftrightarrow QRP$.
   (b) Without making a drawing, list all pairs of parts of $\triangle JKL$ and $\triangle STU$ which correspond with respect to the matching $JKL \longleftrightarrow TSU$. With respect to the matching $KLJ \longleftrightarrow SUT$.
2. Given $\triangle ABC$ and $\triangle LMN$, describe a matching for which
   (a) $\overline{AB}$ and $\overline{MN}$ are corresponding sides;
   (b) $\angle B$ and $\angle M$ are corresponding angles;
   (c) $\overline{AB}$ and $\overline{LM}$ are corresponding sides and $\angle A$ and $\angle M$ are corresponding angles;
   (d) $\overline{AB}$ and $\overline{LM}$ are corresponding sides and $\angle C$ and $\angle N$ are corresponding angles;
   (e) $\angle A$ and $\angle M$ are corresponding angles and $\angle C$ and $\angle N$ are corresponding angles.
3. (a) Give a matching of the vertices of $\triangle ABC$ with those of $\triangle ACD$ such that $\angle A$ of $\triangle ABC$ corresponds with $\angle ADC$ and $\overline{AB}$ corresponds with $\overline{DA}$.
   (b) Give a matching of the vertices of $\triangle ADC$ with those of $\triangle CDB$ such that side $\overline{CD}$ of $\triangle ADC$ corresponds with side $\overline{CD}$ of $\triangle CDB$ but $\angle ACD$ does not correspond with $\angle BCD$.



4. (a) What parts of $\triangle MNP$ and $\triangle JKL$ must be congruent if $MNP \longleftrightarrow KLJ$ is a congruence?
   (b) Suppose that $MNP \longleftrightarrow KLJ$ is not a congruence. Does it follow that $\triangle MNP \not\cong \triangle JKL$? Make a drawing to illustrate your answer.
5. (a) Give all of the matchings of the vertices of $\triangle ABC$ with those of $\triangle ABC$.

Answers for Part A

1. (a) $(\angle A, \angle Q)$, $(\angle B, \angle R)$, $(\angle C, \angle P)$, $(\overline{BC}, \overline{RP})$, $(\overline{CA}, \overline{PO})$, $(\overline{AB}, \overline{QR})$
   (b) $(\angle J, \angle T)$, $(\angle K, \angle S)$, $(\angle L, \angle U)$, $(\overline{KL}, \overline{SU})$, $(\overline{LJ}, \overline{UT})$, $(\overline{JK}, \overline{TS})$; [same]
2. (a) $ABC \longleftrightarrow MNL$ [or: $ABC \longleftrightarrow NML$]
   (b) $ABC \longleftrightarrow LMN$ [or: $ABC \longleftrightarrow NML$]
   (c) $ABC \longleftrightarrow MLN$
   (d) $ABC \longleftrightarrow LMN$ [or: $ABC \longleftrightarrow MLN$]
   (e) $ABC \longleftrightarrow MLN$
3. (a) $ABC \longleftrightarrow DAC$ (b) $ADC \longleftrightarrow BCD$
4. (a) $\angle M \cong \angle K$, $\angle N \cong \angle L$, $\angle P \cong \angle J$, $\overline{NP} \cong \overline{LJ}$, $\overline{PM} \cong \overline{JK}$, $\overline{MN} \cong \overline{KL}$
   (b) No.; [Any drawing which shows two congruent triangles, $\triangle MNP$ and $\triangle JKL$, will do if the vertices are labelled so that $MNP \longleftrightarrow KLJ$ is not a congruence.]
5. (a) $ABC \longleftrightarrow ABC$, $ABC \longleftrightarrow ACB$, $ABC \longleftrightarrow BAC$, $ABC \longleftrightarrow CBA$, $ABC \longleftrightarrow BCA$, $ABC \longleftrightarrow CAB$

- **(b)** Is there at least one of the matchings in (a) which is a congruence? Explain.
- **(c)** Suppose that two of the matchings in (a) are congruences. What can you say about $\triangle ABC$?
- **(d)** Suppose that three of the matchings in (a) are congruences. What can you say about $\triangle ABC$?

6. Suppose that the matching $ABC \longleftrightarrow DEF$ of the vertices of $\triangle ABC$ and $\triangle DEF$ is such that the corresponding sides are congruent.
- **(a)** Show that the corresponding angles are congruent.
- **(b)** Is the given matching a congruence or not? Explain your answer.

7. Suppose that the matching $ABC \longleftrightarrow DEF$ of the vertices of $\triangle ABC$ and $\triangle DEF$ is such that the corresponding angles are congruent. Need the matching be a congruence? Explain your answer.

*

It is a theorem that a matching of the vertices of a first triangle with those of a second is a congruence if and only if each side and each angle of the first triangle is congruent to the corresponding part of the second triangle. There are, however, several more easily applied theorems than this, and we shall collect the most important of these "congruence theorems" in our next theorem. [They are already suggested by the exercises of Part I.] Each of these congruence theorems asserts that a matching is a congruence if each of only three specified parts of the first triangle is congruent to the corresponding part of the second triangle. Consequently, each of these theorems enables us to conclude that certain intervals or angles are congruent once we have established the congruence of other intervals or angles.

In stating the following theorem we shall refer to an angle of a triangle as being *included* between the two sides of the triangle which are subsets of the angle in question, and of a side of a triangle as being *included* between the two angles of the triangle which contain the side in question. For each theorem, we shall also give its conventional nickname – 's.s.s.', 's.a.s.', etc. – in which 's' stands for 'side' and 'a' for 'angle'.

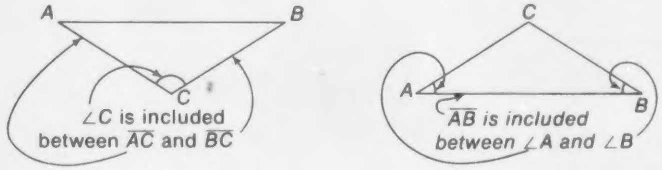

A
B
C
$\angle C$ is included between $\overline{AC}$ and $\overline{BC}$
C
A
B
$\overline{AB}$ is included between $\angle A$ and $\angle B$

Fig. 16–2

Answers for Part A [cont.]

5. (b) Yes.; since $\vec{\mathcal{I}}$ is an isometry the matching $ABC \longleftrightarrow ABC$ is a congruence.

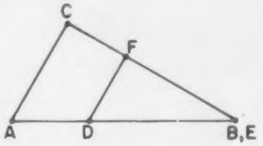
(c) If two of the matchings are congruences then the triangle is isosceles [and may be equilateral]. [If either of the last two matchings is a congruence then the triangle is equilateral.]

(d) If three of the matchings are congruences then the triangle is equilateral [and all six matchings are congruences].

6. (a) This follows from the cosine law and the fact [Theorem 14-28] that congruent intervals have the same measure.

(b) Yes. By the lemma on page 190 [or, by Theorem 14-29] and by Theorem 14-28 it follows that if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$ then there is an isometry which maps A on D, B on E, and C on F. This being the case, the matching $ABC \longleftrightarrow DEF$ is a congruence.

7. The matching need not be a congruence. Consider the following figure, in which $\overleftrightarrow{DF} \parallel \overleftrightarrow{AC}$. Clearly, $ABC \longleftrightarrow DEF$ is a matching for which corresponding angles are congruent but for which corresponding sides are not congruent.



Theorem 16–7 A matching of the vertices of a first triangle with those of a second is a congruence if

(a) each side of the first triangle is congruent to the corresponding side of the second [s.s.s.], or

(b) each of two sides of the first triangle is congruent to the corresponding side of the second and the included angles are congruent [s.a.s.], or

(c) each of two sides of the first triangle and the angle opposite the second side is congruent to the corresponding part of the second triangle *and* the angle opposite the first side is not a supplement of its corresponding angle [s.s.a.], or

(d) each of two angles of the first triangle is congruent to the corresponding angle of the second and the included sides are congruent [a.s.a.], or

(e) each of two angles of the first triangle and the side opposite the second angle is congruent to the corresponding part of the second triangle.

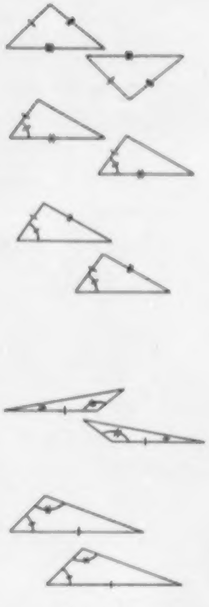


Fig. 16–3

The proof of Theorem 16–7(a) is suggested in the first paragraph of this section; Theorem 16–7(b) can be derived from Theorem 16–7(a) by use of the Cosine Law. [Explain.] As an example of how congruence theorems may be used, suppose that $\triangle ABC$ is isosceles with base AB. Since $CA = CB$, $AB = BA$, and $BC = AC$ it follows by Theorem 16–7(a) that the matching

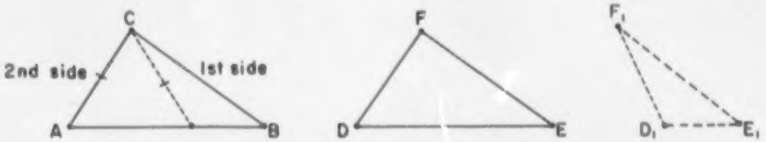
$$ABC \longleftrightarrow BAC$$

is a congruence. Consequently, $\angle A \cong \angle B$. [Show in a similar manner that if, in $\triangle ABC$, $\angle A \cong \angle B$ then $CA = CB$.]

Part B

1. Show how Theorem 16–7(a) follows from Theorem 14–29 and another theorem.
2. Prove Theorem 16–7(b).

The parts of this theorem must, of course, be gone over in class to make sure that students understand them. Parts (a) and (b) are simple enough and are illustrated by Exercises 1 and 2 of Part I on page 262. In fact, part (a) has been proved in Exercise 6(b) of Part A on page 265; and part (b) follows easily from part (a) by the cosine law. Having made these points clear, proceed to part (c), which is one way of resolving the ambiguous case. An example has been given in Exercise 4 of Part I, and figures like the following should help to show what is going on. The matching ABC ⟷ DEF is



one which satisfies the conditions in part (c). The matching ABC ⟷ $D_1E_1F_1$ satisfies all but the last condition in (c). The dashed line in the figure for $\triangle ABC$ shows how the "second side" can occupy a second position so that all but the last part of the condition of part (c) is satisfied. The ambiguous case is discussed again in the exercises of Part D on page 268, leading to Corollary 1 on page 269.

Part (d) of Theorem 16-7 has been foreshadowed in Exercise 6 of Part I, and should offer no difficulty.

Part (e) of Theorem 16-7 — "a.a.s." — is not often taken up since, if one has degree-measure or radian-measure for angles one can easily compute the measure of the third angle of a triangle once you are given the measures of two angles. Doing so reduces a.a.s. to a.s.a. In our treatment we have, as yet, only "cosine-measures", but the same reduction can be carried out by using Theorem 16-6. So, Theorem 16-7(e) follows from Theorem 16-7(d) and Theorem 16-6.

If, in $\triangle ABC$, $\angle A \cong \angle B$ then ABC ⟷ BAC is a congruence by a.s.a., so that CA = CB.

Answers for Part B

1. [See answer for Exercise 6(b) on page 265.]
2. Consider a matching ABC ⟷ PQR for which $\overline{BC} \cong \overline{QR}$, $\overline{CA} \cong \overline{RP}$, and $\angle C \cong \angle R$. It follows that BC = QR, CA = RP, and $\cos \angle C = \cos \angle R$. So, by the cosine law, AB = DE and, hence, $\overline{AB} \cong \overline{DE}$. It now follows by Theorem 16-7(a) that ABC ⟷ PQR is a congruence.

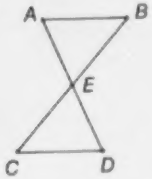
3. Prove Theorem 16–7(c). [*Hint*: Suppose that, in $\triangle ABC$ and $\triangle PQR$, $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$, $\angle A \cong \angle P$, and $\angle C$ and $\angle R$ are not supplementary. Use the Sine Law to show that $\sin \angle C = \sin \angle R$. Use this, the assumption that $\angle C$ and $\angle R$ are not supplementary, and two other theorems to conclude that $\angle B \cong \angle Q$. Then, apply Theorem 16–7(b).]
4. Prove Theorem 16–7(d). [*Hint*: Suppose that, in $\triangle ABC$ and $\triangle PQR$, $\angle A \cong \angle P$, $\angle B \cong \angle Q$, and $\overline{AB} \cong \overline{PQ}$. Conclude that $\angle C \cong \angle R$ and use the Sine Law and Theorem 16–7(b).]
5. Prove Theorem 16–7(e).

Part C

In each of the following, you are given a figure and certain information about it. You are to

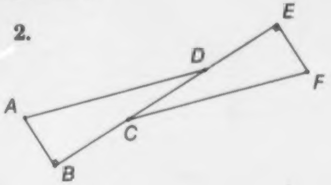
(a) locate two triangles in the figure which you believe to be congruent,
(b) indicate pairs of congruent parts you wish to use to establish a congruence,
(c) tell which congruence theorem you are applying, and
(d) give a matching which is a congruence.

1.



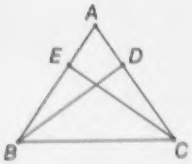
Given: E is the midpoint of both $\overline{BC}$ and $\overline{AD}$

2.



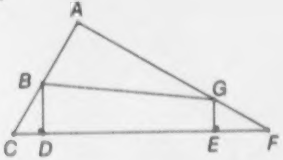
Given: $AB = FE$, $BC = DE$, and $\overleftrightarrow{BE}$ is perpendicular to both $\overline{AB}$ and $\overline{FE}$

3.



Given: $BE = CD$ and $BD = CE$

4.



Given: $\overline{BD}$ and $\overline{GE}$ are perpendicular to $\overline{CF}$, $DC = GE$, and $\angle A$ is a right angle.

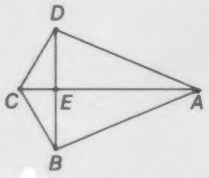
Answers for Part B [cont.]

3. Consider a matching $ABC \longleftrightarrow PQR$ for which $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$, $\angle A \cong \angle P$, and $\angle C$ and $\angle R$ are not supplementary. By the sine law, $\sin \angle C = AB(\sin \angle A/BC) = AB(\sin \angle P/QR) = AB(\sin \angle R/PQ) = \sin \angle R$, since $AB = PQ$. Since $\sin \angle C = \sin \angle R$, $\angle C$ and $\angle R$ are either supplementary or congruent. So, since these angles are not supplementary, $\angle C \cong \angle R$. Since, also, $\angle A \cong \angle P$ it follows by Theorem 16-6 that $\cos \angle B = \cos \angle Q$ and, so, that $\angle B \cong \angle Q$. Since, also, $\overline{AB} \cong \overline{PQ}$ and $\overline{BC} \cong \overline{QR}$ it follows by Theorem 16-7(b) that $ABC \longleftrightarrow PQR$ is a congruence.
4. Consider a matching $ABC \longleftrightarrow PQR$ for which $\angle A \cong \angle P$, $\angle B \cong \angle Q$, and $\overline{AB} \cong \overline{PQ}$. It follows by Theorem 16-6 that $\cos \angle C = \cos \angle R$ and, so, that $\angle C \cong \angle R$. By the sine law, $BC = \sin \angle A(AB/\sin \angle C) = \sin \angle P(PQ/\sin \angle R) = QR$. So, $\overline{BC} \cong \overline{QR}$. Since, also, $\overline{AB} \cong \overline{PQ}$ and $\angle B \cong \angle Q$ it follows by Theorem 16-7(b) that $ABC \longleftrightarrow PQR$ is a congruence. [Alternatively, one can use the method for showing that $\overline{BC} \cong \overline{QR}$ to show, also, that $CA = RP$. Then, Theorem 16-7(a), becomes applicable.]
5. By Theorem 16-6, if two angles of a triangle are congruent to two angles of another then the remaining angles are also congruent. This remark reduces part (e) to part (d).

Answers for Part C

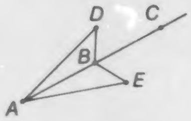
1. (a) $\triangle AEB \cong \triangle DEC$
   (b) $\overline{AE} \cong \overline{DE}$, $\overline{EB} \cong \overline{EC}$, $\angle AEB \cong \angle DEC$
   (c) Theorem 16-7(b) [or: s.a.s.]
   (d) $AEB \longleftrightarrow DEC$
2. (a) $\triangle ABD \cong \triangle FEC$
   (b) $\overline{AB} \cong \overline{FE}$, $\overline{BD} \cong \overline{EC}$, $\angle ABD \cong \angle FEC$
   (c) Theorem 16-7(b) [or: s.a.s.]
   (d) $ABD \longleftrightarrow FEC$
3. (a) $\triangle BEC \cong \triangle CDB$
   (b) $\overline{BE} \cong \overline{CD}$, $\overline{EC} \cong \overline{DB}$, $\overline{CB} \cong \overline{BC}$
   (c) Theorem 16-7(a) [or: s.s.s.]
   (d) $BEC \longleftrightarrow CDB$
4. (a) $\triangle BCD \cong \triangle FGE$
   (b) $\overline{CD} \cong \overline{GE}$, $\angle BCD \cong \angle FGE$ [for, both are complements of $\angle F$], $\angle CDB \cong \angle GEF$
   (c) Theorem 16-7(d) [or: a.s.a.]
   (d) $BCD \longleftrightarrow FGE$

5.



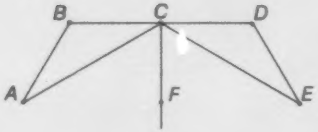
Given: $\overline{AC} \perp \overline{BD}$ at the midpoint E of $\overline{DB}$

6.



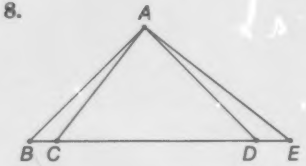
Given: $\overline{AC}$ is the bisector of $\angle DAE$, $AD = AE$

7.



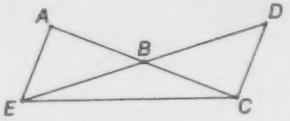
Given: $\overleftrightarrow{CF}$ is the perpendicular bisector of $\overline{BD}$, $\overleftrightarrow{CF}$ is the bisector of $\angle ACE$, and $AC = EC$

8.



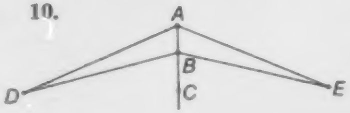
Given: $\triangle ACD$ is isosceles with base $\overline{CD}$, $\overline{AC} \perp \overline{AE}$, and $\overline{AB} \perp \overline{AD}$

9.



Given: B is the midpoint of $\overline{AC}$, and $\angle A \cong \angle DCB$

10.



Given: $\overrightarrow{AB}$ is the angle bisector of $\angle DAE$, and $\overrightarrow{BC}$ is the angle bisector of $\angle DBE$

Part D

1. Draw two noncongruent triangles, $\triangle ABC$ and $\triangle PQR$, such that $\overline{BC} \cong \overline{QR}$, $\overline{CA} \cong \overline{RP}$, and $\angle A \cong \angle P$. [*Hint*: Drawing congruent *right* triangles, $\triangle AB'C$ and $\triangle PQ'R$, may suggest how to proceed. Making $\angle A$ and $\angle P$ about "half of a right angle" will help.]
2. By Theorem 16-7(c), the triangles you drew in Exercise 1 are such that $\angle B$ and $\angle Q$ are supplementary. Show that it is not possible to find triangles like those of Exercise 1 which satisfy the additional condition that $\angle B$ and $\angle Q$ are both acute.

Answers for Part C [cont.]

5. (a) $\triangle CDE \cong \triangle CBE$
   (b) $\overline{CE} \cong \overline{CE}$, $\overline{ED} \cong \overline{EB}$, $\angle CED \cong \angle CEB$
   (c) Theorem 16-7(b) [or: s.a.s.]
   (d) CDE ⟷ CBE

   [Alternatively, $\triangle ADE \cong \triangle ABE$. Bring out in class that both CDE ⟷ CBE and ADE ⟷ ABE are congruences and fish for recognition of the fact that, by s.s.s., it follows that CDA ⟷ CBA is a congruence.]

6. (a) $\triangle BAD \cong \triangle BAE$
   (b) $\overline{BA} \cong \overline{BA}$, $\overline{AD} \cong \overline{AE}$, $\angle BAD \cong \angle BAE$
   (c) Theorem 16-7(b) [or: s.a.s.]
   (d) BAD ⟷ BAE

   [It is worthwhile turning Exercise 6 into an application of Theorem 16-7(c) by changing the assumption that AD = AE to the assumption that BD = BE. Ask students whether it is now the case that BAD ⟷ BAE is a congruence. They should see that an answer to this question requires knowledge concerning $\angle D$ and $\angle E$. The answer is 'Yes.' in case these angles are not supplementary. This last is the case if, as appears from the figure, both are acute, and it is also the case if both are obtuse. Note that it is easy to redraw the figure so that one angle is acute and the other is obtuse. In this case, by Theorem 16-7(c), the angles will be supplementary.]

7. (a) $\triangle ABC \cong \triangle EDC$
   (b) $\overline{BC} \cong \overline{DC}$, $\overline{CA} \cong \overline{CE}$, $\angle BCA \cong \angle DCE$ [on intuitive grounds]
   (c) Theorem 16-7(b) [or: s.a.s.]
   (d) ABC ⟷ EDC

8. (a) $\triangle ABD \cong \triangle AEC$
   (b) $\overline{AD} \cong \overline{AC}$, $\angle ADB \cong \angle ACE$, $\angle DAB \cong \angle CAE$
   (c) Theorem 16-7(d) [or: a.s.a.]
   (d) ABD ⟷ AEC

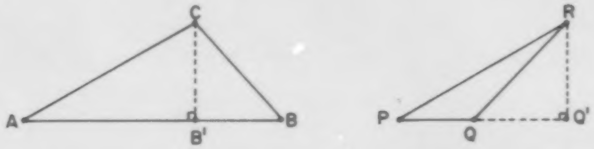
   [Students can now go on to show that $\overline{BC} \cong \overline{ED}$ and $\overline{AB} \cong \overline{AE}$, whence ABC ⟷ AED is a congruence showing that $\triangle ABC \cong \triangle AED$.]

9. (a) $\triangle ABE \cong \triangle CBD$
   (b) $\overline{AB} \cong \overline{CB}$, $\angle EBA \cong \angle DBC$, $\angle EAB \cong \angle DCB$
   (c) Theorem 16-7(d) [or: a.s.a.]
   (d) ABE ⟷ CBD

10. (a) $\triangle ABD \cong \triangle ABE$
    (b) $\overline{AB} \cong \overline{AB}$, $\angle DAB \cong \angle EAB$, $\angle DBA \cong \angle EBA$ [supplements of congruent angles]
    (c) Theorem 16-7(d) [or: a.s.a.]
    (d) ABD ⟷ ABE

Answers for Part D

1.



2. Supplementary angles cannot both be acute. [It is impossible for $\cos\angle B + \cos\angle Q$ to be 0 if both $\cos\angle B$ and $\cos\angle Q$ are positive.] Hence, if $\angle B$ and $\angle Q$ are both acute then $ABC \longleftrightarrow PQR$ is a congruence.

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3. If $\overline{BC}$ is longer than $\overline{CA}$ then $\angle A$ is larger than $\angle B$ and, so, $\angle B$ is acute. Similarly, since $\overline{BC} \cong \overline{QR}$ and $\overline{CA} \cong \overline{RP}$, $\angle Q$ is acute. By Exercise 2, however, in triangles like those of Exercise 1, $\angle B$ and $\angle Q$ cannot both be acute. [So, if $\overline{BC} \cong \overline{QR}$, $\overline{CA} \cong \overline{RP}$, $\angle A \cong \angle P$, and $BC > CA$ then $ABC \longleftrightarrow PQR$ is a congruence.]

4. If neither $\angle B$ nor $\angle Q$ is acute then, in order to be supplementary, they must both be right angles. In this case $ABC \longleftrightarrow PQR$ is a congruence. For since $\angle A \cong \angle P$ and $\angle B \cong \angle Q$ it follows by Theorem 16-6 and Theorem 15-8 that $\angle C \cong \angle R$. The desired conclusion then follows by a.s.a.

5. In case $\angle A$ is a right angle it follows that $\angle P$ is a right angle. In this case $\angle B$ and $\angle Q$ are both acute and, so, are not supplementary. Hence, by s.s.a., $ABC \longleftrightarrow PQR$ is a congruence. [Alternatively, if $\angle A$ and, so, $\angle P$ are right angles then, by the Pythagorean theorem, $AB = PQ$. So, in this case, the triangles are congruent by s.s.s. and, also, by s.a.s.]

Of Corollary 1, part (a) is proved in the answer for Exercise 2 of Part D, part (b) is proved in the answer for Exercise 4, and part (c) is proved in the answer for Exercise 3. Note an addition which could be made to Corollary 1 [it follows from part (c) and earlier theorems]:

(c′) the angle opposite the second side is obtuse.

Corollary 2 is proved in the answer for Exercise 5 of Part D.

3. Is it possible to find triangles like those of Exercise 1 – in particular, noncongruent triangles – which satisfy the additional condition that $\overline{BC}$ is longer than $\overline{CA}$? [*Hint*: What does this additional condition tell you about $\angle A$ and $\angle B$? What follows concerning $\angle P$ and $\angle Q$?] Explain your answer.
4. Is it possible to find triangles like those of Exercise 1 which satisfy the additional condition that neither $\angle B$ nor $\angle Q$ is acute? Explain your answer.
5. Is it possible to find triangles like those of Exercise 1 which satisfy the additional condition that $\angle A$ is a right angle? Explain your answer.

*

The results in Part D can be summarized in two corollaries to Theorem 16–7.

Corollary 1 A matching of the vertices of a first triangle with those of a second is a congruence if each of two sides of the first triangle and the angle opposite the second side is congruent to the corresponding part of the second triangle and
(a) the angle opposite the first side and its corresponding angle are both acute, or
(b) neither the angle opposite the first side nor its corresponding angle is acute, or
(c) the second side is longer than the first.

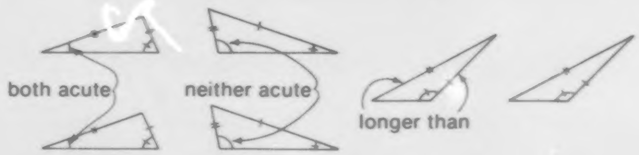


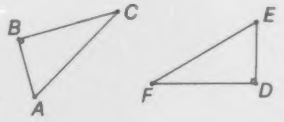
Fig. 16–4

Corollary 2 If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle then the matching for which these are corresponding sides is a congruence [h.*l*.].

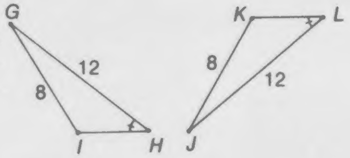
Part E

In each of the following, you are given pictures of two triangles and some information about them. If there is enough information to determine that the triangles are congruent, give a matching of the vertices which is a congruence and cite the theorems which support your answer. If there is not enough information to determine that the triangles are congruent, say so.

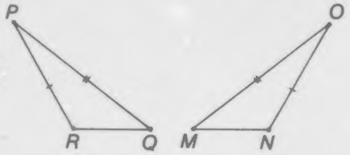
1. Given: Right triangles, $\triangle ABC$ and $\triangle DEF$, with right angles $\angle B$ and $\angle D$; $AC = EF$ and $AB = DE$.



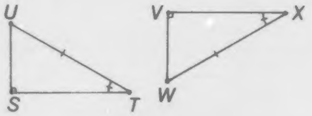
2. Given: $\triangle GHI$ and $\triangle JKL$, with $\angle H \cong \angle L$, and side-measures as in the picture.



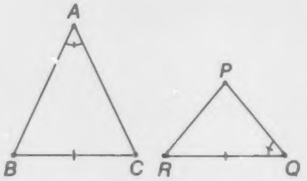
3. Given: $\triangle PQR$ and $\triangle MNO$, with $\angle Q \cong \angle M$, $PR = ON$, and $PQ = OM$.



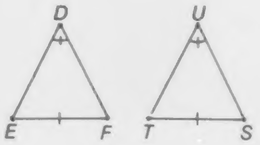
4. Given: Right triangles, $\triangle STU$ and $\triangle VWX$, with right angles $\angle S$ and $\angle V$, $\angle T \cong \angle X$, and $UT = WX$.



5. Given: Isosceles triangles, $\triangle ABC$ and $\triangle PQR$, with bases $\overline{BC}$ and $\overline{QR}$, respectively, $\angle A \cong \angle Q$, and $BC = QR$.



6. Given: Isosceles triangles, $\triangle DEF$ and $\triangle STU$, with bases $\overline{EF}$ and $\overline{ST}$, respectively, $\angle D \cong \angle U$, and $EF = ST$.



Answers for Part E

1. ABC ⟷ EDF is a congruence by Corollary 2 [or: by h.ℓ.].
2. Insufficient information. [If, instead of knowing that $\angle H \cong \angle L$, we know that $\angle I \cong \angle K$ then GHI ⟷ JLK would be a congruence by Corollary 1(c). Or, we could be sure that this matching is a congruence if, in addition to the given information, we knew, for example, that $\angle I$ and $\angle K$ were both obtuse.]
3. Insufficient information. [If, in addition, we knew that $\angle R$ and $\angle N$ are both obtuse, we could conclude that PQR ⟷ OMN is a congruence.]
4. UST ⟷ WVX, by a.s.a. For $\angle U$ and $\angle W$ are congruent because they are complements of congruent angles. [See Part B on page 239.]
5. Insufficient information. [As the figure shows, the triangles can satisfy the given conditions without being congruent. Also, the triangles can satisfy the given conditions and be congruent — but, only if both are equilateral.]
6. DEF ⟷ UTS is a congruence. Let C and V be the feet of the altitudes from D and U, respectively. Since these are also the angle bisectors from D and U it follows as in Exercise 4 that DEC ⟷ UTV is a congruence and, so, that DE = UT. [Halves of congruent angles are congruent by Theorem 15-8 and Exercise 5 of Part C on page 227.] Congruence now follows [since, also, DF = US] by s.a.s. [Note that we have proved a new congruence theorem, somewhat analogous to h.ℓ.: Isosceles triangles are congruent if their bases are congruent and the angles opposite their bases are congruent.]

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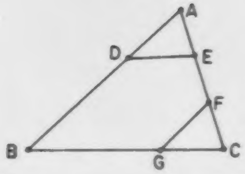
7. PQR ⟷ KLM is a congruence by Corollary 1(a).
8. ABC ⟷ FDE is a congruence by s.a.s.

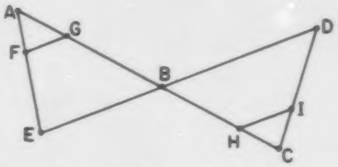
Sample Quiz

Given, in $\triangle ABC$, that $\overline{DE} \parallel \overline{BC}$, $\overline{FG} \parallel \overline{AB}$, and that E and F are points of trisection of $\overline{AC}$, as shown at the right.



1. Give a matching of vertices of triangles ADE and CFG which is a congruence and justify your answer.
2. Are there points H and I on $\overline{AB}$ and $\overline{CB}$, respectively, such that the triangles BHI and ADE are congruent? Justify your answer.

Given that AB = BD, EB = BC, $\overline{FG} \parallel \overline{EB}$, $\overline{HI} \parallel \overline{BD}$, AF = $\frac{1}{3}$AE, and CI = $\frac{1}{3}$CD, as shown in the picture at the right.



3. Give a matching of vertices of triangles ABE and BCD which is a congruence.
4. Give a matching of vertices of triangles AFG and CHI which is a congruence.

Key to Sample Quiz

1. ADE $\longleftrightarrow$ FGC is a congruence by a.s.a., for $\angle DAE \cong \angle GFC$, AE = FC, and $\angle AED \cong \angle FCG$.
2. Yes. Choose H on $\overline{AB}$ and I on $\overline{CB}$ such that BH = AD and BI = CG. Then, $\overline{HI} \parallel \overline{AC}$ and BIH $\longleftrightarrow$ GCF is a congruence by a.s.a.
3. ABE $\longleftrightarrow$ DBC is a congruence by s.a.s. [or, by a.s.a.]
4. AFG $\longleftrightarrow$ ICH is a congruence by a.s.a. [or, by s.a.s.]

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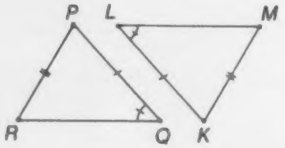
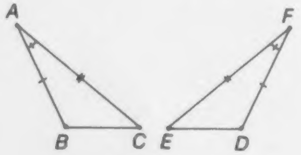
7. Given: $\triangle PQR$ and $\triangle KLM$, with $\angle Q \cong \angle L$, $PQ = KL$, $PR = MK$, and both $\angle R$ and $\angle M$ are acute.



8. Given: $\triangle ABC$ and $\triangle DEF$, with $\angle A \cong \angle F$, $AB = DF$, $CA = FE$, and no information about $\angle B$ and $\angle D$.



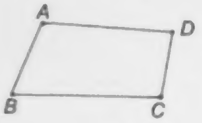
16.03 Parallelograms

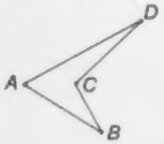
Recall that $PQRS = \overline{PQ} \cup \overline{QR} \cup \overline{RS} \cup \overline{SP}$ and is a quadrilateral if and only if no three of P, Q, R, and S are collinear. The sides of $PQRS$ are the intervals $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, and $\overline{SP}$, and $PQRS$ is simple if and only if no two of its sides intersect. A parallelogram is a quadrilateral each two of whose opposite sides are parallel. According to Theorem 8-16, $ABCD$ is a parallelogram if and only if $\{A, B, C\}$ is noncollinear and $B - A = C - D$. Using what we have learned about sides of lines and about angles and congruence we can now prove a number of useful theorems about parallelograms.

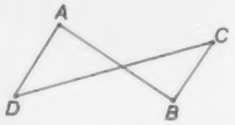
Exercises

Part A

Consider quadrilateral $ABCD$.







1. Show that if $\overline{AB} \cap \overline{CD} \neq \varnothing$ then A and B are on opposite sides of $\overleftrightarrow{CD}$ and C and D are on opposite sides of $\overleftrightarrow{AB}$.
2. Establish the converse of the statement made in Exercise 1.
3. Show that if $C - B$ and $A - D$ have the same sense then $ABCD$ is not simple. [*Hint*: Show that $\overline{AB} \cap \overline{CD} \neq \varnothing$.]
4. Show that if $ABCD$ is a parallelogram then $\angle A$ and $\angle B$ are supplementary, $\angle A$ and $\angle C$ are congruent, and $AD = BC$.
5. Suppose that $ABCD$ is a simple quadrilateral such that $\overline{AD} \parallel \overline{BC}$ and $\overline{AD} \cong \overline{BC}$. Show that $ABCD$ is a parallelogram. [*Hint*: Show that it follows from this assumption that $C - B = D - A$.]

Suggestions for the exercises of section 16.03:

(i) Part A provides good review for use in class.
(ii) Part B can be used for homework.
(iii) The discussion on pages 274-275 and Partc C and D (if used) should have teacher direction. The algebra in these exercises can become unnecessarily involved.
(iv) Parts E and F may be used as homework.

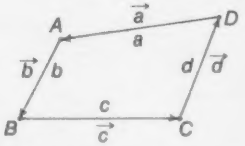
Answers for Part A

[Since ABCD is a quadrilateral, no three of A, B, C, and D are collinear.]

1. [This follows immediately from the noncollinearity remarked on above and Definition 15-3(a).]
2. Suppose that A and B are on opposite sides of $\overleftrightarrow{CD}$ and that C and D are on opposite sides of $\overleftrightarrow{AB}$. Since ABCD is a quadrilateral, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have at most one point in common. Since A and B are on opposite sides of $\overleftrightarrow{CD}$, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ do have a point in common and this point belongs to $\overline{AB}$. Similarly, the same point belongs to $\overline{CD}$. Hence, $\overline{AB} \cap \overline{CD} \neq \emptyset$. [In particular, the quadrilateral ABCD is not simple.]
3. Suppose that C - B and A - D have the same sense. It follows that C - B and D - A have opposite senses and, so, as in Exercise 2 of Part A on page 225, C and D are on opposite sides of $\overleftrightarrow{AB}$. Similarly, since A - D and B - C have opposite senses, it follows that A and B are on opposite sides of $\overleftrightarrow{CD}$. Hence, by Exercise 2, $\overline{AB} \cap \overline{CD} \neq \emptyset$ and, by definition, ABCD is not simple.
4. Suppose that ABCD is a parallelogram and, so, that B - A = C - D. It follows that D - A = C - B and, so, that ∠A and ∠B are consecutive angles formed by a transversal, $\overleftrightarrow{AB}$, of the parallel lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ and, so, are supplementary. [Alternatively, choose unit vectors in the senses of the sides of the angles and compute that $\cos\angle A + \cos\angle B = 0$.] From what has been proved it follows [as an instance] that ∠C and ∠B are supplementary. So, as supplements of the same angle, ∠A and ∠C are congruent. Finally, $AD = \|D - A\| = \|C - B\| = BC$, since D - A = C - B.
5. Since ABCD is simple it follows that C - B and A - D do not have the same sense. But, since $\overline{AD} \parallel \overline{BC}$, C - B and A - D do have the same direction. Hence, C - B and D - A have the same sense. Since, also, $\|C - B\| = BC = AD = \|D - A\|$ it follows [by Exercise 6 of Part B on page 49] that C - B = D - A. Hence, B - A = C - D and quadrilateral ABCD is a parallelogram.

[Note that in Exercises 4 and 5 we have proved that a quadrilateral is a parallelogram if and only if it is simple and some two of its opposite sides are parallel and congruent. We shall state this theorem (on these grounds) as Theorem 16-10 on page 276. Exercise 4 gives various necessary conditions that a quadrilateral be a parallelogram. The sufficiency of some of these will be proved shortly.]

Part B



Consider a simple plane quadrilateral *ABCD*, as shown in the picture. [$\vec{a} = A - D$, $a = \|\vec{a}\|$, etc.] Suppose that $\angle A \cong \angle C$ and $\angle B \cong \angle D$. Show that

1. $\frac{\vec{a}}{a} - \frac{\vec{c}}{c} \neq \vec{0}$ and $\frac{\vec{b}}{b} - \frac{\vec{d}}{d} \neq \vec{0}$ [*Hint*: Use the fact that *ABCD* is simple.];
2. $(\vec{c}, \vec{d})$ is linearly independent and $\{\vec{a}, \vec{b}\} \subseteq [\vec{c}, \vec{d}]$;
3. $\frac{\vec{a} \cdot \vec{b}}{ab} = \frac{\vec{c} \cdot \vec{d}}{cd}$ and $\frac{\vec{b} \cdot \vec{c}}{bc} = \frac{\vec{d} \cdot \vec{a}}{da}$;
4. $\left(\frac{\vec{a}}{a} + \frac{\vec{c}}{c}\right) \cdot \left(\frac{\vec{b}}{b} - \frac{\vec{d}}{d}\right) = 0$ and $\left(\frac{\vec{a}}{a} - \frac{\vec{c}}{c}\right) \cdot \left(\frac{\vec{b}}{b} + \frac{\vec{d}}{d}\right) = 0$. [*Hint*: This is just another way of saying what Exercise 3 says.];
5. $\left(\frac{\vec{a}}{a} - \frac{\vec{c}}{c}\right) \cdot \left(\frac{\vec{a}}{a} + \frac{\vec{c}}{c}\right) = 0$ and $\left(\frac{\vec{b}}{b} - \frac{\vec{d}}{d}\right) \cdot \left(\frac{\vec{b}}{b} + \frac{\vec{d}}{d}\right) = 0$;
6. $\left(\frac{\vec{a}}{a} + \frac{\vec{c}}{c}, \frac{\vec{b}}{b} + \frac{\vec{d}}{d}\right)$ is linearly dependent [*Hint*: Show that if this is not the case then it follows from Exercises 4 and 5 that $(\vec{a}/a) - (\vec{c}/c) = \vec{0}$.];
7. If $\left(\frac{\vec{a}}{a} - \frac{\vec{c}}{c}, \frac{\vec{b}}{b} - \frac{\vec{d}}{d}\right)$ is linearly independent then $\frac{\vec{a}}{a} + \frac{\vec{c}}{c} = \vec{0}$ and $\frac{\vec{b}}{b} + \frac{\vec{d}}{d} = \vec{0}$ and, so, *ABCD* is a parallelogram.

*

In Exercise 4 of Part A you have shown that each two opposite angles of a parallelogram are congruent. In Part B you have shown that if each two opposite angles of a simple plane quadrilateral are congruent, and if another condition [see Exercise 7] is satisfied, then the quadrilateral is a parallelogram. So, you have almost proved:

> **Theorem 16-8** A quadrilateral is a parallelogram if and only if it is simple and plane and each two of its opposite angles are congruent.

To complete the proof of this theorem we need to complete the argument developed in Part B by investigating the assumption that

$$(*) \qquad \left(\frac{\vec{a}}{a} - \frac{\vec{c}}{c}, \frac{\vec{b}}{b} - \frac{\vec{d}}{d}\right) \text{ is linearly dependent.}$$

Do you think that this assumption can be satisfied when *ABCD* is a simple plane quadrilateral whose opposite angles are congruent?

Answers for Part B

1. If $\frac{\vec{a}}{a} - \frac{\vec{c}}{c} = \vec{0}$ then $\vec{a}$ and $\vec{c}$ — that is, A - D and C - B — have the same sense. By Exercise 3 of Part A, the simplicity of ABCD rules out this possibility. Similarly, $\frac{\vec{b}}{b} - \frac{\vec{d}}{d} \neq \vec{0}$.
2. Since ABCD is a quadrilateral, {B, C, D} is noncollinear and, so, $(\vec{c}, \vec{d})$ is linearly independent. Since ABCD is a plane quadrilateral, A ε $\overleftrightarrow{BCD}$ and, so, $\vec{a}$ and $\vec{b}$ belong to $[\overleftrightarrow{BCD}]$. But, the latter is $[\vec{c}, \vec{d}]$.
3. These equations merely say that $-\cos\angle A = -\cos\angle C$ and $-\cos\angle B = -\cos\angle D$. These hold by Theorem 15-8 because $\angle A \cong \angle C$ and $\angle B \cong \angle D$.
4. By Exercise 3,
   $$\frac{\vec{a}\cdot\vec{b}}{ab} + \frac{\vec{b}\cdot\vec{c}}{bc} = \frac{\vec{c}\cdot\vec{d}}{cd} + \frac{\vec{d}\cdot\vec{a}}{da} \text{ and, so, } \left(\frac{\vec{a}}{a} + \frac{\vec{c}}{c}\right)\cdot\frac{\vec{b}}{b} = \left(\frac{\vec{a}}{a} + \frac{\vec{c}}{c}\right)\cdot\frac{\vec{d}}{d}.$$
   This leads at once to the first of the given equations. The second is obtained in a similar manner by subtracting instead of adding. [The procedure of obtaining the equations of this exercise from those of Exercise 3 can be reversed. So, the new equations are, indeed, "just another way of saying what Exercise 3 says."
5. These equations are applications of something long known: If $\vec{u}$ and $\vec{v}$ are unit vectors then $(\vec{u} - \vec{v})\cdot(\vec{u} + \vec{v}) = 0$.
6. Suppose that $\left(\frac{\vec{a}}{a} + \frac{\vec{c}}{c}, \frac{\vec{b}}{b} + \frac{\vec{d}}{d}\right)$ is linearly independent. It follows that there are numbers — say, e and f — such that $\frac{\vec{a}}{a} - \frac{\vec{c}}{c} = \left(\frac{\vec{a}}{a} + \frac{\vec{c}}{c}\right)e + \left(\frac{\vec{b}}{b} + \frac{\vec{d}}{d}\right)f$. Dot multiplying on both sides with '$\frac{\vec{a}}{a} - \frac{\vec{c}}{c}$', and using the results of Exercises 4 and 5, we find that $\left\|\frac{\vec{a}}{a} - \frac{\vec{c}}{c}\right\|^2 = 0$. Since this contradicts Exercise 1 it follows that $\left(\frac{\vec{a}}{a} + \frac{\vec{c}}{c}, \frac{\vec{b}}{b} + \frac{\vec{d}}{d}\right)$ is linearly dependent. [We shall use this result in Part D on page 275.]
7. Suppose that $\left(\frac{\vec{a}}{a} - \frac{\vec{c}}{c}, \frac{\vec{b}}{b} - \frac{\vec{d}}{d}\right)$ is linearly independent. It follows that there are numbers — say, e and f — such that $\frac{\vec{a}}{a} + \frac{\vec{c}}{c} = \left(\frac{\vec{a}}{a} - \frac{\vec{c}}{c}\right)e + \left(\frac{\vec{b}}{b} - \frac{\vec{d}}{d}\right)f$. Dot multiplying on both sides with '$\frac{\vec{a}}{a} + \frac{\vec{c}}{c}$', and using the results of Exercises 4 and 5, we find that $\left\|\frac{\vec{a}}{a} + \frac{\vec{c}}{c}\right\|^2 = 0$. Thus, $\frac{\vec{a}}{a} + \frac{\vec{c}}{c} = \vec{0}$. Similarly, $\frac{\vec{b}}{b} + \frac{\vec{d}}{d} = \vec{0}$. It follows that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ and that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Hence, if $\left(\frac{\vec{a}}{a} - \frac{\vec{c}}{c}, \frac{\vec{b}}{b} - \frac{\vec{d}}{d}\right)$ is linearly independent, ABCD is a parallelogram.

To complete the proof of Theorem 16-8 it is sufficient, from the results of Exercise 7, to show that, under the conditions stated in the preamble to Part B, $\left(\frac{\vec{a}}{a} - \frac{\vec{c}}{c}, \frac{\vec{b}}{b} - \frac{\vec{d}}{d}\right)$ is linearly independent. Attempts to draw a simple plane quadrilateral with congruent opposite angles such that (*) on page 272 is satisfied suggest strongly that it is not. Unfortunately, we know no very simple proof of this fact. This leaves you with two alternatives. Either tell students that Theorem 16-8 is indeed a theorem, and will be taken for such without completing the proof, or complete the proof by going through the exercises of Parts C and D on pages 274 and 275. [The exercises of Part C are merely a review of discussion on the text preceding these exercises. You may wish to skip them and go on to Part D. In any case, don't assign more than one of Exercises 1 - 4 of Part C to one student. They're repetitions.]

Once Theorem 16-8 is accepted the important Theorems 16-9 and 16-11 on page 276 are easy. [Theorem 16-10 has already been proved in Part A.]

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Explanation called for in the text: The non-$\vec{0}$ vectors in question have the same sense if and only if the corresponding unit vectors have the same sense. And, those unit vectors have the same sense if and only if the first and the opposite of the second have opposite senses.

[*Hint*: Try to find a *non*-simple plane quadrilateral whose opposite angles are congruent and which satisfies (*).] We shall be able to show that there is no simple plane quadrilateral whose opposite angles are congruent and which satisfies (*).

Our procedure will be to assume that $ABCD$ is a simple plane quadrilateral whose opposite angles are congruent and which satisfies (*) and to show that this assumption leads to a contradiction.

To begin with, since $ABCD$ is simple we know that $\overline{AB} \cap \overline{CD} = \emptyset$ and $\overline{AD} \cap \overline{BC} = \emptyset$. From an earlier exercise we know that the first of these equations implies that either A and B are not on opposite sides of $\overleftrightarrow{CD}$ or C and D are not on opposite sides of $\overleftrightarrow{AB}$. Since $ABCD$ is a plane quadrilateral, this means that

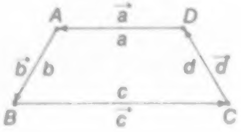
either A and B are on the same side of $\overleftrightarrow{CD}$
or C and D are on the same side of $\overleftrightarrow{AB}$.

Similarly, the fact that $\overline{AD} \cap \overline{BC} = \emptyset$ implies that

either A and D are on the same side of $\overleftrightarrow{BC}$
or B and C are on the same side of $\overleftrightarrow{AD}$.

Using the notation of Part B, to say that A and B are on the same side of $\overleftrightarrow{CD}$ amounts to saying that

$\vec{a} - \vec{d}(\vec{a} \cdot \vec{d})/d^2$ and
$-\vec{c} - \vec{d}(-\vec{c} \cdot \vec{d})/d^2$ have the same sense.



In order to avoid fractions it will help to let $\vec{d_1}$ be the unit vector in the sense of $\vec{d}$. In terms of this our condition becomes:

$$\vec{a} - \vec{d_1}(\vec{a} \cdot \vec{d_1}) \text{ and } -\vec{c} - \vec{d_1}(-\vec{c} \cdot \vec{d_1}) \text{ have the same sense.}$$

or, still more simply, using '$\vec{a_1}$' and '$\vec{c_1}$' for the unit vectors in the senses of $\vec{a}$ and $\vec{c}$:

$$\vec{a_1} - \vec{d_1}(\vec{a_1} \cdot \vec{d_1}) \text{ and } \vec{c_1} - \vec{d_1}(\vec{c_1} \cdot \vec{d_1}) \text{ have } \textit{opposite} \text{ senses.}$$

[Explain.] Finally, since the two vectors in question have the same direction it follows that to say that they have opposite senses amounts to saying that their dot product is negative; that is,

$$\left(\vec{a_1} - \vec{d_1}(\vec{a_1} \cdot \vec{d_1})\right) \cdot \left(\vec{c_1} - \vec{d_1}(\vec{c_1} \cdot \vec{d_1})\right) < 0.$$

On simplifying this we see that, in the plane quadrilateral *ABCD*, *A* and *B* are on the same side of $\overleftrightarrow{CD}$ if and only if

$$\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1) < 0. \tag{1}$$

A discussion like that of the preceding paragraph shows that *C* and *D* are on the same side of $\overleftrightarrow{AB}$ if and only if

$$\vec{c}_1 \cdot \vec{a}_1 - (\vec{c}_1 \cdot \vec{b}_1)(\vec{a}_1 \cdot \vec{b}_1) < 0. \tag{1'}$$

But, since we are assuming that $\angle D \cong \angle B$ and $\angle C \cong \angle A$ it follows that (1′) is satisfied if and only if (1) is satisfied. So, (1) holds if and only if $\overline{AB} \cap \overline{CD} = \emptyset$.

An argument like that in the preceding two paragraphs now shows that $\overline{AD} \cap \overline{BC} = \emptyset$ if and only if

$$\vec{b}_1 \cdot \vec{d}_1 - (\vec{b}_1 \cdot \vec{c}_1)(\vec{d}_1 \cdot \vec{c}_1) < 0.$$

Again, since $\angle B \cong \angle D$, this condition is equivalent to:

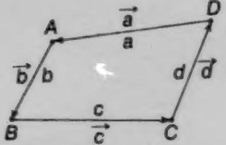
$$\vec{b}_1 \cdot \vec{d}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{d}_1 \cdot \vec{c}_1) < 0 \tag{2}$$

Combining (1) and (2) we see that the plane quadrilateral *ABCD* whose opposite angles are congruent is simple if and only if the numbers

$$\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1) \text{ and } \vec{b}_1 \cdot \vec{d}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1) \tag{3}$$

are both negative. To complete the proof of Theorem 16–8 we shall show that these numbers are opposites—and, so, are not both negative—in the case of our quadrilateral *ABCD*. In Part C you have an opportunity to verify the first statement concerning the numbers in (3); in Part D you may verify the second.

Part C



Suppose that *ABCD* is a plane quadrilateral whose opposite angles are congruent, and let $\vec{a}_1 = \vec{a}/a$, $\vec{b}_1 = \vec{b}/b$, $\vec{c}_1 = \vec{c}/c$, and $\vec{d}_1 = \vec{d}/d$. Show that

1. *A* and *B* are on the same side of $\overleftrightarrow{CD}$ if and only if $\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1) < 0$ [*Hint*: See (1) above.];
2. *C* and *D* are on the same side of $\overleftrightarrow{AB}$ if and only if $\vec{c}_1 \cdot \vec{a}_1 - (\vec{c}_1 \cdot \vec{b}_1)(\vec{a}_1 \cdot \vec{b}_1) < 0$ [*Hint*: See (1) above.];

Answers for Part C

1. [This is merely a repetition of the work on page 273 leading to (1).]
2. [In the argument for (1), merely replace 'A' by 'C', 'B' by 'D', and '$\overleftrightarrow{CD}$' by '$\overleftrightarrow{AB}$'; interchange '$\vec{a}$' and '$\vec{c}$', and replace '$\vec{d}$' by '$\vec{b}$'.]

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3. [In the argument for (1), merely interchange 'B' and 'D', interchange '$\vec{c}$' and '$\vec{d}$', and replace '$\vec{a}$' and '$\vec{b}$'.]
4. [In the argument for Ex. 3, replace 'A' by 'B', 'D' by 'C', and '$\overleftrightarrow{BC}$' by '$\overleftrightarrow{AD}$'; interchange '$\vec{b}$' and '$\vec{d}$' and replace '$\vec{c}$' by '$\vec{a}$'.]
5. Since $\angle D \cong \angle B$, $\vec{d}_1 \cdot \vec{a}_1 = \vec{b}_1 \cdot \vec{c}_1$; since $\angle C \cong \angle A$, $\vec{c}_1 \cdot \vec{d}_1 = \vec{a}_1 \cdot \vec{b}_1$. It follows that the inequation of Exercise 2 is equivalent to that of Exercise 1 and that the inequation of Exercise 4 is equivalent to that of Exercise 3.

Now ABCD is simple if and only if $\overline{AB} \cap \overline{CD} = \emptyset$ and $\overline{AD} \cap \overline{BC} = \emptyset$. It follows from Exercise 2 of Part A that ABCD is simple if and only if it is not the case that

A and B are on opposite sides of $\overleftrightarrow{CD}$ and
C and D are on opposite sides of $\overleftrightarrow{AB}$

and it is not the case that

A and D are on opposite sides of $\overleftrightarrow{BC}$ and
B and C are on opposite sides of $\overleftrightarrow{AD}$.

Since ABCD is a plane quadrilateral and, in particular, has no three of its vertices collinear, it follows that ABCD is simple if and only if it is the case that

A and B are on the same side of $\overleftrightarrow{CD}$ or
C and D are on the same side of $\overleftrightarrow{AB}$

and it is the case that

A and D are on the same side of $\overleftrightarrow{BC}$ or
B and C are on the same side of $\overleftrightarrow{AD}$.

So, by Exercises 1 and 2 and Exercises 3 and 4, ABCD is simple if and only if

$$\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1) < 0 \text{ or } \vec{c}_1 \cdot \vec{a}_1 - (\vec{c}_1 \cdot \vec{b}_1)(\vec{a}_1 \cdot \vec{b}_1) < 0$$

and

$$\vec{b}_1 \cdot \vec{d}_1 - (\vec{b}_1 \cdot \vec{c}_1)(\vec{d}_1 \cdot \vec{c}_1) < 0 \text{ or } \vec{d}_1 \cdot \vec{b}_1 - (\vec{d}_1 \cdot \vec{a}_1)(\vec{b}_1 \cdot \vec{a}_1) < 0.$$

Hence, by the equivalences pointed out in the first paragraph of this answer, ABCD is simple if and only if $\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1)$ and $\vec{b}_1 \cdot \vec{d}_1 - (\vec{b}_1 \cdot \vec{c}_1)(\vec{d}_1 \cdot \vec{c}_1)$ are both negative. But $\vec{b}_1 \cdot \vec{c}_1 = \vec{a}_1 \cdot \vec{d}_1$.

Answers for Part D

1. If p were 0 then q qould be nonzero and $(\vec{b}_1 - \vec{d}_1)q$ would be $\vec{0}$. But, $\vec{b}_1 - \vec{d}_1 \neq \vec{0}$. Hence, $p \neq 0$ and, similarly, $q \neq 0$.
2. (a) For the first result, eliminate '$\vec{b}_1$' from the two displayed equations by multiplying with 'q' on both sides of the first, with 'n' on both sides of the second, and subtracting. For the second result, use a similar procedure to eliminate '$\vec{a}_1$' from the two displayed equations.

Answers for Part D [cont.]

2. (b) Suppose that $mq - np = 0$. It follows from the first equation in part (a) that, since $(\vec{c}_1, \vec{d}_1)$ is linearly independent, $mq + np = 0$. From this and our assumption, mq and np are both zero. Since, by Exercise 1 $q \neq 0$ and $p \neq 0$, it follows that $m = 0$ and $n = 0$. But, this is contrary to the basic assumption concerning m and n. So, $mq - np \neq 0$.

   Suppose that $mq + np = 0$. Since $\{C, D, A\}$ is noncollinear it follows that $(\vec{a}_1, \vec{d}_1)$ is linearly independent and so, as in the immediately preceding argument, that $mq - np = 0$. So, as before, $mq = 0 = np$ which, as before, is impossible. So, $mq + np \neq 0$.

   (c) Solve the equations of part (a) for '$\vec{a}_1$' and '$\vec{b}_1$', respectively.

3. To show that $\vec{a}_1 + \vec{c}_1$ is a multiple of $\vec{c}_1 p + \vec{d}_1 q$, start with the first equation in Exercise 2(c) and adjoin '$+ \vec{c}_1$' to both sides. Easy simplifications show that $\vec{a}_1 + \vec{c}_1 = (\vec{c}_1 p + \vec{d}_1 q)(2n)/(np - mq)$. Similar techniques, using the second equation in Exercise 2(c), show that $\vec{b}_1 + \vec{d}_1 = (\vec{c}_1 p + \vec{d}_1 q)(2m)/(mq - np)$.

4. Techniques like those used in Exercise 3 show that $\vec{a}_1 - \vec{c}_1 = (\vec{c}_1 m + \vec{d}_1 n)/(2q)/(np - mq)$ and $\vec{b}_1 - \vec{d}_1 = (\vec{c}_1 m + \vec{d}_1 n)(2p)/(mq - np)$.

5. Use the results of Exercises 3 and 4 to calculate the products of Exercise 5 of Part B. Since these are 0 it follows that the product of $(\vec{c}_1 m + \vec{d}_1 n) \cdot (\vec{c}_1 p + \vec{d}_1 q)$ by each of the numbers $(4nq)/(np - mq)^2$ and $(4mp)/(mq - np)^2$ is zero. Since it is not the case that nq and mp are both 0 it follows that $(\vec{c}_1 m + \vec{d}_1 n) \cdot (\vec{c}_1 p + \vec{d}_1 q) = 0$. [If $nq = 0 = mp$ then, since $p \neq 0 \neq q$, m and n are both 0. But, they aren't.]

6. Expand the dot product in Exercise 5 and solve for '$\vec{c}_1 \cdot \vec{d}_1$'. [By Exercise 2(b), $mq + np \neq 0$.]

7. By Exercise 2(c)

$$\vec{a}_1 \cdot \vec{c}_1 = \frac{np + mq}{np - mq} - \frac{mp + nq}{mq + np} \cdot \frac{2nq}{np - mq} = \frac{(np + mq)^2 - (mp + nq)(2nq)}{(np - mq)(np + mq)}$$
$$= \frac{n^2p^2 + m^2q^2 - 2n^2q^2}{(np - mq)(np + mq)}$$

$$\vec{b}_1 \cdot \vec{d}_1 = -\frac{mp + nq}{mq + np} \cdot \frac{2mp}{mq - np} + \frac{mq + np}{mq - np} = \frac{2mp(mp + nq) - (np + mq)^2}{(np - mq)(np + mq)}$$
$$= \frac{2m^2p^2 - n^2p^2 - m^2q^2}{(np - mq)(np + mq)},$$

$$\vec{a}_1 \cdot \vec{d}_1 = -\frac{mp + nq}{mq + np} \cdot \frac{np + mq}{np - mq} + \frac{2nq}{np - mq} = \frac{-(mp + nq) + 2nq}{np - mq} = \frac{nq - mp}{np - mq}$$

8. By Exercises 6 and 7

$$\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1) = \frac{n^2p^2 + m^2q^2 - 2n^2q^2}{(np - mq)(np + mq)} + \frac{nq - mp}{np - mq} \cdot \frac{mp + nq}{mq + np}$$
$$= \frac{n^2p^2 + m^2q^2 - n^2q^2 - m^2p^2}{(np - mq)(np + mq)},$$

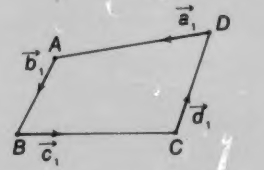
$$\vec{b}_1 \cdot \vec{d}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1) = \frac{2m^2p^2 - n^2p^2 - m^2q^2}{(np - mq)(np + mq)} + \frac{nq - mp}{np - mq} \cdot \frac{mp + nq}{mq + np}$$
$$= \frac{m^2p^2 - n^2p^2 - m^2q^2 + n^2q^2}{(np - mq)(np + mq)}.$$

---

3. *A* and *D* are on the same side of $\overleftrightarrow{BC}$ if and only if $\vec{b}_1 \cdot \vec{d}_1 - (\vec{b}_1 \cdot \vec{c}_1)(\vec{d}_1 \cdot \vec{c}_1) < 0$;
4. *B* and *C* are on the same side of $\overleftrightarrow{AD}$ if and only if $\vec{d}_1 \cdot \vec{b}_1 - (\vec{d}_1 \cdot \vec{a}_1)(\vec{b}_1 \cdot \vec{a}_1) < 0$;
5. *ABCD* is simple if and only if $\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1)$ and $\vec{b}_1 \cdot \vec{d}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1)$ are both negative.

Part D

Suppose that *ABCD* is a simple plane quadrilateral whose opposite angles are congruent. From Exercise 6 of Part B we know that there are numbers *m* and *n*, not both 0, such that



$$(\vec{a}_1 + \vec{c}_1)m + (\vec{b}_1 + \vec{d}_1)n = \vec{0}.$$

Assume also [to complete the proof of Theorem 16–8] that there are numbers *p* and *q*, not both 0, such that

$$(\vec{a}_1 - \vec{c}_1)p + (\vec{b}_1 - \vec{d}_1)q = \vec{0}. \qquad [(*) \text{ on page 272}]$$

1. Show that neither *p* nor *q* is 0. [*Hint*: See Exercise 1 of Part B.]
2. Show that
   (a) $\vec{a}_1(mq - np) + \vec{c}_1(mq + np) + \vec{d}_1(2nq) = \vec{0}$ and
   $\vec{c}_1(2mp) + \vec{b}_1(np - mq) + \vec{d}_1(np + mq) = \vec{0}$,
   (b) $mq - np \neq 0$ and $mq + np \neq 0$. [*Hint*: $(\vec{c}_1, \vec{d}_1)$ and $(\vec{a}_1, \vec{d}_1)$ are linearly independent.],
   (c) $\vec{a}_1 = \vec{c}_1\left(\frac{np + mq}{np - mq}\right) + \vec{d}_1\left(\frac{2nq}{np - mq}\right)$ and
   $\vec{b}_1 = \vec{c}_1\left(\frac{2mp}{mq - np}\right) + \vec{d}_1\left(\frac{mq + np}{mq - np}\right)$.
3. Show that $\vec{a}_1 + \vec{c}_1$ and $\vec{b}_1 + \vec{d}_1$ are multiples of $\vec{c}_1 p + \vec{d}_1 q$. [*Hint*: Use Exercise 2(c).]
4. Show that $\vec{a}_1 - \vec{c}_1$ and $\vec{b}_1 - \vec{d}_1$ are multiples of $\vec{c}_1 m + \vec{d}_1 n$.
5. Show that $(\vec{c}_1 p + \vec{d}_1 q) \cdot (\vec{c}_1 m + \vec{d}_1 n) = 0$. [*Hint*: Use Exercises 3 and 4 and Exercise 5 of Part B.]
6. Show that $\vec{c}_1 \cdot \vec{d}_1 = -(mp + nq)/(mq + np)$.
7. Compute '$\vec{a}_1 \cdot \vec{c}_1$', '$\vec{a}_1 \cdot \vec{d}_1$', and '$\vec{b}_1 \cdot \vec{d}_1$'. [*Hint*: Use Exercise 2(c).]
8. Show that $\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1)$ and $\vec{b}_1 \cdot \vec{d}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1)$ are opposites of each other.
9. Complete the proof of Theorem 16–8. [*Hint*: See Exercise 5 of Part C.]

*

TC 275 (3)

Answers for Part D [cont.]

9. Suppose that ABCD is a simple plane quadrilateral whose opposite angles are congruent. It follows by Exercise 5 of Part C that $\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1)$ and $\vec{b}_1 \cdot \vec{d}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1)$ are both negative. Assuming that $(\vec{a}_1 - \vec{c}_1, \vec{b}_1 - \vec{d}_1)$ is linearly dependent it follows by Exercise 6 of Part D that $\vec{a}_1 \cdot \vec{c}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1)$ and $\vec{b}_1 \cdot \vec{d}_1 - (\vec{a}_1 \cdot \vec{d}_1)(\vec{c}_1 \cdot \vec{d}_1)$ are opposites. Since these numbers cannot be both negative and, also, opposites, it follows that $(\vec{a}_1 - \vec{c}_1, \vec{b}_1 - \vec{d}_1)$ is linearly independent. Hence, by Exercise 7 of Part B, ABCD is a parallelogram. Since a parallelogram is a simple plane quadrilateral whose opposite angles are congruent, this completes the proof of Theorem 16-8.

TC 276 (1)

Proof of Theorem 16-9: A parallelogram is simple and plane and, by Exercise 4 of Part A on page 271, its consecutive angles are supplementary. On the other hand, if the consecutive angles of a quadrilateral are supplementary then its opposite angles will be congruent [because they are supplements of the same angle]. By Theorem 16-8 it follows that if such a quadrilateral is simple and plane then it is a parallelogram.

Restatements of the two parts of Theorem 16-10:

if-part: If a quadrilateral is simple and any two of its opposite sides are parallel and congruent then the quadrilateral is a parallelogram.

only if-part: If a quadrilateral is a parallelogram then each two of its opposite sides are parallel and congruent.

The proof of Theorem 16-10 has already been given on TC271(3). It makes use of Exercises 4 and 5 of Part A.

The only if-part of Theorem 16-11 has been proved in Exercise 4 of Part A.

Answers for Part E

1. The quadrilateral obtained by "folding" a parallelogram along one of its diagonals shows that the word 'plane' cannot be omitted from Theorem 16-11. Continuing the folding process until the parallelogram is flattened out shows that 'simple' cannot be omitted.
2. The sense of the only if-part of the theorem is not changed if 'each' is replaced by 'any'. But it is not the case that if any [that is, some] two opposite sides of a simple plane quadrilateral are congruent then the quadrilateral is a parallelogram. Counterexamples are easy to draw.
3. The only-if part has already been proved in Exercise 4 of Part A on page 271. So, all that remains to be proved is that if each two opposite sides of a simple plane quadrilateral are congruent then the quadrilateral is a parallelogram. It follows immediately by s. s. s. that, given a quadrilateral each two of those opposite sides are congruent, the opposite angles of the quadrilateral are congruent. The desired result now follows by Theorem 16-8.

Now that we have shown that a quadrilateral is a parallelogram if and only if it is simple and plane and has congruent opposite angles it is easy to prove:

> **Theorem 16-9** A quadrilateral is a parallelogram if and only if it is simple and plane and each two of its consecutive angles are supplementary.

[Explain.]

In Part A you established another characterization of parallelograms:

> **Theorem 16-10** A quadrilateral is a parallelogram if and only if it is simple and some two of its opposite sides are parallel and congruent.

Restate the two parts of this theorem [if-part, and only if-part] using 'some' or 'each' in place of 'any'.

There is a fourth theorem like the preceding three:

> **Theorem 16-11** A quadrilateral is a parallelogram if and only if it is simple and plane and each two of its opposite sides are congruent.

As with the other theorems, the only if-part has been established in Part A. The following exercises deal with the if-part.

Part E

1. Show that Theorem 16-11 would become false if either of the words 'simple' or 'plane' were omitted.
2. Does Theorem 16-11 remain a theorem if 'each' is replaced by 'any'?
3. Prove Theorem 16-11. [*Hint*: Use a congruence theorem for triangles.]

Part F

1. Prove:

> **Theorem 16-12** A quadrilateral is a parallelogram if and only if the sum of the squares of its diagonals equals the sum of the squares of its sides.

Answers for Part F

1. Note that the sum of the squares of the measures of the diagonals of ABCD is $(\vec{a} + \vec{b})^2 + (\vec{a} + \vec{d})^2$. Expanding,

$$(\vec{a}+\vec{b})^2 + (\vec{a}+\vec{d})^2 = a^2+b^2+a^2+d^2+2\vec{a}\cdot\vec{b}+2\vec{a}\cdot\vec{d}$$
$$= a^2+b^2+c^2+d^2+(a^2-c^2+2\vec{a}\cdot\vec{b}+2\vec{a}\cdot\vec{d}).$$

Now, since $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}$, $c^2 = a^2 + b^2 + d^2 + 2\vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{d} + 2\vec{d}\cdot\vec{a}$. Hence,

$$a^2 - c^2 + 2\vec{a}\cdot\vec{b} + 2\vec{a}\cdot\vec{d} = -(b^2 + d^2 + 2\vec{b}\cdot\vec{d}) = -(\vec{b} + \vec{d})^2.$$

So, for any quadrilateral ABCD,

$$(\vec{a} + \vec{b})^2 + (\vec{a} + \vec{d})^2 = a^2 + b^2 + c^2 + d^2 - (\vec{b} + \vec{d})^2.$$

Hence, $(\vec{a} + \vec{b})^2 + (\vec{a} + \vec{d})^2 = a^2 + b^2 + c^2 + d^2$ if and only if $\vec{b} = -\vec{d}$—that is, if and only if $B - A = C - D$. This last is the case if and only if ABCD is a parallelogram.

TC 277

2. Suppose that, in quadrilateral ABCD, $\angle A \cong \angle C$ and $B - A$ and $D - C$ have opposite senses. From the latter we know that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and, to show that ABCD is a parallelogram, all that remains is to show that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. This we can deduce from Theorem 15-14 once we have shown that $\angle A$ and $\angle B$ are supplementary and that C and D are on the same side of $\overleftrightarrow{AB}$. [The latter follows at once from the fact that $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$ and, since ABCD is a quadrilateral, $\overleftrightarrow{CD} \neq \overleftrightarrow{AB}$.]

In the notation of this section our assumption can be restated as:

$$(\vec{a}\cdot\vec{b})/(ab) = (\vec{c}\cdot\vec{d})/cd \text{ and } \vec{d} = \vec{b}e, \text{ where } e < 0.$$

In particular, it follows from the second part of our assumption that $d = -be$. Hence,

$$(\vec{a}\cdot\vec{b})/(ab) = [\vec{c}\cdot(\vec{b}e)]/(cd) = [(\vec{c}\cdot\vec{b})/(cb)][(eb)/d] = -(\vec{c}\cdot\vec{b})/(cb)$$

and, so, $\cos\angle A = -\cos\angle B$ and $\angle A$ and $\angle B$ are supplementary.

In statements like (3) both is's should, for grammar's sake, be 'be's. It seems best, however, to follow what is becoming standard practice [however much it hurts a sensitive ear].

* * *

Suggestions for the exercises of section 16.04:

(i) Use Part A for class illustrations.
(ii) After proper samples, Parts B and C may be used for homework.

[*Hint*: Using the notation of earlier exercises, show that for any quadrilateral *ABCD*,

$$(\vec{a} + \vec{b})^2 + (\vec{a} + \vec{d})^2 = a^2 + b^2 + c^2 + d^2 - (\vec{b} + \vec{d})^2.]$$

2. Show that the quadrilateral *ABCD* is a parallelogram if $\angle A \cong \angle C$ and $B - A$ and $D - C$ have opposite senses. [*Hint*: State the assumption algebraically and use it to show that $\angle A$ and $\angle B$ are supplementary and *C* and *D* are on the same side of $\overleftrightarrow{AB}$.]

16.04 Necessary Conditions and Sufficient Conditions

Consider the conditional sentence:

(1) If a quadrilateral is a trapezoid
then the quadrilateral is simple.

By Definition 8-6, (1) is a theorem. And, to say that (1) is a theorem is to say that its antecedent implies its consequent—that is:

(2) 'a quadrilateral is a trapezoid'
implies 'the quadrilateral is simple'.

Thus, knowing that a quadrilateral is a trapezoid is sufficient information to ensure that the quadrilateral is simple. This is sometimes said as follows:

(3) That a quadrilateral is a trapezoid is *a sufficient condition* that the quadrilateral is simple.

Statements (1), (2), and (3) are alternate ways of saying the same thing. Another way of saying the same thing as does (1)—and, so, as do (1)-(3)—is:

(4) A quadrilateral is a trapezoid only
if the quadrilateral is simple.

This may be interpreted as saying that to know that a quadrilateral is a trapezoid it is necessary to know that the quadrilateral is simple. This is sometimes said as follows:

(5) That a quadrilateral is simple is *a necessary condition* that the quadrilateral is a trapezoid.

Statements (4) and (5) are alternate ways of saying the same thing. And, since (4) and (1) are alternate ways of saying the same thing so, then, are the statements (1)-(5).

In general, given a sentence of the form.

$$p \Longrightarrow q$$

sentences of each of the following forms are equivalent to it:

If p then q.
That p is a sufficient condition that q.

p only if q.
That q is a necessary condition that p.

Exercises

Part A

Give four alternate ways of saying each of the following sentences.

1. *ABCD* is convex $\Longrightarrow$ *ABCD* is simple.
2. If *ABCD* is simple then *ABCD* is convex.
3. *ABCD* is equilateral only if *ABCD* is simple.
4. That *ABCD* is equilateral is a sufficient condition that *ABCD* is convex.
5. That $\triangle ABC$ is equilateral is a necessary condition that $\triangle ABC$ is isosceles.
6. That $\triangle ABC$ is equilateral is a sufficient condition that $\triangle ABC$ is isosceles.
7. $\vec{a} \cdot \vec{b} = 0$ if $\vec{a}$ is orthogonal to $\vec{b}$.
8. $\vec{a} \cdot \vec{b} = 0$ only if $\vec{a}$ is orthogonal to $\vec{b}$

*

Note that, in the exercises just completed, Exercise 2 is the converse of Exercise 1, and Exercise 1 is a theorem while Exercise 2 is not. Thus, we have:

(a) That *ABCD* is convex *is* a sufficient condition that *ABCD* is simple.

and:

(b) That *ABCD* is convex *is not* a necessary condition that *ABCD* is simple.

Next, notice that Exercises 7 and 8 are converses of one another and that both are theorems. From Exercise 8, we have:

Answers for Part A

1. If ABCD is convex then ABCD is simple.; That ABCD is convex is a sufficient condition that ABCD is simple.; ABCD is convex only if ABCD is simple.; That ABCD is simple is a necessary condition that ABCD is convex.
2. In the last three sentences above replace the word 'simple' with the word 'convex', and the word 'convex' with the word 'simple', and add: ABCD is simple $\Longrightarrow$ ABCD is convex
3. If ABCD is equilateral then ABCD is simple.; ABCD is equilateral $\Longrightarrow$ ABCD is simple; That ABCD is equilateral is a sufficient condition that ABCD is simple.; That ABCD is simple is a necessary condition that ABCD is equilateral.
4. That ABCD is convex is a necessary condition that ABCD is equilateral.; ABCD is equilateral only if ABCD is convex.; ABCD is equilateral $\Longrightarrow$ ABCD is convex.; If ABCD is equilateral then ABCD is convex.

   [Note that if the statement of Exercise 4 is prefixed by 'If ABCD is a plane quadrilateral then' the result is a theorem.]
5. That $\triangle$ABC is isosceles is a sufficient condition that $\triangle$ABC is equilateral.; If $\triangle$ABC is isosceles then $\triangle$ABC is equilateral.; $\triangle$ABC is isosceles only if $\triangle$ABC is equilateral.; $\triangle$ABC is isosceles $\Longrightarrow$ $\triangle$ABC is equilateral.
6. That $\triangle$ABC is isosceles is a necessary condition that $\triangle$ABC is equilateral.; If $\triangle$ABC is equilateral then $\triangle$ABC is isosceles.; $\triangle$ABC is equilateral only if $\triangle$ABC is isosceles.; $\triangle$ABC is equilateral $\Longrightarrow$ $\triangle$ABC is isosceles.
7. $\vec{a}$ is orthogonal to $\vec{b}$ only if $\vec{a} \cdot \vec{b} = 0$.; $\vec{a}$ is orthogonal to $\vec{b}$ $\Longrightarrow$ $\vec{a} \cdot \vec{b} = 0$.; That $\vec{a} \cdot \vec{b} = 0$ is a necessary condition that $\vec{a}$ is orthogonal to $\vec{b}$.; That $\vec{a}$ is orthogonal to $\vec{b}$ is a sufficient condition that $\vec{a} \cdot \vec{b} = 0$.
8. If $\vec{a} \cdot \vec{b} = 0$ then $\vec{a}$ is orthogonal to $\vec{b}$.; $\vec{a} \cdot \vec{b} = 0 \Longrightarrow \vec{a}$ is orthogonal to $\vec{b}$.; That $\vec{a} \cdot \vec{b} = 0$ is a sufficient condition that $\vec{a}$ is orthogonal to $\vec{b}$.; That $\vec{a}$ is orthogonal to $\vec{b}$ is a necessary condition that $\vec{a} \cdot \vec{b} = 0$.

(c) That $\vec{a}$ is orthogonal to $\vec{b}$ *is* a necessary condition that $\vec{a} \cdot \vec{b} = 0$.

and, from Exercise 7, we have:

(d) That $\vec{a}$ is orthogonal to $\vec{b}$ *is* a sufficient condition that $\vec{a} \cdot \vec{b} = 0$.

We combine these results as follows:

(e) That $\vec{a}$ is orthogonal to $\vec{b}$ is *a necessary and sufficient condition* that $\vec{a} \cdot \vec{b} = 0$.

When one claims that p is a sufficient condition that q, then one is claiming that the sentence:

$$p \longrightarrow q$$

is a theorem. Similarly, claiming that p is a necessary condition that q amounts to the claim that:

$$q \longrightarrow p$$

is a theorem. So, claiming that p is [both] a necessary and sufficient condition that q amounts to claiming that:

$$p \longleftrightarrow q$$

is a theorem. [Why?] For example, to claim (d), above, is to claim that:

$$\vec{a} \text{ is orthogonal to } \vec{b} \longrightarrow \vec{a} \cdot \vec{b} = 0$$

is a theorem; to claim (c), above, is to claim that:

$$\vec{a} \cdot \vec{b} = 0 \longrightarrow \vec{a} \text{ is orthogonal to } \vec{b}$$

is a theorem; to claim (e) is to claim that:

$$\vec{a} \text{ is orthogonal to } \vec{b} \longleftrightarrow \vec{a} \cdot \vec{b} = 0$$

is a theorem. For another example, to claim (b) is to claim that:

$$ABCD \text{ is simple} \longrightarrow ABCD \text{ is convex}$$

is *NOT* a theorem.

Answer to 'Why?':

Claiming that p is both a necessary and sufficient condition that q amounts to claiming that both:

$$p \Longrightarrow q \quad \text{and:} \quad q \Longrightarrow p$$

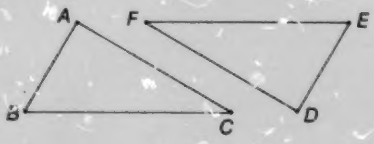
are theorems. But, if both of the latter are theorems then so is:

$$p \Longleftrightarrow q$$

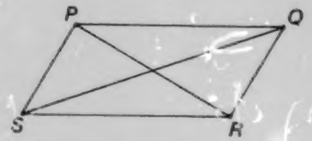
Part B

In each of the following, you are given a picture, a sentence *p*, and some other sentences. For each of the given sentences, you are to determine (i) whether it is a necessary condition that *p* and (ii) whether it is a sufficient condition that *p*. Indicate your choices by writing 'Yes.' or 'No.'. [Remember that in writing 'Yes.' for (i) you are claiming that the sentence '$p \longrightarrow \ldots$' is a theorem.]

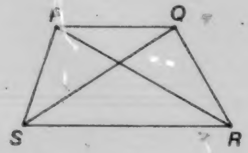
1. p: $AC = DF$ and $BC = FE$
   - **(a)** $ABC \leftrightarrow DEF$ is a congruence
   - **(b)** $ABC \leftrightarrow DFE$ is a congruence
   - **(c)** $ABC \leftrightarrow FED$ is a congruence
   - **(d)** $AB = DE$



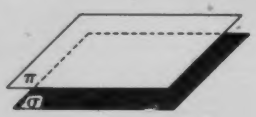
2. p: $PQRS$ is a parallelogram.
   - **(a)** $PQRS$ is a trapezoid.
   - **(b)** $\overline{PR}$ and $\overline{QS}$ bisect each other.
   - **(c)** $\overline{PR}$ and $\overline{QS}$ divide each other in the same ratio.
   - **(d)** $PQRS$ is simple and convex.
   - **(e)** $\angle SPQ \cong \angle SRQ$ and $\angle PSR \cong \angle PQR$.
   - **(f)** $\angle SPQ$ and $\angle PSR$ are supplementary.
   - **(g)** $Q - P$ and $R - S$ have the same sense.



3. p: $PQRS$ is a trapezoid.
   - **(a)** $PQRS$ is a parallelogram.
   - **(b)** $\overline{PR}$ and $\overline{QS}$ bisect each other.
   - **(c)** $\overline{PR}$ and $\overline{QS}$ divide each other in the same ratio.
   - **(d)** $PQRS$ is simple and convex.
   - **(e)** $\angle SPQ \cong \angle SRQ$ and $\angle PSR \cong \angle PQR$.
   - **(f)** $\angle SPQ$ and $\angle PSR$ are supplementary.
   - **(g)** $Q - P$ and $R - S$ have the same sense.



4. p: $\pi \parallel \sigma$
   - **(a)** The lines parallel to π are parallel to σ.
   - **(b)** $\pi \cap \sigma = \varnothing$
   - **(c)** The normals to π are parallel to the normals to σ.



Answers for Part B

1. (a) (i) No. (ii) Yes. (b) (i) No. (ii) No.
   (c) (i) No. (ii) No. (d) (i) No. (ii) No.

2. (a) (i) Yes. (ii) No. (b) (i) Yes. (ii) Yes.
   (c) (i) Yes. (ii) No. (d) (i) Yes. (ii) No.
   (e) (i) Yes. (ii) No. (f) (i) Yes. (ii) No.
   (g) (i) Yes. (ii) No.

   [The answer (ii) for (e) would be 'Yes.' if there were an overall assumption that PQRS is a simple plane quadrilateral. A similar remark applies to (e) (ii) of Exercise 3, below.]

3. (a) (i) No. (ii) Yes. (b) (i) No. (ii) Yes.
   (c) (i) Yes. (ii) Yes. (d) (i) Yes. (ii) No.
   (e) (i) No. (ii) No. (f) (i) No. (ii) No.
   (g) (i) Yes. (ii) No.

   [The answer (i) for (f) would be 'Yes.' if there were an overall assumption that $\overline{PQ} \parallel \overline{SR}$. The answer (ii) for (g) would be 'Yes.' if there were an overall assumption that PQRS is a quadrilateral.]

4. (a) (i) Yes. (ii) Yes. (b) (i) No. (ii) Yes.
   (c) (i) Yes. (ii) Yes.

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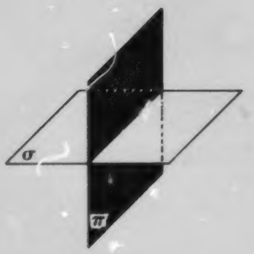
5. (a) (i) No. (ii) Yes. (b) (i) Yes. (ii) No.
   (c) (i) Yes. (ii) Yes. (d) (i) No. (ii) Yes.

   [Answer (ii) for part (a) may raise some question. You might point out that if one line parallel to π is perpendicular to σ then $\pi \perp \sigma$; so, if all lines parallel to π are perpendicular to σ then $\pi \perp \sigma$. But, in addition, since it is impossible that all lines parallel to π be perpendicular to σ it follows [by the law of contradiction] that (a) is a sufficient condition for anything. Similar remarks apply to the answer (ii) for part (d).]

Answer for Part C

Exercise 2, (b); in Exercise 3, (c); in Exercise 4, (a) and (c); in Exercise 5, (c).

5. $p: \pi \perp \sigma$
   - (a) The lines parallel to π are perpendicular to σ.
   - (b) $\pi \cap \sigma \neq \varnothing$
   - (c) The normals to π are perpendicular to the normals to σ.
   - (d) The lines parallel to π are perpendicular to the lines parallel to σ.



Part C

In each of the exercises in Part B, tell which of the given parts are necessary and sufficient conditions that p.

16.05 Rectangles

We begin with a definition:

> **Definition 16-1** A rectangle is a quadrilateral each of whose angles is a right angle.

It is natural to ask whether rectangles are necessarily plane quadrilaterals. Alternately, one might ask whether there are rectangles whose vertices are not coplanar. You will investigate this matter in the next exercises.

Exercises

Part A

1. Draw several rectangles.
2. Try, using sticks or strips of cardboard, to make a model of a rectangle which is not a plane quadrilateral. [One way to make this attempt is to draw two consecutive sides of a quadrilateral on your paper and try to represent the other two sides by two edges of an L-shaped piece of cardboard.]
3. Suppose that $\overline{AB}$ and $\overline{BC}$ are consecutive sides of a rectangle. As you know, there is a unique point D such that $ABCD$ is a parallelogram $D = A + ?$ and $D = C + ?$ Is $ABCD$ a rectangle? Why?
4. In Exercise 3 you probably saw that $ABCD$ is a plane rectangle. Now, let's try to find a point $E \neq D$ such that $ABCE$ is a rectangle. If there is such a point then it must belong to $\overline{C[C \overset{\rightarrow}{-} B]^{\perp}}$. [Why?] To what other plane must such a point E belong?
5. It follows from Exercise 4 that any point E such that $ABCE$ is a rectangle must belong to a certain line. Describe this line.

Suggestions for the exercises of section 16.05:

(i) Parts A and B should be developed under teacher direction.

(ii) Parts C, D, and E are appropriate for homework (except, perhaps, for Exercise 4, Part D). However, these constitute more than a reasonable amount of work for one such assignment.

Answers for Part A

1. [Various answers.]
2. It does not seem to be possible to model a rectangle which is not a plane quadrilateral.
3. $D = A + (C - B) = C + (A - B)$; Yes.; $\angle A$ is a right angle because it and $\angle B$ are supplementary [since they are consecutive angles formed by parallel lines and a transversal] and $\angle B$ is a right angle. A similar argument with '$\angle C$' for '$\angle A$' shows that $\angle C$ is a right angle. Finally, a similar argument based on either the fact that $\angle A$ is a right angle or the fact that $\angle C$ is a right angle shows that $\angle D$ is a right angle.
4. Since all lines through C perpendicular to $\overleftrightarrow{BC}$ are contained in the plane $\overline{C[C \overset{\rightarrow}{-} B]^{\perp}}$, E must belong to this plane. Since all lines through A perpendicular to $\overleftrightarrow{AB}$ are contained in $\overline{A[A \overset{\rightarrow}{-} B]^{\perp}}$, E must belong to this plane as well.
5. E must belong to the line of intersection of $\overline{C[C \overset{\rightarrow}{-} B]^{\perp}}$ and $\overline{A[A \overset{\rightarrow}{-} B]^{\perp}}$.

*

Sample Quiz

In each of the following, write an equivalent sentence of the form 'if ... then ___' and tell whether or not the sentence is a theorem.

1. That a line is parallel to a plane is a sufficient condition that it is parallel to each line in the plane.
2. That a line is perpendicular to a plane is a necessary condition that it is perpendicular to each line in the plane.
3. That a line is perpendicular to a plane is a sufficient condition that it is perpendicular to each line in the plane.
4. That (p and $p \Longrightarrow q$) is a sufficient condition that q.
5. That p is a sufficient condition that (p and q).
6. That p is a necessary condition that (p and q).

Key to Sample Quiz

1. If a line is parallel to a plane then it is parallel to each line in the plane.; Not a theorem.
2. If a line is perpendicular to each line in a plane then the line is perpendicular to the plane.; Theorem.
3. If a line is perpendicular to a plane then it is perpendicular to each line in the plane.; Theorem.
4. If (p and $p \Longrightarrow q$) then q.; Theorem.
5. If p then (p and q).; Not a theorem.
6. If (p and q) then p.; Theorem.

6. Suppose that E is any point on the line described in Exercise 5 other than D. Find the cosine of $\angle AEC$ in terms of AB, BC, and the distance e of E from $\overset{\frown}{ABC}$. [*Hint*: $\triangle ADC$ is a right triangle. Find two other right triangles.]
7. Prove the following:

Lemma Any rectangle is a plane quadrilateral.

*

Using the results of Part A it is easy to prove:

Theorem 16-13 Each rectangle is a parallelogram.

This being the case, all that we know about parallelograms applies to rectangles.

It is also easy to prove:

Theorem 16-14 A parallelogram one of whose angles is a right angle is a rectangle.

and:

Theorem 16-15 A parallelogram is a rectangle if and only if its diagonals are congruent.

Part B

Prove each of the following.

1. Theorem 16-13
2. Theorem 16-14
3. Any rectangle has congruent diagonals.
4. Theorem 16-15.

Part C

In each of the following, decide whether the statement is a theorem or not. Be prepared to justify your answer.

1. That a parallelogram has congruent diagonals is a sufficient condition that it be a rectangle.
2. That a parallelogram has congruent diagonals is a necessary condition that it be a rectangle.
3. A quadrilateral is a rectangle only if its diagonals are congruent.
4. A quadrilateral is a rectangle if its diagonals are congruent.
5. That a quadrilateral has a pair of opposite right angles is a sufficient condition that it be a rectangle.
6. That a quadrilateral has a pair of opposite right angles is a necessary condition that it be a rectangle.
7. In order that a quadrilateral be a rectangle it is necessary that it has two consecutive right angles.

Answers for Part A [cont.]

6. $\triangle ADC$, $\triangle ADE$, and $\triangle CDE$ are all right triangles with right angles at D. Their hypotenuses are the sides of $\triangle AEC$ and, by using them and the cosine law we can find $\cos\angle AEC$. Let $AB = b$ and $BC = c$. Then $(AC)^2 = b^2 + c^2$, $(AE)^2 = c^2 + e^2$, and $(CE)^2 = b^2 + e^2$. Hence, $\cos\angle AEC = [(c^2 + e^2) + (b^2 + e^2) - (b^2 + c^2)]/[2\sqrt{(c^2 + e^2)(b^2 + e^2)}]$ and is, clearly, greater than 0. So, for $E \neq D$, $\angle AEC$ is not a right angle [in fact, it is an acute angle].
7. This follows immediately from the result in Exercise 6. For, as shown there, if ABCE is a quadrilateral with right angles at A, B, C, and E then $E \in \overset{\frown}{ABC}$.

Answers for Part B

1. It was shown in Part A that if ABCD is a rectangle then D is the point such that ABCD is a parallelogram.
2. Since consecutive angles of a parallelogram are supplementary and supplements of right angles are right angles it follows at once that all angles of a parallelogram are right angles if one of them is a right angle. So, by Definition 16-1, a parallelogram one of whose angles is a right angle is a rectangle.
3. The diagonals of a rectangle are the hypotenuses of congruent right triangles and, so, are themselves congruent.
4. The only if-part is taken care of by Exercise 3. So, all that remains is to show that a parallelogram with congruent diagonals is a rectangle. [However, it is easy to prove the "if and only if" at one stroke.] Suppose ABCD is a parallelogram with $DA = BC = a$ and $AB = CD = b$. By the cosine law,

$$(AC)^2 = a^2 + b^2 - 2ab\cos\angle B \text{ and } (BD)^2 = a^2 + b^2 - 2ab\cos\angle A.$$

So, $AC = BD$ if and only if $\cos\angle B = \cos\angle A$. Since, ABCD being a parallelogram, $\angle A$ and $\angle B$ are supplementary it follows that $AC = BD$ if and only if $\cos\angle A = \cos\angle B = 0$. But, a parallelogram is a rectangle if [and only if] one of its angles is a right angle.

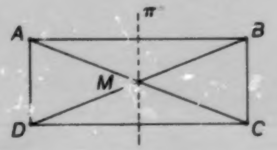
Answers for Part C

1. Theorem. [This is a restatement of the if-part of Theorem 16-15.]
2. Theorem. [This is a restatement of the only if-part of Theorem 16-15.]
3. Theorem. [This is a restatement of the only if-part of Theorem 16-15.]
4. Not a theorem. [Counterexamples are easy to find — draw congruent diagonals which do not bisect each other and then fill in the sides.]
5. Not a theorem. [Counterexamples are easy — begin by drawing a pair of perpendicular sides, then draw the third side not parallel to the second and draw the fourth perpendicular to the third.]
6. Theorem. [This follows at once from Definition 16-1.]
7. Theorem. [This, also, follows at once from Definition 16-1.]

8. In order that a quadrilateral be a rectangle it is sufficient that it has two consecutive right angles.
9. In Exercises 5–8, replace 'quadrilateral' by 'parallelogram' and do the resulting problems.
10. In Exercises 3 and 4, replace 'quadrilateral' by 'convex quadrilateral' and do the resulting problems.
11. That the diagonals of a first rectangle are congruent to the diagonals of a second rectangle is a sufficient condition that the rectangles are congruent.

Part D

Consider the rectangle *ABCD*, pictured at the right. Let π be the perpendicular bisector of $\overline{AB}$ and let f be the reflection in π.



1. Explain why both M and the midpoint of $\overline{CD}$ are points of π.
2. What is the image under f of each of the following?
   (a) A (b) B (c) C (d) D
   (e) $\overline{AB}$ (f) $\overrightarrow{AC}$ (g) $\overline{BC}$ (h) $ABCD$
3. Let l be the intersection of π with the plane of rectangle $ABCD$. l is sometimes called *a line of symmetry* of rectangle *ABCD*.
   (a) Describe another line of symmetry of rectangle *ABCD*.
   (b) Under what conditions is $\overleftrightarrow{AC}$ a line of symmetry of rectangle *ABCD*?

*4. Describe some other figures in the plane of rectangle *ABCD* which have $\overline{AB}$ as a side and l as a line of symmetry.

Part E

Consider rectangle *ABCD*.

1. Given that $AB = 7$ and $BC = 9$, compute the product of the measures of the diagonals *ABCD*.
2. Given that $AB = 24$ and $BC = 10$, compute the distance from B to $\overleftrightarrow{AC}$.
3. In Exercise 2, if E and F are the feet of the perpendiculars from B and D, respectively, to $\overleftrightarrow{AC}$, compute EF and the distance from F to $\overleftrightarrow{DC}$.
4. M and N are the midpoints of $\overline{AD}$ and $\overline{BC}$, respectively, and $MN = 15$. Compute BD.
5. In Exercise 4, if $AB = BC + 6$, compute AB and BC.
6. $BC = a + 2$, $CD = 2a - 3$, and $BD = 7$ for some a. Determine the values of 'a' which satisfy these conditions, and give the lengths of the sides of *ABCD* for each such value of 'a'.
7. Let E be the point of $\overleftrightarrow{BD}$ such that $\overline{AE} \perp \overline{BD}$. Find three triangles, whose vertices are among the points A, B, C, D, and E, such that the angles of these triangles are congruent to the angles of $\triangle AED$.

Answers for Part C [cont.]

8. Not a theorem.
9. All are theorems.
10. [Answers are the same as before.]
11. Not a theorem. [Given any rectangle there is a square whose diagonals are congruent to those of the given rectangle. (And, one can "give" a rectangle which is not a square.)]

Answers for Part D

1. Let P and Q be the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. Since a rectangle is a parallelogram, the diagonals of a rectangle bisect each other and, so, M is the midpoint of $\overline{AC}$. It follows that $M - P = (C - A)/2 - (B - A)/2 = (C - B)/2$. Since $\angle B$ is a right angle, $C - B \in [\pi]$ and, so, $M - P \in [\pi]$. Hence, since $P \in \pi$, $M \in \pi$.

   Since $P = B + (A - B)/2$ and $Q = C + (D - C)/2$, and since $A - B = D - C$ [for, ABCD is a parallelogram], it follows that $Q - P = C - B \in [\pi]$. Hence, since P belongs to π, so does Q.

2. (a) B (b) A (c) D (d) C (e) $\overleftrightarrow{BA}$ (f) $\overleftrightarrow{BD}$ (g) $\overleftrightarrow{AD}$ (h) BACD

   [Parts (a) - (d) are easily justified since π is the perpendicular bisector of $\overleftrightarrow{AB}$ and of $\overleftrightarrow{CD}$. Parts (e) - (h) follow at once by Theorem 14-27.]

3. (a) The line containing the midpoints of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ is a line of symmetry of ABCD.

   (b) $\overleftrightarrow{AC}$ is a line of symmetry of rectangle ABCD if and only if ABCD is equilateral.

   [Proof of (b): $\overleftrightarrow{AC}$ is a line of symmetry if and only if the plane σ which contains $\overleftrightarrow{AC}$ and is perpendicular to the plane of ABCD is the perpendicular bisector of $\overline{BD}$. This is the case if and only if $\overleftrightarrow{BD} \perp \overleftrightarrow{AC}$. Now, if $\overleftrightarrow{BD} \perp \overleftrightarrow{AC}$ then $\overline{AB} \cong \overline{AD}$ by s.a.s. applied to $\triangle AMB$ and $\triangle AMD$. On the other hand, if $\overline{AB} \cong \overline{AD}$ then the same triangles are congruent by s.s.s., and since congruent supplementary angles are right angles it follows that $\overleftrightarrow{BD} \perp \overleftrightarrow{AC}$. So, $\overleftrightarrow{AC}$ is a line of symmetry of rectangle ABCD if and only if $\overline{AB} \cong \overline{AD}$.]

4. In general, the union of any figure in the plane of ABCD with its reflection in π has l as a line of symmetry. More particularly, any isosceles triangle in $\overline{ABC}$ with base $\overline{AB}$, any rectangle in $\overline{ABC}$ with side $\overline{AB}$, and any "isosceles trapezoid" in $\overline{ABC}$ with base $\overline{AB}$, have l as a line of symmetry. ['isosceles trapezoid' is defined on page 290.]

Answers for Part E

[Note. Exercises 5 and 6 involve solution of quadratic equations. You may wish to review this topic before assigning the exercises.]

1. 130 2. 120/13 3. 238/13 [EC = 50/13]; 1440/169
4. 30 [BD = AC = MN·2]
5. 24; 18 [If BC = x then $x^2 + 6x - 432 = 0$.]
6. a = 3.6 [$5a^2 - 8a - 36 = 0$] BC = 5.6, CD = 4.2
7. $\triangle BEA$, $\triangle BAD$, $\triangle DCB$

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Suggestions for the exercises of section 16.06:

(i) To insure a clear understanding of Definition 16-2, Part A should be developed under teacher supervision.

(ii) Parts B, C, and D are appropriate for class or homework assignment.

Answers for Part A

1. [Various answers.]
2. (a) [An impossible task.]
   (b) Since AD = AE + ED and BC = BE + EC it follows that AD + BC = (AE + BE) + (ED + EC) > AB + CD. Since AD = BC = AB = CD, this is impossible. Hence, each rhombus is a simple quadrilateral. [Note that we have proved that, in any nonsimple quadrilateral, the sum of the measures of the intersecting sides is greater than the sum of the measures of the other two sides.]
3. Yes.; by Definition 16-2, Theorem 16-11, and Exercise 2.
4. (a) It is a rhombus. (b) It is a rectangle.
   (c) It is a square. (d) It is a square.

16.06 Rhombuses and Squares

Definition 16-2

(a) A rhombus is a plane quadrilateral all four of whose sides are congruent.

(b) A square is a rectangular rhombus.

Exercises

Part A

1. Draw several rhombuses and draw their diagonals as dashed lines.
2. (a) Try to draw a rhombus which is not a simple quadrilateral.
   (b) Suppose that *ABCD* is a rhombus and that $\overline{AD}$ and $\overline{BC}$ intersect at *E*. Obtain a contradiction. [*Hint*: Compare *AD* + *BC* with *AB* + *CD*.]
3. Is a rhombus a parallelogram? Explain your answer.
4. What can you say about
   (a) a parallelogram with two consecutive congruent sides?
   (b) a parallelogram with two consecutive congruent angles?
   (c) a rhombus with two consecutive congruent angles?
   (d) a rectangle with two consecutive congruent sides?
5. Show that each diagonal of a rhombus is contained in the perpendicular bisector of the other. What does this tell you about the diagonals of a rhombus?
6. Prove:
   That a parallelogram is a rhombus is a necessary and sufficient condition that its diagonals are perpendicular.
7. Suppose that each diagonal of a plane quadrilateral is contained in the perpendicular bisector of the other.
   (a) Show that the quadrilateral is a rhombus.
   (b) Give an example to show that the word 'plane' is needed in the hypothesis in order to prove (a).
8. Show that
   (a) the midpoints of the sides of a rectangle are the vertices of a rhombus;
   (b) the midpoints of the sides of a rhombus are the vertices of a rectangle.
9. (a) Under what conditions is the rhombus in Exercise 8(a) a square?
   (b) Under what conditions is the rectangle in 8(b) a square?

*

Answers for Part A [cont.]

5. Consider rhombus ABCD and diagonal $\overline{BD}$. By Definition 16-2, each of A and C is equidistant from B and D. Hence, A and C — and, with them, $\overline{AC}$ — is contained in the perpendicular bisector of $\overline{BD}$. Consequently, each diagonal of a rhombus is contained in the perpendicular bisector of the other. In particular, the diagonals of a rhombus are subsets of perpendicular lines and, so, are perpendicular to each other.

6. After Exercise 5, all we need show is that a parallelogram whose diagonals are perpendicular is a rhombus. Using our earlier notation for quadrilateral ABCD — that $A - D = \vec{a}$, $B - A = \vec{b}$, $C - B = \vec{c}$, and $D - C = \vec{d}$ — the directions of the diagonals are those of $\vec{a} + \vec{b}$ and $\vec{b} + \vec{c}$. In a parallelogram, $\vec{c} = -\vec{a}$ and, so, its diagonals are perpendicular if and only if $(\vec{a} + \vec{b}) \cdot (\vec{b} - \vec{a}) = 0$. As we know, the latter is the case if and only if $\|\vec{a}\| = \|\vec{b}\|$ — that is, if and only if the parallelogram is a rhombus. [Note that in this proof we have given a new solution for Exercise 5.]

7. (a) Consider quadrilateral ABCD. Since $\overline{AC}$ is contained in the perpendicular bisector of $\overline{BD}$ it follows that $\overline{AC} \perp \overline{BD}$. Since {A, B, C, D} is coplanar it follows that $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect. It follows, since $\overline{AC}$ is contained in the perpendicular bisector of $\overline{BD}$, that $\overline{AC}$ contains the midpoint of $\overline{BD}$. Similarly, $\overline{BD}$ contains the midpoint of $\overline{AC}$. Since $\overline{AC}$ and $\overline{BD}$ bisect each other, ABCD is a parallelogram. So, since $\overline{AC} \perp \overline{BD}$, ABCD is a rhombus.

   (b) To obtain a counterexample, "fold" a rhombus along one of its diagonals.

8. (a) One solution is to use the Pythagorean theorem, and the fact that a rectangle is a plane quadrilateral, to show that Definition 16-2(a) is satisfied. Another is to use the fact that, since a rectangle is a quadrilateral, the midpoints of its sides are vertices of a parallelogram and then show that the diagonals of this parallelogram are perpendicular.

   (b) The midpoints of the sides of a rhombus are vertices of a parallelogram. The sides of this parallelogram are parallel to the diagonals of the rhombus. Hence, consecutive sides are perpendicular.

9. (a) The rhombus of Exercise 8(a) is a square if and only if the diagonals of the rectangle are perpendicular. This is the case if and only if the rectangle is a square. [See Definition 16-2(b) and Exercise 6.]

   (b) The rectangle in 8(b) is a square if and only if the rhombus is a square.

Some of the results from Part A are summarized in the following theorems:

Theorem 16–16 Each rhombus is a parallelogram.

Theorem 16–17 Any parallelogram two of whose consective sides are congruent is a rhombus.

Theorem 16–18 A parallelogram is a rhombus if and only if its diagonals are perpendicular.

Corollary. A rectangle is a square if and only if its diagonals are perpendicular.

Theorem 16–19 A plane quadrilateral is a rhombus if and only if each of its diagonals is contained in the perpendicular bisector of the other.

There is one more handy theorem concerning rhombuses which we state in:

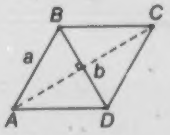
Theorem 16–20 A quadrilateral is a rhombus if and only if its diagonals are contained in the bisectors of its angles.

Part B

1. Prove that a diagonal of a rhombus is contained in the bisectors of the angles at its end points.
2. Prove that any quadrilateral whose diagonals are contained in the bisectors of its angles is a rhombus. [*Hint*: Note that such a quadrilateral is a plane quadrilateral and use a congruence theorem for triangles to prove that it is equilateral.]
3. Make use of Exercises 1 and 2 to establish Theorem 16–20.
4. (a) Find two lines of symmetry for a rhombus.
   (b) Describe a rhombus which has four lines of symmetry.

Part C

Suppose that *ABCD* is a rhombus, that $AB = a$ and that $BD = b$, as shown in the picture at the right.



1. Compute each of the following in terms of 'a' and 'b'.
   (a) cos $\angle BAD$ (b) cos $\angle ABC$ (c) AC
   (d) cos $\angle BAC$ (e) sin $\angle BAC$ (f) sin $\angle ABD$

Answers for Part B

1. In rhombus ABCD, $A = C + [(D - C) + (B - C)]$. But, since $\|D - C\| = \|B - C\|$, $(D - C) + (B - C)$ is in the sense of the bisector of $\angle C$. Hence, $\overline{AC}$ is a subset of this bisector. [Theorem 15-15]. So, each diagonal of a rhombus is contained in the bisectors of the angles at its end points.
2. Suppose given quadrilateral ABCD such that $\overline{AC}$ is a subset of the bisector of $\angle A$ and of the bisector of $\angle C$ and $\overline{BD}$ is a subset of the bisector of $\angle B$ and of the bisector of $\angle D$. Since the bisector of $\angle B$ is contained in the plane of $\angle B$ — that is, the plane $\overleftrightarrow{ABC}$ — and since $\overline{BD}$ is a subset of this angle bisector it follows that $D \in \overleftrightarrow{ABC}$ and, so, that ABCD is a plane quadrilateral. Now, it follows by a.s.a. that $ABC \longleftrightarrow ADC$ and $BCD \longleftrightarrow BAD$ are congruences. So, $AB = AD$, $BC = DC$, and $BC = BA$. Hence, the plane quadrilateral ABCD is equilateral and so, Definition 16-2, is a rhombus.
3. [Obvious.]
4. (a) The lines containing the diagonals of a rhombus are lines of symmetry.
   (b) A square has four lines of symmetry.

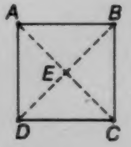
Answers for Part C

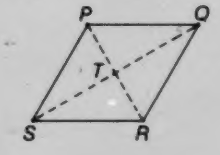
1. (a) $1 - b^2/(2a^2)$ (b) $b^2/(2a^2) - 1$ (c) $\sqrt{4a^2 - b^2}$
   (d) $\sqrt{4a^2 - b^2}/(2a)$ (e) $b/(2a)$ (f) $\sqrt{4a^2 - b^2}/(2a)$

2. Give a relation between the values for 'a' and 'b' for which
   (a) $\angle BAD$ is a right angle; (b) $\angle ABC$ is obtuse;
   (c) $\angle ABC$ is acute; (d) $\triangle ABC$ is equilateral.
3. Let E be the foot of the perpendicular from B to $\overleftrightarrow{AD}$.
   (a) Compute BE in terms of 'a' and 'b'.
   (b) Show that $BE \cdot AD = \frac{1}{2}AC \cdot BD$.
   (c) Show that $d(C, \overleftrightarrow{AB}) = BE$.

Part D

In each of the following you are given some information about the square $ABCD$ or the rhombus $PQRS$, which are pictured here.





Do the required computations.

1. $AB = 8$. Compute BD and $\sin \angle BDC$.
2. $PQ = 12$ and $PT = 5$. Compute QS and $\cos \angle PSR$.
3. $\cos \angle QSR = \frac{1}{2}$ and $QS = 20$. Compute PR, PQ, and $\sin \angle SPQ$.
4. $BD = 15$. Compute $AB \cdot BC$ and the perimeter of $ABCD$.
5. Compute the ratio of BC to BD.
6. Show that $SQ/PR = \cos \angle PSQ/\cos \angle SPR$.
7. (a) Show that $\cos \angle SPQ = 2(\cos \angle SPR)^2 - 1$ and $\cos \angle PSR = 2(\cos \angle PSQ)^2 - 1$.
   (b) Make use of the results in Exercises 6 and 7(a) to show that

$$\frac{SQ}{PR} = \sqrt{\frac{1 + \cos \angle PSR}{1 + \cos \angle SPQ}}.$$

   (c) Show that

$$\frac{SQ}{PR} = \sqrt{\frac{1 + \cos \angle PSR}{1 - \cos \angle PSR}}.$$

16.07 Other Special Quadrilaterals

In the last two sections we studied some of the properties of rectangles, rhombuses, and squares. We know that any rectangle has congruent diagonals and since a square is a rectangle, a square has congruent diagonals. We also know that a rhombus has perpendicular diagonals and, since a square is a rhombus, a square has perpendicular diagonals.

Answers for Part C [cont.]

2. (a) $b = a\sqrt{2}$ (b) $b < a\sqrt{2}$ (c) $b > a\sqrt{2}$ (d) $b = a\sqrt{3}$
3. (a) $b\sqrt{4a^2 - b^2}/(2a)$
   (b) Both are equal to $b\sqrt{4a^2 - b^2}/2$.
   (c) Let F be the foot of the perpendicular to $\overleftrightarrow{AB}$. Then $d(C, \overleftrightarrow{AB}) = CF$. By Theorem 16-7(c) and Theorem 15-13, $BFC \longleftrightarrow AEB$ is a congruence. So, $CF = BE$.

Answers for Part D

1. $8\sqrt{2}$; $1/\sqrt{2}$ [or: $\sqrt{2}/2$]
2. $2\sqrt{119}$; $47/72$
3. $20\sqrt{3}$; 20; $\sqrt{3}/2$
4. $225/2$; $60/\sqrt{2}$ [or: $30\sqrt{2}$]
5. $1/\sqrt{2}$ [or: $\sqrt{2}/2$]
6. $\cos \angle PSQ = ST/SP$ and $\cos \angle SPR = PT/SP$. So, $\cos \angle PSQ/\cos \angle SPR = ST/PT = SQ/PR$.
7. (a) By Exercise 5 of Part C on page 227, and Theorem 16-20, $\cos \angle SPR = \sqrt{(1 + \cos \angle SPQ)/2}$. Solve for '$\cos \angle SPQ$'. A similar argument yields the other conclusion.
   (b) Obvious [from Exercises 6 and 7(a)].
   (c) From (b) and the fact that $\angle SPQ$ and $\angle PSR$ are supplementary. [A rhombus is a parallelogram, and consecutive angles of a parallelogram are supplementary.]

The introduction and the definitions of classificatory terms — such as 'trapezoid', 'parallelogram', and 'rectangle' — must be justified by the theorems which result. We have chosen our terminology with this in view. We shall define 'isosceles trapezoid' in such a way to exclude parallelograms because in this way we get better theorems. For example, isosceles trapezoids and rectangles turn out to be precisely those quadrilaterals which are isodiagonal. Orthodiagonal quadrilaterals have many of the properties of kites [which we allow to be nonplanar] and the latter, of course, have some of the properties of rhombuses.

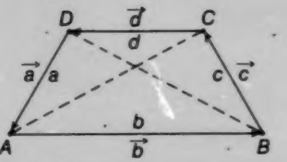
Suggestions for use of exercises of section 16.07:
(i) Part A may be used as class exercises.
(ii) Parts B and C are appropriate for homework.
(iii) Part D and the discussion preceding it should be teacher directed.
(iv) The Exploration Exercises on page 293 may be assigned for homework and then discussed in class.

Quadrilaterals which have congruent diagonals are sometimes called *isodiagonal quadrilaterals*. And, quadrilaterals which have perpendicular diagonals are sometimes called *orthodiagonal quadrilaterals*. Thus, rectangles and, so, squares are isodiagonal quadrilaterals, and rhombuses and squares are orthodiagonal quadrilaterals. However, there are isodiagonal quadrilaterals which are not rectangles, and there are orthodiagonal quadrilaterals which are not rhombuses. Such quadrilaterals will be among those which we shall study in this section.

Exercises

Part A

1. Draw several quadrilaterals which have perpendicular diagonals but are not rhombuses. Try to draw at least one which is not a plane quadrilateral. [*Hint*: Draw the diagonals first. You should be able to find three "really different" kinds of orthodiagonal quadrilaterals.]
2. Draw--or, make a model of—a quadrilateral each of whose sides is perpendicular to the opposite side. What appears to be the case with respect to the diagonals of the quadrilateral you just drew [or, made]?
3. Given quadrilateral *ABCD*, as pictured at the right, show that
   (a) $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{d}$;
   [*Hint*: $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}$.]
   (b) $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) = 0$ if and only if $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{d}$.



4. Prove that if each side of a quadrilateral is perpendicular to the opposite side then the diagonals of the quadrilateral are perpendicular.
5. Suppose that *ABCD* is a quadrilateral such that $\overline{AB} \perp \overline{CD}$ and $\overline{AC} \perp \overline{BD}$. What can you say about $\overline{BC}$ and $\overline{AD}$? Explain.
6. In the quadrilateral of Exercise 3, show that

$$(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) = \frac{(b^2 + d^2) - (a^2 + c^2)}{2}.$$

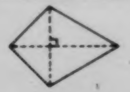
[*Hint*: Build on the result in Exercise 3.]

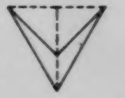
7. Use the result of Exercise 6 to show that the diagonals of a rhombus are perpendicular.

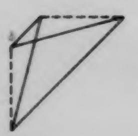
*

Answers for Part A

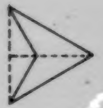
1.

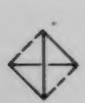






2.





[The first figure is a plane arrowhead. The second is intended to show the edges of a tetrahedron whose opposite edges are perpendicular.]

It appears that if each side of a quadrilateral is perpendicular to the opposite side then the quadrilateral's diagonals are also perpendicular. [See Exercise 4.]

3. (a) $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot (\vec{a} + \vec{b} + \vec{c})$
   $= \vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{d}$

   (b) This follows at once from part (a).

4. Using the figure and results of Exercise 3 it follows that if $\overline{AD} \perp \overline{BC}$ and $\overline{CD} \perp \overline{AB}$ then $\vec{a} \cdot \vec{c} = 0 = \vec{b} \cdot \vec{d}$, and, so, $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) = 0$ and $\overline{AC} \perp \overline{BD}$. Hence, if each side of a quadrilateral is perpendicular to the opposite side then the diagonals of the quadrilateral are perpendicular.

5. $\overline{AD} \perp \overline{BC}$; Given four points, no three of which are collinear, one may describe just three quadrilaterals, ABCD, ACBD, ABDC, with these points as vertices. [One of the intervals $\overline{AC}$, $\overline{AB}$, and $\overline{AD}$ must be a diagonal, and any one of them may be.] The result of Exercise 4 applies to each of these quadrilaterals. In particular, it applies to ABDC. In this case it says, of quadrilateral ABCD, that if it is such that a pair of opposite sides are perpendicular and its diagonals are perpendicular then the other opposite sides are perpendicular.

6. Note that $\vec{a} \cdot \vec{c} = [(\vec{a} + \vec{c}) \cdot (\vec{a} + \vec{c}) - (a^2 + c^2)]/2$ and $\vec{b} \cdot \vec{d} = [(\vec{b} + \vec{d}) \cdot (\vec{b} + \vec{d}) - (b^2 + d^2)]/2$. So, by Exercise 3(a), and noting that $(\vec{a} + \vec{c}) \cdot (\vec{a} + \vec{c}) = (\vec{b} + \vec{d}) \cdot (\vec{b} + \vec{d})$, yields the desired conclusion.

7. If ABCD is a rhombus then $a^2 = b^2 = c^2 = d^2$ and, so, $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) = 0$. So, since $\vec{a} + \vec{b}$ and $\vec{b} + \vec{c}$ are in the directions of the diagonals of ABCD, it follows that the diagonals of ABCD are perpendicular.

One consequence of the result obtained in Exercise 6 is worth stating. We do this in:

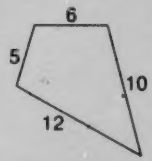
> **Theorem 16–21** The sum of the squares of two opposite sides of a quadrilateral equals the sum of the squares of the other two sides if and only if the diagonals of the quadrilateral are perpendicular.

Notice that Theorem 16–21 gives us a necessary and sufficient condition to employ in determining whether a given quadrilateral is orthodiagonal.

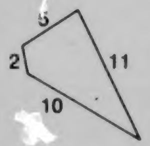
Part B

1. In each of the following, determine whether the quadrilateral pictured has perpendicular diagonals.

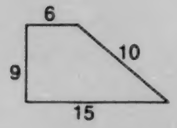
(a)



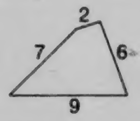
(b)



(c)



(d)



2. In each part, you are given the measures of three consecutive sides of an orthodiagonal quadrilateral. Compute the measure of the fourth side in each case.

(a) 6, 8, 10 (b) 6, 14, 7 (c) 6, 7, 14
(d) 9, 9, 6 (e) 9, 6, 9 (f) 8, 9, 10

3. Suppose that, with respect to an orthonormal coordinate system, A, B, C, and D have coordinates as follows:

A: (5, 8, 3) B: (6, 12, 2) C: (4, 9, 12) D: (4, 1, 3)

Show that $ABCD$ has perpendicular diagonals. [*Hint*: Consider $C - A$ and $D - B$.]

4. Suppose that A, B, and C have coordinates (2, 1, 3), (−1, 5, 7), and (4, 3, 5), respectively, with respect to an orthonormal coordinate system.
   (a) Describe the coordinates of all points D such that $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$.
   (b) Describe all points D such that $ABCD$ is an orthodiagonal quadrilateral.

Answers for Part B

1. (a) No. $[5^2 + 10^2 \neq 6^2 + 12^2]$ (b) Yes. $[5^2 + 10^2 = 2^2 + 11^2]$
   (c) No. $[9^2 + 10^2 \neq 6^2 + 15^2]$ (d) Yes. $[2^2 + 9^2 = 6^2 + 7^2]$

2. (a) $6\sqrt{2}$ (b) [impossible] (c) $\sqrt{183}$
   (d) 6 (e) $\sqrt{126}$ (f) $\sqrt{83}$

3. The components of $C - A$ are $(-1, 1, 9)$ and those of $D - B$ are $(-2, -11, 1)$. Since $-1 \cdot -2 + 1 \cdot -11 + 9 \cdot 1 = 0$ it follows that $C - A$ and $D - B$ are orthogonal. Hence, the diagonals of $ABCD$ are perpendicular. [The problem can also be solved by finding the sums of the squares of the measures of opposite sides. But, this requires much more computation.]

4. (a) The coordinates of such points are all the solutions of '$x + y + z = 11$'.
   (b) These are the points whose coordinates have been found in part (a) and which do not belong to $\overleftrightarrow{AC} \cup \{B\}$.

Definition 16-3 A kite is a quadrilateral which has two consecutive sides congruent and the sides opposite these congruent.

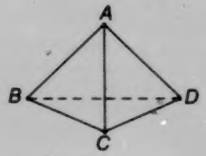
One example of a kite is a rhombus. Is there a rectangular kite? Are there kites which are not plane quadrilaterals?

Part C

1. Draw some kites which are not rhombuses.
2. Show that any kite is a simple quadrilateral.
3. Make use of Theorem 16-21 to show that a kite is an orthodiagonal quadrilateral.
4. Show that at least one diagonal of a kite is contained in the perpendicular bisector of the other.
5. Prove:

Theorem 16-22 A quadrilateral is a kite if and only if one of its diagonals is contained in the perpendicular bisector of the other.

6. Consider the pyramid pictured at the right. Suppose that each of the triangles whose vertices are among the points *A*, *B*, *C*, and *D* is equilateral.
   (a) How many such triangles are there?
   (b) What can you say about the segments which contain the sides of these triangles?
   (c) Consider quadrilateral *ABCD*. What are its diagonals? Show that they are perpendicular.
   (d) Give the other quadrilaterals whose vertices are *A*, *B*, *C*, and *D*. Show that each is an orthodiagonal quadrilateral.



7. Consider each of the quadrilaterals discussed in Exercise 6.
   (a) Is each of them a kite?
   (b) Give a proof that the perpendicular bisector of $\overline{CD}$ contains $\overline{AB}$.

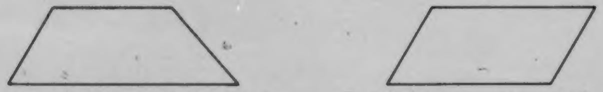
*

Recall that, in this book, a trapezoid is a simple quadrilateral with [at least] two parallel sides. In particular, parallelograms are trapezoids. [In some books trapezoids are defined to be simple quadrilaterals with exactly two parallel sides.] Any pair of parallel sides of a trapezoid is called a pair of bases of the trapezoid. Opposite sides of a trapezoid which are not parallel are called *legs* of the trapezoid. Note that, while a trapezoid can have two pairs of bases [and must have at

The only rectangular kites are squares. There are [under our definition] many kites which are not plane figures. [Plane kites do not seem to have any interesting properties which are not shared by all kites.]

Answers for Part C

1. [Various answers.]
2. If a kite is not simple it must be plane. That a plane kite must be simple follows by the same argument we used to prove that a rhombus must be simple. [See answer for Exercise 2(b) of Part A on page 284.] [There is an alternative proof which, of course, applies equally well to rhombuses. Suppose that ABCD is a quadrilateral such that AB = BC and $\overline{AB} \cap \overline{CD} \neq \emptyset$. (Draw a figure.) It follows that ∠ACD is smaller than ∠ACB, ∠ACB ≅ ∠CAB, and ∠CAB is smaller than ∠CAD. So, ∠ACD is smaller than ∠CAD and, hence, AD < CD. In particular, a nonsimple quadrilateral cannot be a kite.]
3. If ABCD is a kite then, with our usual notation either a = b and c = d or a = d and b = c. In either case, $a^2 + c^2 = b^2 + d^2$ and, so, the quadrilateral is orthodiagonal.
4. Let ABCD be a kite such that AB = BC and AD = DC. Since each of B and D is equidistant from A and C, $\overline{BD}$ is contained in the perpendicular bisector of $\overline{AC}$. So, at least one diagonal of a kite is contained in the perpendicular bisector of the other diagonal.
5. Quadrilateral ABCD is a kite if and only if either AB = BC and and AD = DC or DA = AB and DC = CB. This is the case if and only if either both B and D are equidistant from A and C or both A and C are equidistant from D and B — that is, if and only if either $\overline{BD}$ is contained in the perpendicular bisector of $\overline{AC}$ or $\overline{AC}$ is contained in the perpendicular bisector of $\overline{BD}$.
6. (a) 4
   (b) They are congruent.
   (c) $\overline{AC}$ and $\overline{BD}$; This follows by Theorem 16-21.
   (d) ACBD, ACDB; In each case Theorem 16-21 applies.
7. (a) All are kites.
   (b) Since CA = AD and CB = BD, both A and B are equidistant from C and D. Hence, the perpendicular bisector of $\overline{CD}$ contains both A and B and, being a plane, it also contains $\overline{AB}$.



Trapezoid with one pair of legs (nonparallel opposite sides) and one pair of bases.

Parallelogram – trapezoid with two pairs of bases and no legs.

Fig. 16–6

least one pair of bases], a trapezoid has at most one pair of legs [and may have none]. Two angles of a trapezoid which contain one of its bases are called *a pair of base angles* of the trapezoid. [So, for example, any two consecutive angles of a parallelogram are a pair of base angles of the parallelogram.] Note that, given a pair of base angles of a trapezoid, the angles opposite these are also a pair of base angles.

In particular, a trapezoid which is not a parallelogram has just two pairs of base angles and if the angles of either pair are congruent, so are those of the other pair. [Explain.] A parallelogram, on the other hand, has four pairs of base angles and if the angles of one of these pairs are congruent then all four angles are congruent. [Explain. What kind of angles are the angles of a parallelogram which has congruent base angles?]

A trapezoid which has congruent legs is called *an isosceles trapezoid*. Draw some pictures of isosceles trapezoids and explain why no parallelogram is an isosceles trapezoid.

For the remainder of this section we shall investigate isodiagonal trapezoids – that is, trapezoids whose diagonals are congruent. We already know some examples of isodiagonal trapezoids. [What are these examples?] In general, we shall be looking for necessary [and sufficient] conditions that a trapezoid be isodiagonal.

In order to establish results concerning isodiagonal trapezoids, it is convenient to make use of some results from Chapter 8 which concern ratios. To begin with, we know that if $ABCD$ is a trapezoid then its diagonals $\overline{AC}$ and $\overline{BD}$ intersect at a point O such that

$$(O - B) : (D - B) = (O - A) : (C - A)$$

and

$$(D - O) : (D - B) = (C - O) : (C - A).$$

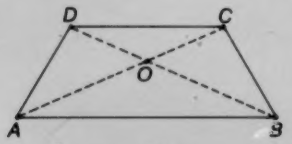
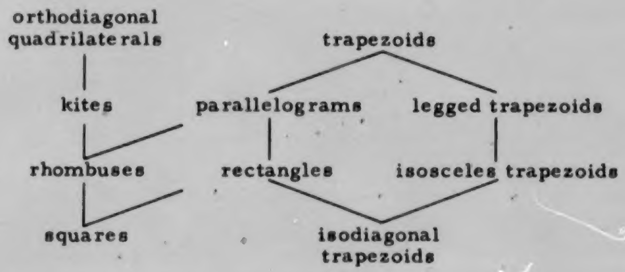


Fig. 16–7

Note that a trapezoid has legs if and only if it is not a parallelogram. So, the simple quadrilaterals with exactly two parallel sides are, precisely, the "legged" trapezoids. Such a trapezoid has just two bases and, consequently, two pairs of base angles. Assuming that the bases are $\overline{AB}$ and $\overline{CD}$ and that the trapezoid is ABCD, it follows by Theorem 15-13 that $\angle A$ and $\angle D$ are supplementary and that $\angle B$ and $\angle C$ are supplementary. [For more detail, see answer for Exercise 1 of Part D on page 292.] Hence, if the base angles $\angle A$ and $\angle B$ are congruent, so are the base angles $\angle C$ and $\angle D$.

Since base angles are consecutive and since consecutive angles of a parallelogram are supplementary it follows that a parallelogram with a pair of congruent base angles has two consecutive right angles. So, since the opposite angles are congruent, all its angles are right angles.

Since a parallelogram does not, by definition, have legs, no parallelogram is an isosceles trapezoid.



Rectangles are isodiagonal trapezoids. [We shall see very shortly that the only other isodiagonal trapezoids are the isosceles trapezoids.]

Suppose that trapezoid ABCD is isodiagonal — that is, suppose that AC = BD. It follows from (1) that AO = BO and OD = OC. Hence, since the vertical angles ∠AOD and ∠BOC are congruent, the matching AOD ⟷ BOC is a congruence by s.a.s. So, AD = BC. From this it follows by s.s.s. that the matchings DAB ⟷ CBA and BCD ⟷ ADC are congruences. So, ∠A ≅ ∠B and ∠C ≅ ∠D.

It follows that an isodiagonal trapezoid which has legs — that is, which is not a parallelogram — has congruent legs and, so, is an isosceles trapezoid. And, as we have proved, any isodiagonal parallelogram has congruent base angles. Since a parallelogram with two consecutive angles congruent is a rectangle it follows that an isodiagonal trapezoid which is a parallelogram is a rectangle.

Note that we have shown that an isodiagonal trapezoid is either an isosceles trapezoid [if it is legged] or a rectangle [if it is a parallelogram]. Since a rectangle is an isodiagonal trapezoid it is natural to ask whether an isosceles trapezoid is isodiagonal. We now turn our attention to this question. Its affirmative answer completes Theorem 16-23.

Suppose that, in trapezoid ABCD, $\overline{AD}$ and $\overline{BC}$ are congruent legs. It follows from (2) that AP = BP and PC = PD. Since ∠APC = ∠BPD it follows [s.a.s.] that APC ⟷ BPD is a congruence. In particular, AC = BD. So, an isosceles trapezoid is isodiagonal.

By Theorem 16-15, a rectangle is isodiagonal.

Suppose, in Figure 16-8, that ∠A and ∠B of the legged trapezoid ABCD are congruent. It follows that △APB is isosceles and, so, that AP = BP. From this and the first equation in (2) it follows that AD = BC. Hence, by s.a.s., ADB ⟷ BCA is a congruence and, in particular, BD = AC.

It has just been shown that a legged trapezoid with congruent base angles is isodiagonal. [We assumed that the base angles ∠A and ∠B were congruent, but we showed earlier that if either pair of base angles are congruent then both pairs are congruent.] On the other hand, if a pair of base angles of a parallelogram [that is, of a nonlegged trapezoid] are congruent then the parallelogram is a rectangle [since consecutive angles of a parallelogram are supplementary] and, so, is isodiagonal [by Theorem 16-15]. Hence, any trapezoid which has a pair of congruent base angles is isodiagonal. Since we have seen previously [in connection with Figure 16-7] that any pair of base angles of an isodiagonal trapezoid are congruent we now have Theorem 16-24.

Now that we have distances to talk about, we see that this implies that

$$(1) \qquad BO/BD = AO/AC \text{ and } OD/BD = OC/AC.$$

Using (1) it is easy to show that if *ABCD* is isodiagonal then $AD = BC$. [Explain.] From this it follows that if *ABCD* is isodiagonal then $\angle A \cong \angle B$ and $\angle C \cong \angle D$. [Explain.] So, an isodiagonal trapezoid which is not a parallelogram is isosceles, and any isodiagonal trapezoid has congruent base angles. [Explain.] In particular, an isodiagonal trapezoid which is a parallelogram is a rectangle. [Explain.]

In the previous paragraph, we established several necessary conditions that a trapezoid be isodiagonal. To obtain corresponding sufficient conditions that a trapezoid be isodiagonal we need additional results on ratios. If *ABCD* is a trapezoid which is not a parallelogram then $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect in a point *P*.

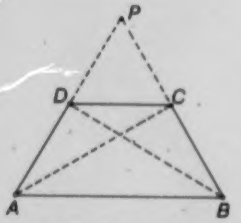


Fig. 16–8

Since *ABCD* is not a parallelogram $(C - D) : (B - A) \neq 1$ and we may as well assume that this ratio is between 0 and 1. In this case *D*, for example, divides the interval from *A* to *P* in a ratio between 0 and 1 and *ABCD* is as shown in Fig. 16–8. From our knowledge of ratios we also know that

$$(P - A) : (D - A) = (P - B) : (C - B)$$
$$\text{and}$$
$$(P - D) : (D - A) = (P - C) : (C - B);$$

in particular we see that

$$(2) \qquad AP/AD = BP/BC \text{ and } DP/AD = CP/BC.$$

Considering $\triangle PDB$ and $\triangle PCA$ in the light of (2), we see that an isosceles trapezoid is isodiagonal. [Explain.]

Since a rectangle is also isodiagonal [Why?] we have:

Theorem 16-23 A trapezoid is isodiagonal if and only if it is either isosceles or a rectangle.

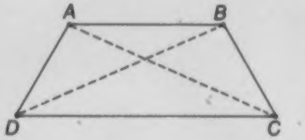
and:

Corollary A parallelogram is isodiagonal if and only if it is a rectangle.

We have seen earlier that any pair of base angles of an isodiagonal trapezoid are congruent. Using (2) we can prove that if any pair of base angles of a trapezoid which is not a parallelogram are congruent then the trapezoid is isodiagonal. And, if a pair of base angles of a parallelogram are congruent then the parallelogram is a rectangle. [Explain.] So, we have:

Theorem 16-24 A trapezoid is isodiagonal if and only if any pair of its base angles are congruent.

Part D



Suppose that *ABCD* is a trapezoid with bases $\overline{AB}$ and $\overline{CD}$, as pictured at the right.

1. Given that the pair of base angles ∠*C* and ∠*D* are congruent, show that ∠*A* and ∠*B* are congruent.
2. Suppose that *ABCD* is isodiagonal. Give arguments to support the following:
   (a) $AD = BC$ (b) $\angle A \cong \angle B$ (c) $\angle C \cong \angle D$
3. Assume that *ABCD* is isosceles—that is, that $AD = BC$. Prove that $AC = BD$.
4. Prove the following:
   (a) If a pair of consecutive angles of a parallelogram are congruent then the parallelogram is a rectangle.
   (b) An isodiagonal trapezoid which is not a parallelogram is isosceles.
   (c) An isodiagonal parallelogram is a rectangle.
   (d) A trapezoid which is not a parallelogram but which has a pair of congruent base angles is isodiagonal.
5. Using the results in Exercises 1-4, prove:
   (a) Theorem 16-23 (b) Theorem 16-24

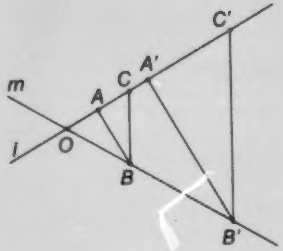
Answers for Part D

[These exercises merely review the steps taken in the text to prove Theorems 16-23 and 16-24.]

1. Since trapezoid ABCD, with bases $\overline{AB}$ and $\overline{CD}$, is simple it follows from Exercise 3 of Part A on page 271 that C - D and A - B do not have the same sense. [Use the instance of Exercise 3 obtained by interchanging 'B' and 'D'.] Since these translations are non-$\vec{0}$ and have the same direction it follows that they have opposite senses. So, C - D and B - A have the same sense and D - C and A - B have the same sense. It follows from Theorem 15-13 [consecutive angles] that ∠D and ∠A are supplementary and that ∠C and ∠B are supplementary. Since supplements of congruent angles are congruent it follows that if ∠C ≅ ∠D then ∠B ≅ ∠A.
2. [This is answered in the text referring to Figure 16-7 and the related commentary.]
3. [This is answered in the text referring to Figure 16-8 and the related commentary.]
4. (a) Since consecutive angles of a parallelogram are supplementary, if a pair of consecutive angles are congruent then a pair of consecutive angles are right angles. Since opposite angles of a parallelogram are congruent it follows that all angles are right angles.
   (b) [Answered in the text and commentary concerning Figure 16-7.]
   (c) [Ditto.]
   (d) [See text and commentary relating to Theorem 16-24.]
5. [See text and commentary.]

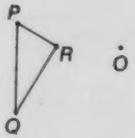
Exploration Exercises

1. Suppose that lines l and m intersect at O, that A and C are on l and B is on m, and that A', C', and B' are such that
   $A' - O = (A - O)3$,
   $C' - O = (C - O)3$,
   and $B' - O = (B - O)3$
   as shown in the picture at the right.



   (a) Show that the matching $ABC \leftrightarrow A'B'C'$ of the vertices of $\triangle ABC$ with those of $\triangle A'B'C'$ is such that corresponding angles are congruent.
   (b) Is the matching $ABC \leftrightarrow A'B'C'$ a congruence? Explain your answer.
   (c) Given the matching discussed in (a) and (b), what relation exists between corresponding sides of the triangles $\triangle ABC$ and $\triangle A'B'C'$?

2. Copy the picture of $\triangle PQR$ and point O, as shown in the picture at the right.



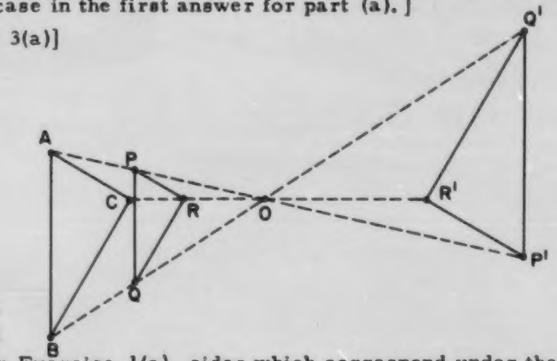
   (a) On your picture, locate P', Q', and R' so that O is on each of the segments $\overleftrightarrow{PP'}$, $\overleftrightarrow{QQ'}$, and $\overleftrightarrow{RR'}$, and $PO/OP' = QO/OQ' = RO/OR' = \frac{1}{2}$.
   (b) Given the matching $PQR \leftrightarrow P'Q'R'$, show that the corresponding angles of $\triangle PQR$ and $\triangle P'Q'R'$ are congruent.
   (c) What is the ratio of corresponding sides?
   (d) Is the matching $PQR \leftrightarrow P'Q'R'$ a congruence? Explain.

3. In your picture for Exercise 2, locate points A, B, and C so that $P \in \overleftrightarrow{OA}$, $Q \in \overleftrightarrow{OB}$, and $R \in \overleftrightarrow{OC}$, and $OP/PA = OQ/QB = OR/RC = \frac{3}{2}$.
   (a) Given the matching $PQR \leftrightarrow ABC$, show that corresponding angles are congruent and that corresponding sides are not congruent.
   (b) Consider the matching $P'Q'R' \leftrightarrow ABC$. Are corresponding angles congruent? Are corresponding sides congruent? Explain your answers.

Answers for Exploration Exercises

1. (a) Since $(A' - O):(A - O) = (B' - O):(B - O)$ and $(B' - O):(B - O) = (C' - O):(C - O)$ it follows that $\overleftrightarrow{A'B'} \parallel \overleftrightarrow{AB}$ and $\overleftrightarrow{B'C'} \parallel \overleftrightarrow{BC}$ [see Theorem 8-3(b)]. Since, also, $\overleftrightarrow{A'C'} = \overleftrightarrow{AC}$ it follows that $A'B'/AB = B'C'/BC = C'A'/CA = 3$ [see Theorem 8-11(a)]. It follows by the cosine law [and Theorem 15-8] that, under the matching $ABC \longleftrightarrow A'B'C'$, corresponding angles of $\triangle ABC$ and $\triangle A'B'C'$ are congruent. [Alternatively, having proved that $\overleftrightarrow{A'B'} \parallel \overleftrightarrow{AB}$ and $\overleftrightarrow{B'C'} \parallel \overleftrightarrow{BC}$, one may note that, since B and B' are on the same side of ℓ, it follows by Theorem 15-13 [corresponding angles] that $\angle A$ of $\triangle ABC$ and $\angle A'$ of $\triangle A'B'C'$ are congruent and that $\angle C$ of $\triangle ABC$ and $\angle C'$ of $\triangle A'B'C'$ are congruent. The congruence of $\angle C$ and $\angle C'$ then follows by Theorems 16-6 and 15-8.]
   (b) No. Corresponding sides are not congruent.
   (c) Corresponding sides are proportional. [This is shown to be the case in the first answer for part (a).]

2. (a) [and 3(a)]



   (b) As in Exercise 1(a), sides which correspond under the matching $PQR \longleftrightarrow P'Q'R'$ are parallel and, so, are proportional. Hence, by the cosine law [and Theorem 15-8], corresponding angles are congruent.
   (c) 1/2
   (d) No. Corresponding sides are not congruent.

3. [See figure for Exercise 2(a).]
   (a) [As in Exercises 1 and 2.]
   (b) Yes [congruence of angles is transitive].; No [the ratio of corresponding sides is 6/5].

16.08 Similar Triangles

Intuitively, geometric figures are *similar* if they have the same shape, whether or not they have the same size. More precisely, a first figure is similar to a second if and only if it can be mapped onto the second by a mapping f such that, for some positive real number m,

$$d(f(X), f(Y)) = d(X, Y)m, \tag{1}$$

for all points X and Y. Such mappings are called *similitudes*. Notice that each isometry is a similitude. The simplest examples of similitudes other than isometries are the *uniform stretchings*—or *shrinkings*—about a point O:

$$g(X) = O + (X - O)m, \quad m > 0 \tag{2}$$

We shall show that each similitude (1) is the resultant of a uniform stretching (2) followed by an isometry.

To do so, suppose that f is as in (1), so that, for each X and Y,

$$\|f(X) - f(Y)\| = \|X - Y\|m,$$

where $m > 0$. It is easy to see that the mapping g described by (2) has an inverse and that, for each X,

$$g^{-1}(X) = O + (X - O)/m.$$

It follows that the resultant $f \circ g^{-1}$ is such that, for each X,

$$[f \circ g^{-1}](X) = f(O + (X - O)/m).$$

Hence,

$$\begin{aligned}
&\|[f \circ g^{-1}](X) - [f \circ g^{-1}](Y)\| \\
&= \|f(O + (X - O)/m) - f(O + (Y - O)/m)\| \\
&= \|(O + (X - O)/m) - (O + (Y - O)/m)\|m \\
&= \|(X - O)/m - (Y - O)/m\|\,m \\
&= \|((X - O) - (Y - O))/m\|\,m \\
&= \|(X - Y)/m\|\,m \\
&= \|X - Y\|.
\end{aligned}$$

In short, the mapping $f \circ g^{-1}$ is an isometry—say, h—and, so, $f = h \circ g$. [Explain.]

To show that g^{-1} as described in the text is actually the inverse of g [and, incidentally, to show that g has an inverse] we compute '$g \circ g^{-1}$' and '$g^{-1} \circ g$':

$$[g \circ g^{-1}](X) = g(O + (X - O)/m) = O + [(X - O)/m]m = O + (X - O) = X$$

$$[g^{-1} \circ g](X) = g^{-1}(O + (X - O)m) = O + [(X - O)m]/m = O + (X - O) = X$$

Having shown as in the text that $f \circ g^{-1} = h$, where h is an isometry it follows that $f = f \circ [g^{-1} \circ g] = [f \circ g^{-1}] \circ g = h \circ g$.

* * *

Suggestions for use of the exercises of section 16.08:

(i) Parts A and B, and the associated text discussions should be developed by the teacher. Be sure to have the students graph several figures and their images under uniform stretchings or shrinkings.

(ii) Parts C and D may be assigned for homework.

(iii) Part E may be completed in class.

Summarizing the preceding we have:

Definition 16-4 f is a similitude of $\mathscr{E}$ if and only if f is a mapping of $\mathscr{E}$ onto itself such that, for some $m > 0$, $\forall_X \forall_Y\, d(f(X), f(Y)) = d(X, Y)m$.

Definition 16-5 A first figure is similar to a second if and only if there is a similitude of $\mathscr{E}$ which maps the first figure onto the second.

Theorem 16-25 Each similitude of $\mathscr{E}$ is the resultant of a uniform stretching about any given point O followed by an isometry.

[A uniform stretching is a mapping g as described in (2).]

Theorems concerning similitudes which are like Theorems 14-26 and 14-27 for isometries are easily proved. It is sufficient to prove corresponding theorems for uniform stretchings and then make use of Theorem 16-25. [Explain.] In particular, any similitude maps any angle onto an angle, and any triangle onto a triangle, mapping vertices on vertices. We shall not trouble to prove these theorems but you may make use of them in the following exercises.

Exercises

Part A

1. Draw a triangle, mark a point O, and draw the image of the triangle under the uniform stretching about O which doubles distances.
2. Show that a uniform stretching maps any angle onto a congruent angle.
3. Explain how you know that similar angles are congruent.
4. Show that if there is a similitude f which maps $\triangle ABC$ onto $\triangle A'B'C'$ in such a way that $f(A) = A'$, $f(B) = B'$, and $f(C) = C'$ then

$$AB/A'B' = BC/B'C' = CA/C'A'.$$

[In short, the side measures AB, BC, and CA are proportional to $A'B'$, $B'C'$, and $C'A'$.]

*

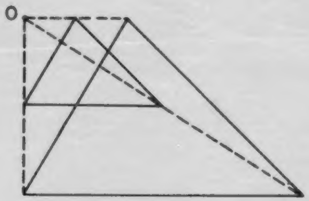
By Theorem 16-25, anything — say, coplanarity — which is preserved by uniform stretchings and, also, by isometries, must be preserved by similitudes. For example, if uniform stretchings map planes onto planes then, since isometries also map planes onto planes, the same is true of similitudes. For another example, if uniform stretchings map intervals onto intervals, mapping end points on end points then, since isometries do the same, so do similitudes.

It is also important to note that similitudes form a group under function composition. To see this note, first, that the set of similitudes is closed under function composition. This follows at once from Definition 16-4 and the fact that the product of two positive numbers is positive. It also follows from the definition [and the fact that the reciprocal of a positive number is positive] that the inverse of a similitude is a similitude. Finally, the identity mapping $\mathcal{I}$ is obviously a similitude.

Having shown that similitudes constitute a group it follows — just as for isometries [see the beginning of section 14.05] — that similarity of geometric figures is a reflexive, symmetric, and transitive relation. [For an application of this, see Part D on page 298.]

Answers for Part A

1.



2. This follows at once from the cosine law and Theorem 15-8.
3. Angles are similar if and only if there is a similitude which maps one onto the other. Since a similitude is the resultant of a uniform stretching followed by an isometry, and since mappings of both these kinds map angles onto congruent angles, it follows that a similitude maps each angle onto a congruent angle. Hence, similar angles are congruent.
4. By Definition 16-4 if f is a similitude which maps A on A', B on B', and C on C' then there is a number — say, m — which is positive and is such that $A'B'/AB = B'C'/BC = C'A'/CA = m$. Hence, if there is such a similitude then $AB/A'B' = BC/B'C' = CA/C'A'$.

We have said earlier that a matching

$$ABC \leftrightarrow A'B'C'$$

of the vertices of two triangles is a congruence if there is an isometry which maps A on A', B on B', and C on C'. We shall now say that such a matching is a *similarity* if there is a similitude which maps A on A', B on B', and C on C'.

In Exercise 4 of Part A we have seen that if a matching of the vertices of one triangle with those of a second is a similarity then corresponding sides are proportional. The converse of this is the s.s.s. similarity theorem:

> **Theorem 16–26** A matching of the vertices of one triangle with those of a second is a similarity if corresponding sides are proportional.

The proof is easy. Suppose that $ABC \leftrightarrow A'B'C'$ is a matching such that

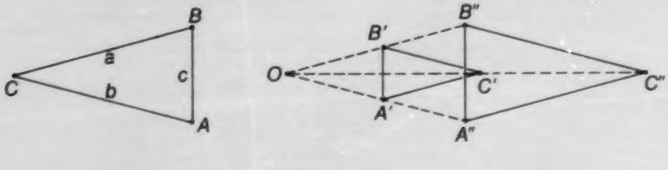


Fig. 16–9

$a/a' = b/b' = c/c' = m$, say. Choose a point O and let A'', B'', and C'' be the images of A', B', and C', respectively, under the uniform stretching about O which multiplies all distances by m. Since $a'' = a'm = a$, $b'' = b'm = b$, and $c'' = c'm = c$ it follows that the matching $ABC \leftrightarrow A''B''C''$ is a congruence. So, the matching $ABC \leftrightarrow A'B'C'$ is a similarity.

Some of the importance of Theorem 16–26 is due to a consequence of Exercise 3 of Part A:

> **Theorem 16–27** If a matching of the vertices of one triangle with those of a second is a similarity then corresponding angles are congruent and corresponding sides are proportional.

The same sort of proof which gave us the s.s.s. similarity theorem also gives us the s.a.s. similarity theorem:

> **Theorem 16–28** A matching of the vertices of one triangle with those of a second is a similarity if two sides of the first triangle are proportional to the corresponding sides of the second and the included angles are congruent.

We could also prove an a.s.a. similarity theorem but we can do better:

> **Theorem 16–29** A matching of the vertices of one triangle with those of a second is a similarity if two angles of the first triangle are congruent to the corresponding angles of the second.

This is the a.a. similarity theorem.

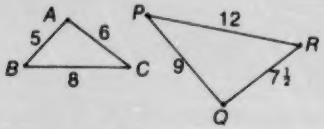
Part B

1. Prove the s.a.s. similarity theorem.
2. Prove the a.a. similarity theorem. [*Hint:* Note that, by an earlier theorem, if two angles of one triangle are congruent to two angles of the other then the remaining angles are also congruent. Then, use the Sine Law and the s.s.s. similarity theorem.]

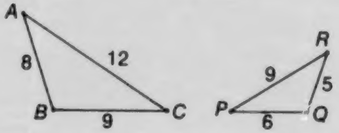
Part C

In each of the following, you are given pictures of two triangles and some information about them. You are to decide whether the triangles are similar. Be prepared to justify your answer.

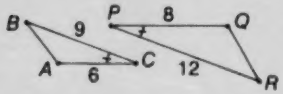
1.



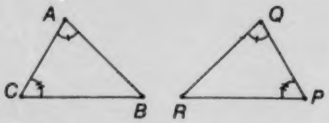
2.



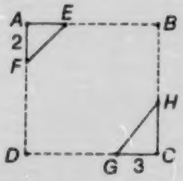
3. $\angle C \cong \angle P$



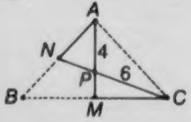
4. $\angle A \cong \angle Q$ and $\angle C \cong \angle P$



5. $ABCD$ is a square; $\overline{EF} \parallel \overline{GH}$



6. $\overline{AM}$ and $\overline{CN}$ are medians of $\triangle ABC$



Answers for Part B

1. Suppose that, in $\triangle ABC$ and $\triangle A'B'C'$, $A'B'/AB = B'C'/BC = m$ and $\angle B' \cong \angle B$. By the cosine law [and Theorem 15-8]
$$(C'A')^2 = (A'B')^2 + (B'C')^2 - 2(A'B')(B'C')\cos\angle B'$$
$$= [(AB)^2 + (BC)^2 - 2(AB)(BC)\cos\angle B]m^2 = (CA)^2m^2$$
and, so, $C'A'/CA = m$. It follows, now, by the s.s.s. similarity theorem that $ABC \longleftrightarrow A'B'C'$ is a similarity.

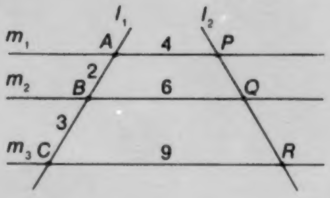
2. Suppose that, in $\triangle ABC$ and $\triangle A'B'C'$, $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$. It follows by Theorem 16-6 that $\angle C \cong \angle C'$. It follows that $\sin\angle A' = \sin\angle A$, $\sin\angle B' = \sin\angle B$, and $\sin\angle C' = \sin\angle C$. Also, by the sine law,
$$\frac{B'C'}{\sin\angle A'} = \frac{C'A'}{\sin\angle B'} = \frac{A'B'}{\sin\angle C'}, \text{ and}$$
$$\frac{\sin\angle A}{BC} = \frac{\sin\angle B}{CA} = \frac{\sin\angle C}{AB}.$$
From these results it follows at once that $B'C'/BC = C'A'/CA = A'B'/AB$. Hence, by the s.s.s. similarity theorem, $ABC \longleftrightarrow A'B'C'$ is a similarity.

Answers for Part C

1. The triangles are similar. [$ABC \longleftrightarrow QRP$ is a similarity by s.s.s.]
2. The triangles are not similar. [Taking account of the order of the lengths of the sides, the only possible similarity is $ABC \longleftrightarrow RQP$. But, since $5/8 \neq 6/9$, sides which correspond under this matching do not have proportional measures.]
3. The triangles are similar. [$ACB \longleftrightarrow QPR$ is a similarity by s.a.s.]
4. The triangles are similar. [$ACB \longleftrightarrow QPR$ is a similarity by a.a.]
5. The triangles are similar. [Let P be the point of intersection of $\overleftrightarrow{EF}$ and $\overleftrightarrow{CD}$. Then $\angle EPD \cong \angle AEF$ and $\angle EPD \cong \angle CGH$. Hence, by a.a., $AEF \longleftrightarrow CGH$ is a similarity.]
6. The triangles are not similar. [Very difficult to justify. Clearly there is no similarity in which P is matched with itself. To show that there is no similarity in which either A or N is matched with P, proceed as follows: Let $k = \cos\angle APN = \cos\angle CPM$. By the cosine law $(AN)^2 = 25 - 24k$ and $(MC)^2 = 40 - 24k$. Using the cosine law, $\cos\angle A = (4 - 3k)/\sqrt{25 - 24k}$, $\cos\angle N = (3 - 4k)/\sqrt{25 - 24k}$, $\cos\angle C = (3 - k)/\sqrt{10 - 6k}$, and $\cos\angle M = (1 - 3k)/\sqrt{10 - 6k}$. Suppose that there is a similarity in which A is matched with P. It follows that $\cos\angle A = k$ and, so, that $k = 2/3$. In this case $\cos\angle N = 1/9$, and $\cos\angle P = 2/3 = \cos\angle A$. Since $\cos\angle C = 7/(3\sqrt{6})$ the triangles are not similar in this case. Finally, suppose that there is a similarity in which N is matched with P. It follows that $\cos\angle N = k$ and, so, that $k = 3/8$. In this case $\cos\angle A = 23/32$ and $\cos\angle P = 3/8 = \cos\angle N$. Since $\cos\angle C = 21/(4\sqrt{31})$ the triangles are not similar in this case. Consequently, no matching of the vertices of $\triangle APN$ and $\triangle CPM$ is a similarity.]

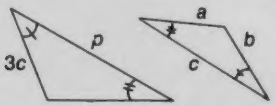
Part D

1. Draw a rectangle $ABCD$ such that $AB = 3$ and $BC = 4$.
   (a) Draw a quadrilateral $PQRS$ such that $ABCD \leftrightarrow PQRS$ is a similarity and $PQ = 6$. How long are $\overleftrightarrow{QR}$ and $\overleftrightarrow{QS}$?
   (b) Draw a quadrilateral $KLMN$ such that $ABCD \leftrightarrow KLMN$ is a similarity and $LM = 6$. How long are $\overleftrightarrow{KL}$ and $\overleftrightarrow{KM}$?
   (c) Show that $PQRS$ and $KLMN$ are similar and give the ratio of corresponding sides.
2. Suppose that $m_1 \parallel m_2 \parallel m_3$, that l_1 intersects these lines in A, B, and C, and that l_2 intersects these lines in P, Q, and R, as shown in the picture at the right. Determine which of the following are similar trapezoids.

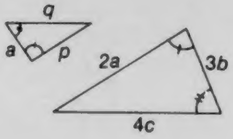


   (a) $ABQP$ and $BCRQ$
   (b) $ABQP$ and $ACRP$
   (c) $BCRQ$ and $ACRP$
3. Suppose that, in $\triangle ABC$ and $\triangle PQR$, $ABC \leftrightarrow PQR$ is a similarity.
   (a) Given that $AB = 5$, $AC = 4$, $\cos \angle A = \frac{4}{5}$, and $QR = 10$, compute BC, $\sin \angle R$, PQ, and QR.
   (b) Given that $\cos \angle A = -\frac{3}{5}$, $\sin \angle B = \frac{5}{13}$, $BC = 26$, and $PQ = 10$, compute PR and QR, and tell which is the largest angle in $\triangle PQR$.
   (c) Given that $\cos \angle C = \frac{1}{2}$ and $AC = BC = PQ = 5$, compute AB, PR, and QR.
4. In each of the following parts, you are given two similar figures and some information about them. You are to express 'p' and 'q' in terms of 'a', 'b', and 'c'.

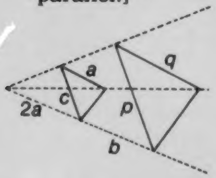
(a)



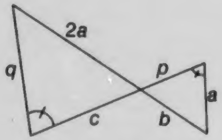
(b)



(c) [Corresponding sides are parallel.]



(d)



Answers for Part D

1. (a), (b) [In part (a) PQRS is, of course, a 6 by 8 rectangle and, in part (b), KLMN is a 9/2 by 6 rectangle. Actually, any such rectangles will do. If, however, you wish to make sure that ABCD ⟷ PQRS and ABCD ⟷ KLMN are similarities it is easiest to choose a point O in the plane of ABCD — the point A is a good choice — and obtain PQRS and KLMN from ABCD by the uniform stretchings with "stretching factors" 2 and 3/2, respectively. For more on this point see below.]

   (c) Suppose that f and g are similitudes which map ABCD onto PQRS and KLMN, respectively. Then, since the inverse of a similitude is a similitude and the resultant of two similitudes is a similitude, $g \circ f^{-1}$ is a similitude. Since this similitude maps PQRS onto KLMN it follows that PQRS and KLMN are similar. Since the stretching factor for f^{-1} is 1/2 and that for g is 3/2, the stretching factor for $g \circ f^{-1}$ is 3/4. So, the ratio of the measure of a side of PQRS to the corresponding side of KLMN is 4/3.

*

The procedure suggested for part (a) of Exercise 1 gives one a 6 by 8 rectangle similar to PQRS. To show that any 6 by 8 rectangle is similar to PQRS it is sufficient, in view of the transitivity of similarity, to show that any two 6 by 8 rectangles are similar. It is not difficult to go further and prove that any two rectangles of the same dimensions are congruent. To do so, suppose that EFGH and TUVW are rectangles such that EF = TU and FG = UV. Since, by the Pythagorean theorem, GE = VT it follows that there is an isometry which maps E on T, F on U, and G on V. This isometry maps $\overleftrightarrow{EFG}$ onto $\overleftrightarrow{TUV}$ and maps the line $\overleftrightarrow{EH}$ on $\overleftrightarrow{FEG}$ through E perpendicular to $\overleftrightarrow{EF}$ onto the line $\overleftrightarrow{TW}$ of $\overleftrightarrow{TUV}$ through T perpendicular to $\overleftrightarrow{TU}$. Similarly, it maps $\overleftrightarrow{GH}$ onto $\overleftrightarrow{VW}$. Hence it maps the point H of intersection of $\overleftrightarrow{EH}$ and $\overleftrightarrow{GH}$ on the point W of intersection of $\overleftrightarrow{TW}$ and $\overleftrightarrow{VW}$. Since the isometry maps E, F, G, and H on T, U, V, and W it follows by Theorem 14-27 that it maps EFGH onto TUVW. [A similar argument is needed for Exercise 4(a) of Part D on page 306. So, it may be well to go through this argument in class. With an occasional leading question, students should be able to do most of the work.]

*

Answers for Part D [cont.]

2. (a) Similar. (b) Not similar. (c) Not similar.
3. (a) BC = 3, $\sin\angle R = 1$, PR = 40/3, PQ = 50/3

   (b) From the given information AB = 33/2 and AC = 25/2. So, the ratio of similitude is 33/20. Thus, PR = $\frac{20}{33}\cdot\frac{25}{2}$ = 250/33 and RQ = 520/33. Also, ∠P is the largest angle of △PQR, since ∠P is obtuse.

   (c) AB = 5, PR = 5, QR = 5

Answers for Part D [cont.]

4. (a) $p = 3c^2/b$, $q = 3ac/b$ $[p/c = 3c/b = q/a]$

(b) $p = 2a^2/(3b)$, $q = 4ac/(3b)$

(c) $p = (2a + b)c/(2a)$, $q = (2a + b)/2$

(d) $p = bc/(2a)$, $q = 2a^2/b$

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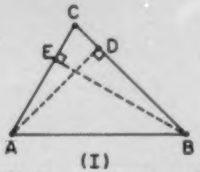
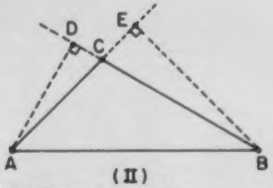
5. (a), (b), (c)



(I)



(II)

In case (I) the right triangles, △ADC and △BEC, share ∠C and are similar by a.a.

In case (II) the right triangles, △ADC and △BEC, have congruent vertical angles at C and, so, are similar by a.a. In both cases the similarity is ADC ⟷ BEC. In particular, AC/BC = AD/BE and, so, AC·BE = BC·AD.

(d) The statement means that, in any triangle, the product of any side by the altitude to that side is the same as the product of any other side by the altitude to that side. [To complete the proof of the statement it is necessary to consider a third case, in addition to (I) and (II) above, in which ∠C is a right angle. This case is easy to settle since AD = AC and BE = BC. It should be recalled at this point that the theorem proved here has been proved previously by vector methods and listed as Theorem 14-13.]

Answers for Part E

1. (a) ABD ⟷ CBA (b) ACD ⟷ BCA (c) ABD ⟷ CAD

2. (a) $BD/BA = AB/CB$. So $BD = c^2/a$.

(b) $DC/AC = AC/BC$. So $DC = b^2/a$.

(c) $AD/CA = AB/CB$. So $AD = bc/a$.

[Note that these results have been obtained, by vector methods, in Exercise 2 of Part G on page 162.]

3\.

	AB	AC	BC	BD	DC	AD	sin∠B	sin∠C
(a)	8	10	$2\sqrt{41}$	$32/\sqrt{41}$	$50/\sqrt{41}$	$40/\sqrt{41}$	$5/\sqrt{41}$	$4/\sqrt{51}$
(b)	9	12	15	27/5	48/5	36/5	4/5	3/5
(c)	$2\sqrt{5}$	$4\sqrt{5}$	10	2	8	4	$2/\sqrt{5}$	$1/\sqrt{5}$
(d)	$4\sqrt{13}$	$6\sqrt{13}$	26	8	18	12	$3/\sqrt{13}$	$2/\sqrt{13}$
(e)	20	15	25	16	9	12	3/5	4/5
(f)	12	$6\sqrt{5}$	18	8	10	$4\sqrt{5}$	$\sqrt{5}/3$	2/3
(g)	$4\sqrt{3}$	4	8	6	2	$2\sqrt{3}$	1/2	$\sqrt{3}/2$
(h)	21	$9\sqrt{23}$	48	147/16	621/16	$63\sqrt{23}/16$	$3\sqrt{23}/16$	7/16

5. Draw a triangle, △*ABC*, such that ∠*C* is not a right angle, and draw its altitudes $\overline{AD}$ and $\overline{BE}$, from *A* and *B*, respectively.

(a) Show that △*ADC* and △*BEC* are similar.

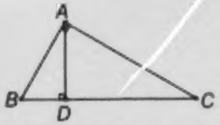
(b) Show that $AC \cdot BE = BC \cdot AD$.

(c) There are two cases to consider—that in which ∠*C* is an acute angle and that in which ∠*C* is an obtuse angle. Repeat (a) and (b) for the other case.

(d) Explain the statement:

> In any triangle, the product of a side by the altitude to that side is constant.

Part E



Consider right triangle, △*ABC*, with hypotenuse $\overline{BC}$ and altitude $\overline{AD}$, as shown in the picture at the right.

1. Give matchings of vertices which are similarities for each of the following pairs of triangles.

(a) △*ABD*, △*ABC* **(b)** △*ACD*, △*ABC* **(c)** △*ABD*, △*ADC*

2. Let $AB = c$, $BC = a$, and $CA = b$. Make use of your results in Exercise 1 to show each of the following:

(a) $BD = c^2/a$ **(b)** $DC = b^2/a$ **(c)** $AD = bc/a$

3. Complete the table.

	AB	*AC*	*BC*	*BD*	*DC*	*AD*	sin ∠*B*	sin ∠*C*
(a)	8	10						
(b)		12	15					
(c)				[illegible]	8			
(d)				8		12		
(e)		15					$\frac{3}{5}$	
(f)			18					$\frac{2}{3}$
(g)				6			$\frac{1}{2}$	
(h)	21							$\frac{7}{16}$

16.09 Areas of Triangular and Quadrangular Regions

It can be proved that any triangle, and any *simple plane* quadrilateral, is the *boundary* of a region of its plane called the *interior* of the triangle or quadrilateral.

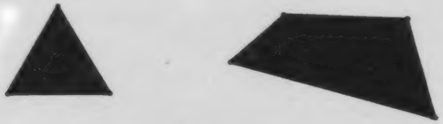


Fig. 16–10

We shall call the union of a triangle [or simple plane quadrilateral] and its interior a *triangular* [or *quadrangular*] *region.*

It can also be proved that it is possible to assign numerical area-measures to such regions. This can be done by, first, assigning such measures to triangular regions and then assigning as area-measure to a quadrangular region the sum of the area-measures of triangular regions into which it can be "cut up". What needs to be proved [and what can be proved] is that, for example, with an appropriate definition of area-measure for triangular regions, the sum of the area-measures of the triangular regions T_1 and T_2 is the same as the sum of the area-measures of T_3 and T_4.

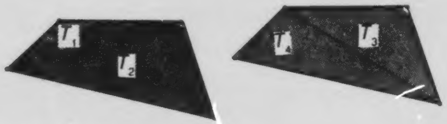


Fig. 16–11

The two intuitively obvious theorems mentioned in the preceding two paragraphs are quite difficult to prove and, although we shall make use of them in this section, we shall not attempt to prove them in this course.

As you probably know, an appropriate area-measure for a triangular region is obtained by taking one-half the product of the measure of any side of its triangular boundary by the measure of the altitude to that side. That this is an appropriate area-measure for a tri-

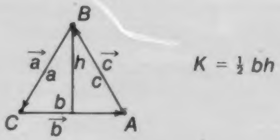


Fig. 16–12

As indicated by the 'It can be proved's we shall not attempt a rigorous development of the theory of area-measure for polygonal regions. Vector methods do make possible a rather simple treatment, but we do not have the space [nor you the time] to carry it out. In treating area-measure somewhat intuitively we all but follow current practice at this level.

* * *

Suggestions for the exercises of section 16.09:

(i) Use Part A for class illustrations.
(ii) Part B may be used for homework.
(iii) Part C and the discussion following may be used for supervised class exercises.
(iv) Part D may be assigned as homework.

*

Sample Quiz

Here is a list of descriptive terms which apply to geometric figures:

(a) equilateral
(b) quadrilateral
(c) equiangular
(d) trapezoidal
(e) rectangular
(f) rhombic
(g) perpendicular diagonals
(h) congruent diagonals
(i) bisecting diagonals
(j) a pair of parallel sides

For each of the geometric figures given below, tell which of the descriptive terms applies to it.

1. Square 2. Rhombus 3. Rectangle 4. Trapezoid

Key to Sample Quiz

1. [All apply]
2. (a), (b), (d), (f), (g), (i), (j)
3. (b), (c), (d), (e), (h), (i), (j)
4. (b), (d), (j)

angular region stems from the fact that, by Theorem 14–13, the number obtained in this way is the same no matter which side and corresponding altitude one chooses. This motivates us to adopt the following:

> Definition 16–6 The area-measure of a triangular region is one-half the product of any side by the altitude to that side.

From exercises in Parts F and G on page 161 and in Part B on page 256, there are several other ways to express the area-measure of the triangular region bounded by $\triangle ABC$. For example:

$$\begin{aligned} K &= \tfrac{1}{2}\sqrt{a^2b^2 - (\vec{a}\cdot\vec{b})^2} \\ &= \tfrac{1}{2}ab\sin\angle C \\ &= \sqrt{s(s-a)(s-b)(s-c)},\text{ where } s = (a+b+c)/2. \end{aligned}$$

[Note, in the second of these formulas that $a \sin \angle C$ is the measure of the altitude to the side $\overline{CA}$.]

We can now assign area-measures to trapezoidal regions. Given such a region with bases whose measures are b_1 and b_2 and with altitude measure h, it is clear from Fig. 16–13 that its area-measure, K, is one-half the product of the sum of the measures of its bases by its altitude. And it is easy to obtain from this, as special cases, formulas for the area-measures of regions bounded by parallelograms and rectangles. [Do so.] Another way of computing the area-measures of regions bounded by a parallelogram is suggested by this figure:

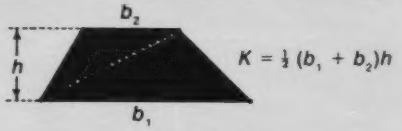


Fig. 16–13

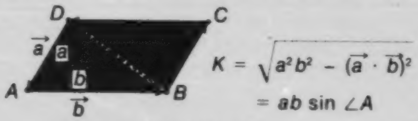


Fig. 16–14

In this connection, it is worth recalling that, by one of our most useful theorems, $a^2b^2 - (\vec{a}\cdot\vec{b})^2 = 0$ if and only if $(\vec{a}, \vec{b})$ is linearly dependent.

Note that, in Fig. 16–14, $(\vec{a}, \vec{b})$ is linearly dependent if and only if the parallelogram "collapses" into a segment. Notice, also, that the formula given in Fig. 16–14 can be used to obtain special formulas for the area-measure of a rectangular region and of a square region. [How?]

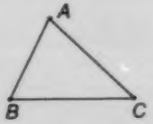
Finally, the area-measure of a region bounded by a rhombus can be computed from the measures of its diagonals. [Do so. To what more inclusive class of plane regions does this formula apply?] And, this may be used to obtain, as a special case, a formula for the area-measure of a square region in terms of the measure of a diagonal of the square. [Give the formula.]

Exercises

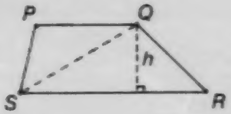
Part A

In each of the following, you are given a figure and some information about it. Do the indicated computations. $K(ABC)$ is the area-measure of the triangular region bounded by $\triangle ABC$, $K(PQRS)$ is the area-measure of the quadrangular region bounded by $PQRS$, etc.

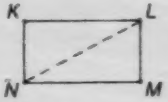
1. In $\triangle ABC$, $AB = 26$, $BC = 28$, and $\cos\angle B = \frac{5}{13}$. Compute AC and $K(ABC)$.
2. In $\triangle ABC$. $\sin\angle A = \frac{4}{5}$, $AC = 12$, and $AB = 15$. Compute $K(ABC)$.
3. In $\triangle ABC$, $AB = CB = 15$ and $AC = 12$. Compute $K(ABC)$ and $\sin \angle A$.



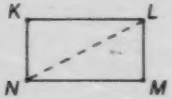
4. In trapezoid $PQRS$, with bases $\overline{PQ}$ and $\overline{RS}$, $PQ = 12$, $SR = 18$, and $h = 7$. Compute $K(PQRS)$.
5. In trapezoid $PQRS$, $PQ = 15$, $h = 9$, and $K(PQRS) = 72$. Compute SR.



6. Given that trapezoid $PQRS$ is isosceles and that $PS = QR = 5$, $PQ = 3$, and $RS = 11$, compute $K(PQRS)$ and QS.
7. In rectangle $KLMN$, $KL = 12$ and $\cos \angle KLN = \frac{3}{5}$. Compute $K(KLMN)$.



8. In rectangle $KLMN$, $KN = 20$ and $LN = 30$. Compute $K(KLMN)$.
9. Given that $KLMN$ is a square and $LN = 10$, compute KN and $K(KLMN)$.



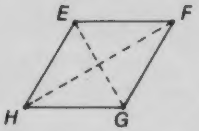
For a rectangular [and, so, in particular, for a square region] $\angle A$ is a right angle and, so, $\sin \angle A = 1$. Hence, for a rectangular region $K = ab$ [and, for a square region, $K = a^2$].

For a rhombus and, more generally, for any region bounded by any simple orthodiagonal quadrilateral, the area-measure is one-half the product of the measures of its diagonals. In particular, for a square with diagonals of measure d, $K = d^2/2$.

Answers for Part A

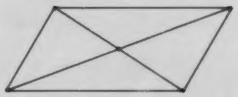
1. $AC = 30$, $K(ABC) = 336$
2. $K(ABC) = 60$
3. $K(ABC) = 18\sqrt{21}$, $\sin \angle A = \sqrt{21}/5$
4. $K(PQRS) = 105$
5. $SR = 1$
6. $K(PQRS) = 21$, $QS = \sqrt{58}$
7. $K(KLMN) = 192$
8. $K(KLMN) = 200\sqrt{5}$
9. $KN = 5\sqrt{2}$, $K(KLMN) = 50$

10. In rhombus *EFGH*, $EG = 10$ and $FH = 15$. Compute $K(EFGH)$.
11. In rhombus *EFGH*, $EG = 12$ and $\triangle EHG$ is equilateral. Compute $K(EFGH)$ and *FH*.
12. In rhombus *EFGH*, $FH = 15$ and $\cos \angle HGF = -\frac{2}{3}$. Compute $K(EFGH)$ and *EG*.



Part B

1. Derive formulas for the altitude and area-measure of a triangular region bounded by an equilateral triangle whose side-measure is *s*.
2. Given a rhombus whose diagonals measure d_1 and d_2, where $d_1 \geq d_2$, derive formulas for the following:
   (a) the area-measure of the region bounded by the rhombus
   (b) the cosine of the smaller angles of the rhombus
   (c) the sine of the larger angles of the rhombus
3. Given a nonrectangular parallelogram with side-measures *a* and *b* and whose shorter diagonal has measure *c*, derive formulas for the following:
   (a) the cosine of the smaller angles of the parallelogram
   (b) the sine of the larger angles of the parallelogram
   (c) the area-measure of the region bounded by the parallelogram
4. Prove each of the following:
   (a) A median of a triangle separates the triangular region into two regions with the same area-measure.
   (b) An angle bisector of a triangle separates the triangular region into two regions whose area-measures are in the same ratio as the sides of the triangle which are contained in the given angle.
   (c) Given any point on a side of a triangle, the interval whose end points are that point and the vertex opposite its side separates the triangular region into two regions whose area-measures are in the same ratio as that in which the point divides the side of the triangle.
5. Consider the four regions into which the diagonals of a parallelogram divide the region bounded by the parallelogram. Show that these four regions have the same area-measure.



6. Suppose that *PQRS* is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$ and whose diagonals intersect in the point *M*. Prove the following:
   (a) The ratio of the areas of triangular regions *MPQ* and *MRS* is PQ^2/RS^2.
   (b) The area-measures of the regions bounded by $\triangle MPS$ and $\triangle MQR$ are the same.

Answers for Part A [cont.]

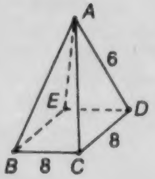
10. $K(EFGH) = 75$
11. $K(EFGH) = 72\sqrt{3}$, $FH = 12\sqrt{3}$
12. $K(EFGH) = 45\sqrt{5}/2$, $EG = 3\sqrt{5}$

Answers for Part B

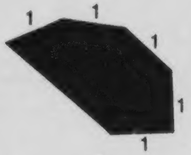
1. $h = \sqrt{s^2 - (s/2)^2} = s\sqrt{3}/2$, $K = (s/2)/h = s^2\sqrt{3}/4$
2. (a) $K = \left[\frac{1}{2}d_1(\frac{d_2}{2})\right]2 = d_1d_2/2$
   (b) By the Pythagorean theorem, the side-measure of the rhombus is $\sqrt{d_1^2 + d_2^2}/2$. So, by the cosine law, the cosine of the smaller angles is $(d_1^2 - d_2^2)/(d_1^2 + d_2^2)$.
   (c) The sine of the larger angle is the same as the sine of the smaller angle [since the angles are supplementary]. Computing the latter from its cosine [which we found in part (b)] results in $2d_1d_2/(d_1^2 + d_2^2)$ as the sine of the larger angle of the rhombus.
3. (a) By the cosine law, the cosine of the smaller angles is $(a^2 + b^2 - c^2)/(2ab)$.
   (b) The sine of the larger angles is the sine of the smaller angles and this is $\sqrt{s(s - a)(s - b)(s - c)}/(2ab)$.
   (c) $2\sqrt{s(s - a)(s - b)(s - c)}$
4. (a) A median of a triangle separates the triangular region into two triangular regions with congruent bases and the same altitude.
   (b) An angle bisector of a triangle separates the triangular region into two triangular regions which have the same altitude and the measures of whose bases are proportional to the measures of the sides of the triangle which are contained in the given angle [Theorem 15-17].
   (c) The interval in question separates the triangular region into two triangular regions which have the same altitude and the measures of whose bases have the given ratio.
5. This follows from Exercise 4(a) and the fact that the diagonals of a parallelogram bisect each other.
6. (a) $K(MPQ) \cdot 2 = PQ \cdot PM \cdot \sin \angle QPR$ and $K(MRS) \cdot 2 = RS \cdot RM \cdot \sin \angle SRP$. Since $\angle QPR \cong \angle SRP$ [alternate interior angles] it follows that $K(MPQ)/K(MRS) = (PQ \cdot PM)/(RS \cdot RM)$. But, $PM/RM = PQ/RS$ [Theorem 8-17(b)]. So, $K(MPQ)/K(MRS) = PQ^2/RS^2$. [That $\angle QPR$ and $\angle SRP$ are, actually, alternate angles follows by Theorem 8-16(a).]
   (b) By Exercise 4(c), $K(MPS)/K(MRS) = PM/MR$ and $K(MQR)/K(MRS) = QM/MS$. Since $PM/MR = PQ/RS = QM/MS$ it follows that $K(MPS) = K(MQR)$.

Part C

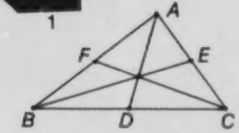
1. The four faces of the pyramid at the right are regions bounded by congruent isosceles triangles whose legs have length 6. The base *BCDE* of the pyramid is a square region with side-measure 8. Compute the total area-measure of the five faces of the pyramid.



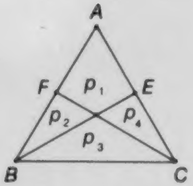
2. Given the pyramid described in Exercise 1, compute the area-measure of the region bounded by $\triangle ABD$ and the distance from *A* to the plane of *BCDE*.
3. Compute the area-measure of the shaded region pictured at the right.



4. In the triangle, $\triangle ABC$, pictured at the right, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are the medians of the triangle. Show that the six "small" triangular regions have the same area-measure.

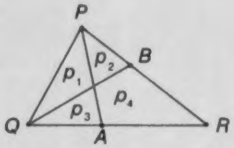


5. Given $\triangle ABC$, let *l* be the line through *A* and parallel to $\overleftrightarrow{BC}$. Show that each triangle $\triangle PBC$, where $P \in l$, has the same area-measure as does $\triangle ABC$.
6. In $\triangle ABC$ pictured at the right, $\overline{BE}$ and $\overline{CF}$ are angle bisectors, and $AB = 6$, $BC = 4$, and $CA = 5$. Also, p_1, p_2, p_3, and p_4 are the area-measures of the indicated regions. Verify each of the following, where *k* is the area-measure of $\triangle ABC$.



(a) $AE = 3$ and $EC = 2$ (b) $AF = \frac{10}{3}$ and $FB = \frac{8}{3}$
(c) $p_1 + p_2 = \frac{3}{5}k$ (d) $p_2 + p_3 = \frac{4}{9}k$
(e) $p_2 = \frac{2}{5} \cdot K(BFC)$ (f) $p_2 = \frac{8}{45}k$
(g) $p_3 = \frac{4}{15}k$ (h) $p_1 = \frac{19}{45}k$

7. In $\triangle PQR$ pictured at the right, $\overline{PA}$ and $\overline{QB}$ are medians and p_1, p_2, p_3, and p_4 are the area-measures of the indicated regions. Verify the following, given that the area-measure of $\triangle PQR$ is *k*.



(a) $p_2 = p_3$ (b) $p_1 = 2p_2$
(c) $p_1 = \frac{1}{3}k$ (d) $p_4 = p_1$

Answers for Part C

1. $64 + 32\sqrt{5}$ [$= 32(2 + \sqrt{5})$]
2. $K(ABD) = 8\sqrt{2}$, $d(A, \overleftrightarrow{BCD}) = 2$
3. $(1 + \sqrt{2} + \sqrt{3} + \sqrt{4})/2 = (3 + \sqrt{2} + \sqrt{3})/2$
4. Let M be the point of intersection of the medians. By Exercise 4(a) it follows that $K(BMD) = K(DMC) = a$, say, that $K(CME) = K(EMA) = b$, and that $K(AMF) = K(FMB) = c$. For the same reason, $K(CBE) = K(ABE)$ and $K(CAD) = K(BAD)$ and, so, that $2a + b = b + 2c$ and $a + 2b = 2c + a$. From these last results it follows that $a = c$ and that $b = c$. Hence, $a = b = c$ and all six triangular regions have the same area-measure.
5. For any choice of $P \in \ell$, $\triangle PBC$ and $\triangle ABC$ have the same base and congruent altitudes. Hence, they have the same area-measure. [To show that A and any point $P \in \ell$ are at the same distance from $\overleftrightarrow{BC}$ note, first, that $P = \vec{a} + \vec{u}u$, where $\vec{u}$ is the unit vector in $[C - B]^{+}$ and $\vec{a} = A - B$. By earlier results [Theorem 14-16(a) or the Pythagorean theorem] the square of the altitude from A of $\triangle ABC$ is $a^2 - (\vec{a} \cdot \vec{u})^2$, where $a = \|\vec{a}\|$, and, for the same reason the square of the altitude from P of $\triangle PBC$ is $\|\vec{a} + \vec{u}u\|^2 - [(\vec{a} + \vec{u}u) \cdot \vec{u}]^2$. Simplifying this last after expanding it to '$a^2 + u^2 + 2(\vec{a} \cdot \vec{u})u - [(\vec{a} \cdot \vec{u})^2 + u^2 + 2(\vec{a} \cdot \vec{u})u]$' we see that $\triangle ABC$ and $\triangle PBC$ have congruent altitudes from A and P, respectively. The result just proved is often stated as: Parallel lines are everywhere equidistant.]
6. (a) By Theorem 15-17, $AE = 5(6/10) = 3$ and $EC = 5(4/10) = 2$.
   (b) Similarly, $AF = 6(5/9) = 10/3$ and $FB = 6(4/9) = 8/3$.
   (c) Since $AE = AC(3/5)$, $p_1 + p_2 = 3k/5$ [Exercise 4(c) of Part B].
   (d) Since $BF = BA(4/9)$, $p_2 + p_3 = 4k/9$.
   (e) Let M be the point of intersection of the angle bisectors. Since $BF = 8/3$ and $BC = 4$, $FM/FC = (8/3) \div (8/3 + 4) = 2/5$. So, $p_2 = 2K(BFC)/5$.
   (f) Since $BF/BA = (8/3)6 = 4/9$, $K(BFC) = 4k/9$. Combining this with the result of part (e) it follows that $p_2 = 8k/45$.
   (g) By (d), $p_2 + p_3 = 4k/9$ and by (f) $p_2 = 8k/45$. So, $p_3 = 4k/9 - 8k/45 = 4k/15$.
   (h) By (c) and (f), as in part (g), $p_1 = 19k/45$.
7. (a) Since $p_1 + p_2 = k/2$ and $p_1 + p_3 = k/2$ it follows that $p_2 = p_3$.
   (b) Since the point of intersection of the medians divides the median from Q to B in 2:1 it follows that $p_1 = 2p_2$.
   (c) Since $p_1 + p_2 = k/2$ and $p_2 = p_1/2$ it follows that $p_1 = k/3$.
   (d) Since $p_4 + p_3 = p_1 + p_2$ and $p_2 = p_3$ it follows that $p_4 = p_1$.

*

The ratio of similitude of $\triangle DEF$ to $\triangle ABC$ is /k.

*

Consider $\triangle ABC$ and $\triangle DEF$, shown in the picture at the right. Given that $ABC \longleftrightarrow DEF$ is a similarity, we know that the corresponding sides are proportional [and, that the corresponding angles are congruent] so that, for some $k > 0$, $AB = kDE$, $BC = kEF$, and $AC = kDF$. The number k is sometimes called *the ratio of similitude* of $\triangle ABC$ to $\triangle DEF$. [What, then, is the ratio of similitude of $\triangle DEF$ to $\triangle ABC$?]

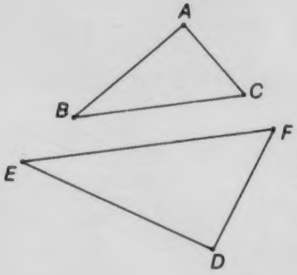


Fig. 16–13

The area-measure, K_1, of triangular region ABC is $\frac{1}{2}AB \cdot AC \cdot \sin \angle A$, and the area-measure, K_2, of triangular region DEF is $\frac{1}{2}DE \cdot DF \cdot \sin \angle D$. Given that $ABC \longleftrightarrow DEF$ is a similarity, we know that $\sin \angle A = \sin \angle D$ and that there is a ratio of similitude—say, k—of $\triangle ABC$ to $\triangle DEF$. Thus,

$$\begin{aligned} K_1 &= \tfrac{1}{2}AB \cdot AC \cdot \sin \angle A \\ &= \tfrac{1}{2} \cdot kDE \cdot kDF \cdot \sin \angle D && \text{[Why?]} \\ &= k^2 \cdot [\tfrac{1}{2}DE \cdot DF \cdot \sin \angle D] && \text{[Why?]} \\ &= k^2 K_2 && \text{[Why?]} \end{aligned}$$

This tells us that if $ABC \longleftrightarrow DEF$ is a similarity—that is, if $\triangle ABC$ is similar to $\triangle DEF$—then the ratio of the area-measure, K_1, of triangular region ABC to the area-measure, K_2, of triangular region DEF is k^2, where k is the ratio of similitude of $\triangle ABC$ to $\triangle DEF$. For short, $K_1 : K_2 = k^2$.

For convenience, we summarize this result in:

> **Theorem 16–30** The ratio of the area-measure of a given triangle to that of a similar triangle is the square of the ratio of similitude of the given triangle to the second triangle.

Part D

1. Suppose that $ABC \longleftrightarrow PQR$ is a similarity between $\triangle ABC$ and $\triangle PQR$.
   (a) Given that $AC = 7$, $BC = 11$, and $QR = 14$, what is the ratio of similitude of $\triangle ABC$ to $\triangle PQR$? What is the ratio of the area-measure of $\triangle PQR$ to $\triangle ABC$?

*

Sample Quiz

1. An altitude of one equilateral triangle has the same measure as a side of another equilateral triangle. What is the ratio of their areas? Of their perimeters?
2. If three squares have side-measures 3, 4, and 12, respectively, what is the side-measure of a fourth square whose area is the sum of the area-measures of the three given squares?
3. Given $\triangle ABC$ with $AB = 10$, $BC = 15$, and $\cos \angle B = -3/5$. Compute the area-measure of $\triangle ABC$ and the measure of the altitude from A.
4. In $\triangle ABC$, the medians $\overline{AM}$ and $\overline{CN}$ intersect in O. P is the mid-point of $\overline{AC}$ and $\overline{MP}$ intersects $\overline{CN}$ in Q. If the area-measure of $\triangle ABC$ is 1, compute the area-measures of each of the following regions.

   (a) CPQ (b) CQM (c) OQM
   (d) APQO (e) AON (f) MONB

Key to Sample Quiz

1. $4/3$; $2/\sqrt{3}$ 2. 13 3. 60; 8
4. (a) 1/8 (b) 1/8 (c) 1/24 (d) 5/24 (e) 1/6 (g) 1/3

Answers for Part D

1. (a) 11/14; 196/121

TC 306 (1)

(b) K(ABC) = 10, K(PQR) = 45/8
(c) $1/\sqrt{2}$ [By the cosine law, $PQ = 9\sqrt{2}$.], $K(PQR) = 9\sqrt{119}/4$, $K(ABC) = 9\sqrt{119}/8$

2. [Note that $LPQ \longleftrightarrow LKM$ is a similarity by a.a. In fact, the uniform stretching about L with stretching factor 9/7 is a similitude which, since it maps L on L, P on K, and Q on M, maps $\triangle LPQ$ onto $\triangle LKM$.]
   (a) 7/9; 49/81
   (b) 10/3; 4/9
   (c) $1/\sqrt{2}$; $1/(\sqrt{2} - 1)$ [or: $\sqrt{2}/2$, $1 + \sqrt{2}$]

Answers for Part D [cont.]

3. (a) Since ∠PQD ≅ ∠MCQ, ∠MCQ is complementary to ∠MCB, and ∠CMB is complementary to ∠MCB, it follows that ∠PQD ≅ ∠CMB. Since ∠D and ∠B are both right angles it follows by a.a. that PQD ⟷ CMB is a similarity.
   (b) 1/3
   (c) 1/9
4. (a) Suppose that ABCD and PQRS are squares with side-measures s_1 and s_2, respectively, where $s_1 < s_2$. Let g be the uniform stretching about A with stretching ratio s_2/s_1 — that is, for each point X, let

$$g(X) = A + (X - A)(s_2/s_1).$$

Suppose that g maps A, B, C, and D on K, L, M, and N, respectively. [So, for example, K = A and L = A + (B - A) (s_2/s_1).] Since g maps intervals on intervals, mapping end points on end points, it follows that g maps ABCD onto KLMN. Since g preserves perpendicularity and multiplies distances by s_2/s_1 it follows that KLMN is a square of side-measure s_2. We shall now find an isometry which maps KLMN onto PQRS. Having done so, h∘g will be a similitude which maps ABCD on PQRS and we shall have proved that ABCD and PQRS are similar, the ratio of similitude of ABCD to PQRS being s_1/s_2.

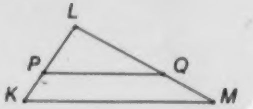
Since KL = s_2 = PQ and NK = s_2 = SP it follows, since ∠K and ∠P are right angles, that NL = SQ. It follows that there is an isometry h which maps N on S, K on P, and L on Q. Since h is an isometry it maps $\overleftrightarrow{NKL}$ onto $\overleftrightarrow{SPQ}$ and preserves perpendicularity. Hence, h maps $\overleftrightarrow{NM}$ onto $\overleftrightarrow{SR}$ and maps $\overleftrightarrow{LM}$ onto $\overleftrightarrow{QR}$. So, h maps the point M of intersection of $\overleftrightarrow{NM}$ and $\overleftrightarrow{LM}$ on the point R of intersection of $\overleftrightarrow{SR}$ and $\overleftrightarrow{QR}$. Since h is an isometry which maps K on P, L on Q, M on R, and N on S, h maps KLMN onto PQRS.

   (b) s_1/s_2
   (c) $(s_2/s_1)^2$
5. 256, 36 [$k_2 - k_1 = 220$, $k_2/k_1 = 64/9$]
6. K(ABCD) = 32, K(MNPQ) = 16 [Students should not take it for granted that MNPQ is a square. One way of showing that it is is to begin by showing that it is a rhombus with side-measure 4 and then noting that, since its diagonals are parallel to the sides of ABCD, it is orthodiagonal. Students can, of course, by-pass the proof of the squareness of MNPQ by noting, as above, that it is orthodiagonal and that each of its diagonals has measure $8/\sqrt{2}$.]

(b) Given that $AB = 8$, $BC = 12$, $\sin \angle B = \frac{5}{13}$, and $PQ = 6$, compute the area-measures of triangular regions *ABC* and *PQR*.

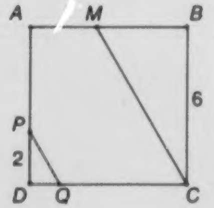
(c) Given that $RP = 6$, $RQ = 9$, $\cos \angle R = -\frac{5}{13}$, and $AB = 9$, compute the ratio of similitude of $\triangle ABC$ to $\triangle PQR$ and the area-measures of the triangular regions *ABC* and *PQR*.

2. Consider $\triangle KLM$ with $\overleftrightarrow{PQ} \parallel \overleftrightarrow{KM}$, as shown in the picture at the right. Answer the following questions.



   (a) Given that $LP = 7$ and $PK = 2$, what is the ratio of similitude of $\triangle LPQ$ to $\triangle LKM$? What is the ratio of the area-measures of *LPQ* and *LKM*?
   (b) Given that $KL = 10$ and the ratio of similitude of $\triangle LPQ$ to $\triangle LKM$ is $\frac{3}{5}$, what is *KP*? What is the ratio of the area-measures of *LPQ* and *LKM*?
   (c) Given that the ratio of the area-measures of *LPQ* and *LKM* is $\frac{4}{9}$, what is *LP* : *LK*, and what is *LP* : *PK*?

3. Consider the square *ABCD* shown at the right. $AB = 6$, *M* is the midpoint of $\overline{AB}$, $PD = 2$, and $\overline{PQ} \parallel \overline{MC}$.



   (a) Show that $\triangle PQD$ is similar to $\triangle CMB$.
   (b) What is the ratio of similitude of $\triangle PQD$ to $\triangle CMB$?
   (c) What is the ratio of the area-measures of *PQD* and *CMB*?

4. Suppose that two squares have sides whose measures are s_1 and s_2, respectively, and that $s_1 < s_2$.
   (a) Show that the squares are similar.
   (b) What is the ratio of similitude of the smaller to the larger square?
   (c) What is the ratio of the area-measures of the larger to the smaller of the squares?

5. The difference in the area-measures of two squares is 220 and the ratio of similitude of the larger to the smaller of the squares is $\frac{8}{3}$. Compute the area-measures of the squares.

6. Let *M*, *N*, *P*, and *Q* be the midpoints of consecutive sides of square *ABCD*. Given that the measure of a diagonal of *ABCD* is 8, compute the area-measures of *ABCD* and *MNPQ*.

16.10 Chapter Summary

Vocabulary Summary

- the cosine law
- congruence
- sufficient condition
- rectangle
- square
- isodiagonal
- isosceles trapezoid
- similar figures
- boundary
- quadrangular region
- the sine law
- congruent figures
- necessary condition
- rhombus
- kite
- orthodiagonal
- similitude
- similarity
- triangular region
- area-measure
- exterior angle
- congruence theorems
  - s.s.s.
  - s.a.s.
  - s.s.a.
  - a.s.a.
- uniform stretching
- similarity theorems
  - s.a.s.
  - a.a.

Definitions

16-1. A rectangle is a quadrilateral each of whose angles is a right angle.

16-2. **(a)** A rhombus is a plane quadrilateral all four of whose sides are congruent.
(b) A square is a rectangular rhombus.

16-3. A kite is a quadrilateral which has two consecutive sides congruent and the sides opposite these congruent.

16-4. f is a similitude of $\mathscr{E}$ if and only if f is a mapping of $\mathscr{E}$ onto itself such that, for some $m > 0$, $\forall_X \forall_Y\ d\Big(f(X), f(Y)\Big) = d(X, Y)m$.

16-5. A first figure is similar to a second if and only if there is a similitude of $\mathscr{E}$ which maps the first figure onto the second.

16-6. The area-measure of a triangular region is one-half the product of any side by the altitude to that side.

Other Theorems

16-1. [The Projection Theorem] In $\triangle ABC$,
(a) $a \cos \angle B + b \cos \angle A = c$, and
(b) $\cos \angle A = \dfrac{c - a \cos \angle B}{b}$.

Corollary 1. No two angles of a triangle are supplementary.

Corollary 2. Each triangle has at least two acute angles.

Corollary 3. An exterior angle of a triangle is larger than each of the angles of the triangle which are opposite it.

Corollary 4. A triangle is isosceles with a given side as base if and only if the angles of the triangle at the end points of the given side are congruent.

Corollary 5. A triangle is equilateral if and only if it is equiangular.

Corollary 6. If two sides of a triangle are not congruent then the angle opposite the longer side is larger than the angle opposite the shorter side.

Corollary 7. If two angles of a triangle are not congruent then the side opposite the larger angle is longer than the side opposite the smaller angle.

Corollary 8. If each side of one angle intersects both sides of another angle then the angles are not supplementary.

16-2. [The Cosine Law] In $\triangle ABC$,
(a) $c^2 = a^2 + b^2 - 2ab \cos \angle C$, and
(b) $\cos \angle C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

16-3. If a, b, and c are positive numbers such that

$$|a - b| < c < a + b$$

then there is a triangle whose side measures are a, b, and c.

16-4. [The Sine Law] In $\triangle ABC$,

$$\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c}.$$

16-5. In a right triangle, the cosine of an acute angle is the quotient of the adjacent leg by the hypotenuse, and the sine of an acute angle is the quotient of the opposite leg by the hypotenuse.

16-6. In $\triangle ABC$, $\cos \angle C = -(\cos \angle A \cos \angle B - \sin \angle A \sin \angle B)$, and $\sin \angle C = \sin \angle A \cos \angle B + \cos \angle A \sin \angle B$.

16-7. A matching of the vertices of a first triangle with those of a second is a congruence if
(a) each side of the first triangle is congruent to the corresponding side of the second [s.s.s.], or
(b) each of two sides of the first triangle is congruent to the corresponding side of the second and the included angles are congruent [s.a.s.], or
(c) each of two sides of the first triangle and the angle opposite the second side is congruent to the corresponding part of the second triangle *and* the angle opposite the first side is not a supplement of its corresponding angle [s.s.a.], or

(d) each of two angles of the first triangle is congruent to the corresponding angle of the second and the included sides are congruent [a.s.a.], or

(e) each of two angles of the first triangle and the side opposite the second angle is congruent to the corresponding part of the second triangle.

Corollary 1. A matching of the vertices of a first triangle with those of a second is a congruence if each of two sides of the first triangle and the angle opposite the second side is congruent to the corresponding part of the second triangle and

(a) the angle opposite the first side and its corresponding angle are both acute, or

(b) neither the angle opposite the first side nor its corresponding angle is acute, or

(c) the second side is longer than the first.

Corollary 2. If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle then the matching with respect to which these are corresponding sides is a congruence.

16-8. A quadrilateral is a parallelogram if and only if it is simple and plane and each two of its opposite angles are congruent.

16-9. A quadrilateral is a parallelogram if and only if it is simple and plane and each two of its consecutive angles are supplementary.

16-10. A quadrilateral is a parallelogram if and only if it is simple and some two of its opposite sides are parallel and congruent.

16-11. A quadrilateral is a parallelogram if and only if it is simple and plane and each two of its opposite sides are congruent.

16-12. A quadrilateral is a parallelogram if and only if the sum of the squares of its diagonals equals the sum of the squares of its sides.

Lemma. Any rectangle is a plane quadrilateral.

16-13. Each rectangle is a parallelogram.

16-14. A parallelogram one of whose angles is a right angle is a rectangle.

16-15. A parallelogram is a rectangle if and only if its diagonals are congruent.

16-16. Each rhombus is a parallelogram.

16-17. Any parallelogram two of whose consecutive sides are congruent is a rhombus.

16-18. A parallelogram is a rhombus if and only if its diagonals are perpendicular.

Corollary. A rectangle is a square if and only if its diagonals are perpendicular.

16-19. A plane quadrilateral is a rhombus if and only if each of its diagonals is contained in the perpendicular bisector of the other.

16-20. A quadrilateral is a rhombus if and only if its diagonals are contained in the bisectors of its angles.

16-21. The sum of the squares of two opposite sides of a quadrilateral equals the sum of the squares of the other two sides if and only if the diagonals of the quadrilateral are perpendicular.

16-22. A quadrilateral is a kite if and only if one of its diagonals is contained in the perpendicular bisector of the other.

16-23. A trapezoid is isodiagonal if and only if it is either isosceles or a rectangle.

Corollary. A parallelogram is isodiagonal if and only if it is a rectangle.

16-24. A trapezoid is isodiagonal if and only if any pair of its base angles are congruent.

16-25. Each similitude of $\mathscr{E}$ is the resultant of a uniform stretching about any given point O followed by an isometry.

16-26. A matching of the vertices of one triangle with those of a second is a similarity if corresponding sides are proportional.

16-27. If a matching of the vertices of one triangle with those of a second is a similarity then corresponding angles are congruent and corresponding sides are proportional.

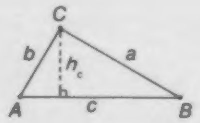
16-28. A matching of the vertices of one triangle with those of a second is a similarity if two sides of the first triangle are proportional to the corresponding sides of the second and the included angles are congruent.

16-29. A matching of the vertices of one triangle with those of a second is a similarity if two angles of the first triangle are congruent to the corresponding angles of the second.

16-30. The ratio of the area-measure of a given triangle to that of a similar triangle is the square of the ratio of similitude of the given triangle to the second triangle.

Area Formulas

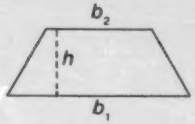
Triangular region



$$\begin{aligned} K &= (h_c \cdot c)/2 \\ &= \sqrt{a^2b^2 - (\vec{a}\cdot\vec{b})^2}/2 \\ &= \tfrac{1}{2}ab \sin \angle C \\ &= \sqrt{s(s-a)(s-b)(s-c)}, \\ &\qquad s = (a+b+c)/2 \end{aligned}$$

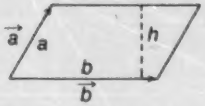
Trapezoidal region

$$K = h(b_1 + b_2)/2$$



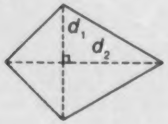
Region bounded by a parallelogram

$$K = bh = \sqrt{a^2b^2 - (\vec{a} \cdot \vec{b})^2}$$



Region bounded by a simple orthodiagonal plane quadrilateral

$$K = (d_1d_2)/2$$



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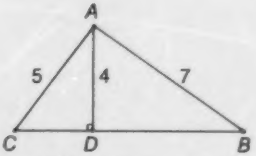
Chapter Test

1. True or false?
   - **(a)** That two angles of a triangle are complements is a sufficient condition that the triangle is a right triangle.
   - **(b)** That two angles of a first triangle are congruent to the corresponding angles of a second triangle under a matching of vertices is a necessary condition that the triangles be similar.
   - **(c)** That the angles of a first quadrilateral are congruent to the corresponding angles of a second quadrilateral under a matching of vertices is a sufficient condition that the quadrilaterals be similar.
   - **(d)** That the angles of a first quadrilateral are congruent to the corresponding angles of a second quadrilateral under a matching of vertices is a necessary condition that the quadrilaterals be similar.
   - **(e)** There is no triangle which has an exterior angle which is supplementary to an opposite interior angle.
   - **(f)** There is a triangle which has an exterior angle which is complementary to an opposite interior angle.
   - **(g)** There is a triangle which has an exterior angle which is congruent to an opposite interior angle.
   - **(h)** If a trapezoid has perpendicular diagonals, then it is an isosceles trapezoid.

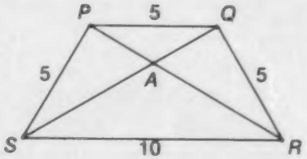
Answers for Chapter Test

1. (a) True.
   (b) True.
   (c) False. [Consider two rectangles only one of which is a square.]
   (d) True.
   (e) False. [Consider any isosceles triangle.]
   (f) True. [Consider any triangle which has an acute exterior angle which is larger than half of a right angle.]
   (g) False. [See Corollary 3 of Theorem 16-1.]
   (h) False.
2. (a) 87/100 (b) $\sqrt{2431}/100$ (c) $-61/160$ (d) $3\sqrt{2431}/4$
3. (a), (c), (d)
4. $PR = \sqrt{2}/2$; Since $QR > PR$, $\angle P$ is larger than $\angle Q$ and, since $\angle P$ is acute, so is $\angle Q$.; $\sin\angle R = (\sqrt{2} + \sqrt{6})/4 = PQ$ [Since $QR/\sin\angle P = 1$ it follows by the sine law that $PR = \sin\angle Q = \sqrt{2}/2$ and $PQ = \sin\angle R$. Since $\angle P$ and $\angle Q$ are both acute, $\cos\angle P = 1/2$ and $\cos\angle Q = \sqrt{2}/2$. So, Theorem 16-6 can be used to compute $\sin\angle R$ and, so, PQ.]
5. (a) $\sqrt{33}/7$ (b) 3/5 (c) $2\sqrt{33}$ (d) $3 + \sqrt{33}$
6. (a) $ABE \longleftrightarrow CDE$ is a congruence by s.a.s. [$EA = EC$, $\angle AEB \cong \angle CED$, $BE = DE$ since the diagonals of a parallelogram bisect each other and vertical angles are congruent.] and, also, by s.s.s. [$AB = CD$ since opposite sides of a parallelogram are congruent.]
   (b) $\triangle ABE$ and $\triangle CDE$ have the same area-measure since they are congruent. So, for the same reason, do $\triangle BCE$ and $\triangle DAE$. Each of $\triangle ABE$ and $\triangle BCE$ has area-measure half that of $\triangle ABC$. [One can repeat this last argument to show that $\triangle BCE$ and $\triangle CDE$ have the same area-measure and that $\triangle CDE$ and $\triangle DAE$ have the same area-measure. In this way one does not need the result obtained in part (a).]
7. (a) $PAQ \longleftrightarrow RAS$ is a similarity by s.a.s. For, since $SQ = RP$ and each is divided in 10 : 5 by A, $AP = AR/2$ and $AQ = AS/2$. Also, the vertical angles at A are congruent.
   (b) 1/4 (c) 2 (d) $75\sqrt{3}/4$
8. (a) 135/2 (b) 144 (c) $40\sqrt{3}$
   (d) 121/2 (e) $48\sqrt{3}$ (f) 84

2. Given $\triangle ABC$ with $BC = 8$, $AC = 10$, and $AB = 15$. Compute each of the following.
   (a) $\cos \angle A$ (b) $\sin \angle A$ (c) $\cos \angle C$ (d) area-measure of $\triangle ABC$
3. For which of the following is it possible to have triangles whose side-measures are the given numbers.
   (a) 6, 6, 2 (b) 4, 5, 9 (c) 2, $\sqrt{3}$, 1 (d) $\sqrt{2}$, $\sqrt{3}$, 1
4. In $\triangle PQR$, $\sin \angle Q = \sqrt{2}/2$, $\sin \angle P = \sqrt{3}/2$, $QR = \sqrt{3}/2$, and $\angle P$ is acute. Compute PR. Show that $\angle Q$ is acute, and compute $\sin \angle R$ and PQ.
5. Given $\triangle ABC$ with altitude $\overline{AD}$, and $AB = 7$, $AC = 5$, and $AD = 4$, as shown at the right. Compute the following.
   (a) $\cos \angle B$ (b) $\sin \angle CAD$
   (c) area-measure of $\triangle BAD$ (d) BC



6. Suppose that the diagonals of parallelogram $ABCD$ intersect in the point E.
   (a) Give a matching of the vertices of $\triangle ABE$ with those of $\triangle DEC$ which is a congruence, and justify your answer.
   (b) Show that the triangles AED, AEB, BEC, and CED have the same area-measure.
7. Suppose that $PQRS$ is an isosceles trapezoid with bases $\overline{PQ}$ and $\overline{RS}$, that diagonals $\overline{PR}$ and $\overline{QS}$ intersect in the point A, and that $PQ = 5$, $RS = 10$, and $PS = 5 = QR$, as shown in the picture at the right.

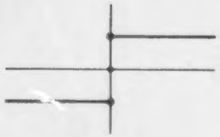


   (a) Give a matching of the vertices of $\triangle APQ$ with those of $\triangle SAR$ which is a similarity, and justify your answer.
   (b) What is the ratio of the area-measure of $\triangle APQ$ to that of $\triangle SAR$?
   (c) What is the ratio of the area-measure of $\triangle APS$ to that of $\triangle APQ$?
   (d) Compute the area-measure of the region bounded by $PQRS$.
8. In each of the following, you are given some information about a figure. Compute the area-measure of the region bounded by that figure.
   (a) Rhombus whose diagonals have measures 9 and 15.
   (b) Square whose sides have measure 12.
   (c) Parallelogram two of whose adjacent sides have measures 8 and 10 and one of whose angles has cosine-value $-\frac{1}{2}$.
   (d) Isosceles triangle whose legs have measure 11 and whose base angles have cosine-value $\sqrt{2}/2$.
   (e) Equilateral triangle whose medians have measure 12.
   (f) Triangle whose side-measures are 13, 14, and 15.

Background Topic

Part A

In these exercises we shall study a function called *the signum function* or, for short 'sgn'. [Read 'sgn' as you would 'sig'.] Its graph is shown in the figure at the right, and it may be defined by:



$$(*) \qquad \operatorname{sgn}(x) = \begin{cases} 1, \text{ for } x > 0 \\ 0, \text{ for } x = 0 \\ -1, \text{ for } x < 0 \end{cases}$$

Another possible definition is suggested by Exercise 1.

1. Show that, for each real number x,

$$(**) \qquad \operatorname{sgn}(x) = x/|x|.$$

[*Hint*: Recall that $|0| = 0$ and that $/0$ is some real number although we have not specified which.]

2. Compute each of the following, using (*).

(a) $\operatorname{sgn}(5)$ (b) $\operatorname{sgn}(-\frac{5}{2})$ (c) $\operatorname{sgn}(-\sqrt{3}/2)$
(d) $\operatorname{sgn}(\sqrt{3}/2)$ (e) $\operatorname{sgn}(6) \cdot \operatorname{sgn}(-5)$ (f) $\operatorname{sgn}(13) \cdot \operatorname{sgn}(\frac{1}{13})$
(g) $\operatorname{sgn}(-5) \cdot \operatorname{sgn}(-\frac{1}{5})$ (h) $-12 \cdot \operatorname{sgn}(-12)$ (i) $17 \cdot \operatorname{sgn}(17)$
(j) $5 \operatorname{sgn}(17 - 12)$ (k) $5 \operatorname{sgn}(12 - 17)$ (l) $5 \cdot \operatorname{sgn}(17 \cdot 12)$

3. Verify your answers in Exercise 2 by using (**) of Exercise 1.

4. Prove each of the following:

(a) $a = \operatorname{sgn}(a)|a|$ (b) $\operatorname{sgn}(-a) = -\operatorname{sgn}(a)$
(c) $\operatorname{sgn}(ab) = \operatorname{sgn}(a) \cdot \operatorname{sgn}(b)$
(d) $(\operatorname{sgn}(a))^2 = 1$ for $a \neq 0$, and $(\operatorname{sgn}(a))^2 = 0$ for $a = 0$.
(e) $1 - (\operatorname{sgn}(a))^2 = 0$, for $a \neq 0$, and $1 - (\operatorname{sgn}(a))^2 = 1$ for $a = 0$.

5. Show that, for each x,

$$\operatorname{sgn}(x - a) = \begin{cases} 1, \text{ for } x > a \\ 0, \text{ for } x = a. \\ -1, \text{ for } x < a \end{cases}$$

*6. Use the signum function to describe a function f such that

$$f(x) = \begin{cases} 0, \text{ for } x < a \\ \frac{1}{2}, \text{ for } x = a. \\ 1, \text{ for } x > a \end{cases}$$

Part B

In these exercises we shall study another "peculiar" function called *the integral part function.* The integral part of a given real number is the greatest integer which is less than or equal to the given number. For example, the integral part of 4.6 is 4, and the integral part of −1.5 is −2. What is the integral part of 3? Of 17.01? Of −16.2?

Answers for Part A

1. If $x > 0$ then $\operatorname{sgn}(x) = 1 = x/x = x/|x|$; if $x < 0$ then $\operatorname{sgn}(x) = -1 = x/-x = x/|x|$; finally, $\operatorname{sgn}(0) = 0 = 0 \cdot /0 = 0/0$.

2. (a) 1 (b) −1 (c) −1 (d) 1 (e) −1 (f) 1
(g) 1 (h) 12 (i) 17 (j) 5 (k) −5 (ℓ) = 5

3. (a) $5/|5| = 5/5 = 1$
(b) $(-5/2)/|-5/2| = (-5/2)/(5/2) = -1$ etc.

4. (a) For $a \neq 0$ this follows from (**); also, $0 = 0 \cdot |0|$.
(b) By (**), $\operatorname{sgn}(-a) = -a/|-a| = -(a/|a|) = -\operatorname{sgn}(a)$.
(c) By (**), $\operatorname{sgn}(ab) = (ab)/|ab| = (ab)/(|a||b|) = (a/|a|)(b/|b|) = \operatorname{sgn}(a) \cdot \operatorname{sgn}(b)$.
(d) $(\operatorname{sgn}(a))^2 = 1^2$, 0, or $(-1)^2$ according as $a = 1$, 0, or −1. And $1^2 = 1 = (-1)^2$.
(e) This follows at once from part (d).

5. By (*), $\operatorname{sgn}(x - a) = 1$ for $x - a > 0$, $= 0$ for $x - a = 0$, and $= -1$ for $x - a < 0$. And, $x - a > 0$ if and only if $x > a$, $x - a = 0$ if and only if $x = a$, and $x - a < 0$ if and only if $x < a$.

6. Let, for each x, $f(x) = [\operatorname{sgn}(x) + 1]/2$.

The integral part function can be characterized by a defining principle like those we used on page 5 to introduce absolute valuing [(2) on page 5] and square rooting [(3) on page 5]. To do so we introduce the notation '[[. . .]]', which is to be read as the 'the integral part of. . .', and adopt the defining principle:

$$(1) \qquad [[a]] \in I \text{ and } [[a]] \le a < [[a]] + 1$$

[Study (1) to see that it does mean, intuitively, that $[[a]]$ is the integer which is the greatest of those integers which are not greater than the number a.]

1. Draw a graph of the integral part function for arguments between -3 and 3.
2. Prove that (1) does characterize the integral part function by proving:

$$(2) \qquad (b \in I \text{ and } b \le a < b + 1) \longrightarrow b = [[a]]$$

[*Hint*: Suppose that $b \in I$ and $b \le a < b + 1$. Use this and (1), together with a theorem about integers proved at the end of Chapter 15 to show that $[[a]] \le b$ and $b \le [[a]]$.]

3. Use (2) and a theorem about integers to prove:

$$(3) \qquad b \in I \longrightarrow [[a + b]] = [[a]] + b$$

4. **(a)** Show that, for any $a \in \mathscr{R}$ and for $b > 0$, there exists an integer q and a number r such that

$$a = qb + r \text{ and } 0 \le r < b.$$

[*Hint*: Apply (1) with 'a/b' for 'a'.]

(b) What is q [in terms of 'a' and 'b']? What is r?

*

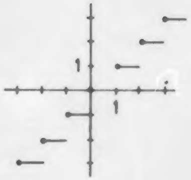
Once we have adopted (1) we are committed to the belief that, given any real number a, there is an integer – say, p – such that $p \le a < p + 1$. It is of some importance that this can be proved to be the case if, in addition to our postulates (Nn_1) – (Nn_3) and (I), we adopt:

$$(C) \qquad \exists_x (x \in Nn \text{ and } x > a)$$

What (C) says is that there is no real number which is greater than or equal to all of the nonnegative integers. Another way of putting this is to say that the set Nn has no upper bound. Briefly, an upper bound of set S of real numbers is a number which is greater than or equal to each member of S. [For example, each positive number and 0 is an upper bound of the set of all negative numbers.] A greatest member of

Answers for Part B

1.



2. Suppose that $b \in I$ and $b \le a < b + 1$. Since $\llbracket a \rrbracket \le a < \llbracket a \rrbracket + 1$ it follows that $b < \llbracket a \rrbracket + 1$ and that $\llbracket a \rrbracket < b + 1$. Since $\llbracket a \rrbracket$, as well as b, belongs to I it follows that $b \le \llbracket a \rrbracket$ and that $\llbracket a \rrbracket \le b$. So, $b = \llbracket a \rrbracket$. Hence, if $b \in I$ and $b \le a < b + 1$ then $b = \llbracket a \rrbracket$.

3. Since $\llbracket a \rrbracket \le a < \llbracket a \rrbracket + 1$ it follows that $\llbracket a \rrbracket + b \le a + b < (\llbracket a \rrbracket + b) + 1$. Assuming that b is an integer it follows, since $\llbracket a \rrbracket \in I$, that $\llbracket a \rrbracket + b \in I$. So, by (2) $\llbracket a \rrbracket + b = \llbracket a + b \rrbracket$. Hence, if $b \in I$ then $\llbracket a + b \rrbracket = \llbracket a \rrbracket + b$. [Note that, as examples will show, it is not in general the case that $\llbracket a + b \rrbracket = \llbracket a \rrbracket + \llbracket b \rrbracket$.]

4. (a) By (1), $\llbracket a/b \rrbracket \in I$ and $\llbracket a/b \rrbracket \le a/b < \llbracket a/b \rrbracket + 1$. So, with $q = \llbracket a/b \rrbracket$ then $q \in I$ and $qb \le a < qb + b$. So, with $r = a - qb$, $a = qb + r$ and $0 \le r < b$.

   (b) $q = \llbracket a/b \rrbracket$; $r = a - \llbracket a/b \rrbracket b$.

a set S is a member of the set which is, also, an upper bound of the set. [For example, the set of all negative numbers has no greatest member; but the set of all nonpositive numbers has 0 as its greatest member.]

To justify our introduction of the greatest integer function we shall use (C) and other properties of integers to prove another important theorem about integers:

Theorem Each nonempty set of integers which has an upper bound has a greatest member.

[Notice that this theorem does not remain true if 'integers' is replaced by 'real numbers'.] Aside from (C) we shall use two theorems proved at the end of Chapter 15:

$$(*) \qquad (a \in I \text{ and } b \in I) \longrightarrow b - a \in I$$
$$(**) \qquad (a \in I \text{ and } b \in I) \longrightarrow [b < a \longrightarrow b \leq a - 1]$$

Proof of Theorem: Suppose that S is a set of integers which has an upper bound—say, a—but has no greatest member. [We shall show that $S = \varnothing$.] By (C) there is a nonnegative integer—say, n—which is greater than a and, so, is an upper bound of S. Since S has no greatest member it follows that $n \notin S$. Hence, each member of S is less than n. Let S' be the set of numbers obtained by subtracting n from the members of S. By (*), the members of S' are integers and, clearly, each member of S' is less than 0. Suppose that b is an integer such that each member of S is less than b. It follows by (**) that each member of S' is less than or equal to $b - 1$ and, since S' has no greatest member, each member of S' is less than $b - 1$. It follows by induction that, for any nonpositive integer b, each member of S' is less than b. Since S' consists of nonpositive integers it follows that each member of S' is less than itself. So, since no number is less than itself, S' has no members. Since $S' = \varnothing$ it is clear that, also, $S = \varnothing$.

It is now easy to justify our introduction of the greatest integer function. We know by (C) that, for any real number a there is a nonnegative integer—say, p—such that $p > -a$. So, there is an integer—for example, $-p$—which is less than a. It follows that the set S of all integers less than or equal to a is nonempty and, of course, has a as an upper bound. Hence, by the theorem, there is a greatest integer—say, q—which is less than or equal to a. Since q is the greatest such integer, $q + 1 \nleq a$—that is, $a < q + 1$. All that remains is to define the integral part of a to be this integer q.

It remains only to be said that (C), which we have adopted as a new postulate concerning the real numbers will turn out to be a consequence of our final set of postulates for $\mathscr{R}$. In addition to the latter we need only $(Nn_1) - (Nn_3)$ and (I) to establish all properties of integers.

Chapter Seventeen
Circles

17.01 Spheres and Circles

Given a point C and a positive real number r, the set of all points which are at a distance r from C is called *the sphere with center C and radius r*. [In some books, what we call a sphere is called a spherical surface. We shall use 'ball' to refer to the set of all points "inside" or on a sphere.] We adopt the following definition:

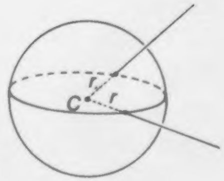


Fig. 17–1

Definition 17–1 The sphere with center C and radius $r > 0$ is $\{X: \|X - C\| = r\}$.

Notice that, by Theorem 14–6, a center of a sphere belongs to the perpendicular bisector of any segment whose endpoints belong to the sphere. Note, also, that each half-line with vertex C contains exactly one point of the sphere with center C and a given radius. [Explain.] Using this, it can be shown that no sphere is a subset of a plane and, so, that each sphere contains four noncoplanar points. A stronger result is:

Theorem 17–1 Four noncoplanar points belong to one and only one sphere.

[One proof of this theorem makes use of Exercise 4 of Part C on page 319. In proving the theorem you will also show that a sphere has only one center.]

Note that it is not immediately clear that the sphere with center C and radius r is different from the sphere with center D and radius s if $C \neq D$ or $r \neq s$. In Exercise 5 of Part A it will be shown that a sphere has a unique center and from this it is obvious that a sphere has a unique radius. After that we shall be entitled to speak of *the* center and *the* radius of a given sphere. [We shall also speak of *a radius* of a sphere when we wish to refer to an interval whose end points are the center and some point of the given sphere.]

Given a half-line with vertex C there is one and only one point of this half-line at a given nonzero distance from C. More specifically, for $r > 0$, $C + \vec{u}r$ is the only point of $C[\vec{u}]^{+}$ at the distance r from C.

* * *

Suggestions for the exercises of section 17.01:

(i) Part A should be directed by the teacher.

(ii) Parts B and C may be used for homework, to be followed by class discussion.

(iii) Part D may be used as supervised class exercises.

(iv) Part E is appropriate for homework after appropriate examples. You should discuss these exercises carefully since they illustrate many of the common applications of spheres.

Exercises

Part A

1. (a) Describe the set of all points equidistant from two points – say, A and B.
   (b) Describe the set of all points equidistant from three noncollinear points – say, A, B, and C.
   (c) Tell how to locate the spheres which contain three noncollinear points A, B, and C. [One "locates" a sphere by locating its center and giving its radius.]
2. Show that no sphere is a subset of a plane.
3. Given a sphere, is there a plane which is a subset of it? Explain your answer.
4. Prove Theorem 17–1. [*Hint*: Consider four noncoplanar points – say, P, A, B, and C – and the perpendicular bisectors π_A, π_B, and π_C of $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$, respectively. Why are these three planes not all parallel to the same line?]
5. Can a sphere have more than one center? Explain your answer.
6. Can a sphere contain three collinear points? Explain your answer.

*

Given a plane π, a point C in π, and a positive real number r, the set of all points for π which are at a distance r from C is called *the circle of π with center C and radius r*. We adopt this definition:

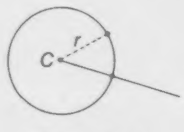


Fig. 17–2

> Definition 17–2 The circle of π with center $C \in \pi$ and radius $r > 0$ is $\{X\colon X \in \pi \text{ and } \|X - C\| = r\}$.

As in the case of spheres, it is easy to see that a center of a circle is contained in the perpendicular bisector of any segment whose end points belong to the circle. Also, it is easy to show that

(a) each half-line with vertex C and in a given plane π contains exactly one point of the circle of π with center C and a given radius,
(b) no circle is a subset of a line, and
(c) each circle contains three noncollinear points.

Analogous to Theorem 17–1 is:

> Theorem 17–2 Three noncollinear points belong to one and only one circle.

Answers for Part A

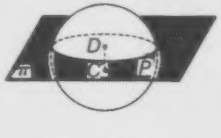
1. (a) The set of all points equidistant from A and B is the perpendicular bisector of $\overleftrightarrow{AB}$ — that is, it is the plane perpendicular to $\overleftrightarrow{AB}$ which contains the midpoint of $\overleftrightarrow{AB}$.
   (b) The set of all points equidistant from A, B, and C is the intersection of the perpendicular bisector of $\overleftrightarrow{AB}$ and the perpendicular bisector of $\overleftrightarrow{AC}$. Given that $\{A, B, C\}$ is noncollinear it follows that the two perpendicular bisectors are not parallel and, so, that their intersection is a line.
   (c) A sphere contains the noncollinear points A, B, and C if and only if its center belongs to the line of intersection of the planes referred to in part (b) and its radius is the distance from its center to, say, A.
2. Let $\mathcal{K}$ be a sphere with center C and radius r, and let π be any plane. Let $(\vec{a}, \vec{b}, \vec{c})$ be a basis for $\mathcal{T}$ consisting of vectors of norm r. If $\mathcal{K} \subseteq \pi$ then $C + \vec{a}$, $C + \vec{b}$, $C + \vec{c}$, and $C - \vec{a}$ all belong to π, from which it follows that $(\vec{b} - \vec{a}, \vec{c} - \vec{a}, -\vec{a}2)$ is linearly dependent. By Theorem 10-18 [or by direct application of the definition of linear independence] this is not the case. So, $\mathcal{K} \not\subseteq \pi$. [Note that it follows from this result — and, in the proof, we have shown explicitly — that any sphere contains four noncoplanar points.]
3. If A and B are points of the sphere with center C and radius r then $d(A, B) \leq d(A, C) + d(B, C) = 2r$. However, for any $r > 0$ there are points in any plane whose distance is greater than 2r. Hence, no sphere has a plane for a subset.
4. Consider four noncoplanar points, P, A, B, and C, and let π_A, π_B, and π_C be the perpendicular bisectors of $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$, respectively. If these planes were parallel to a line — say, ℓ — then ℓ would be perpendicular to $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ and A, B, and C would all belong to the plane through P perpendicular to ℓ. Since this is not the case, it follows that π_A, π_B, and π_C are not all parallel to any line and, so, by Exercise 4 of Part C on page 164, these planes have exactly one point — say, D — in common. The point D is the only point which is equidistant from P, A, B, and C and, so, is the only point which can be the center of a sphere containing those points. Hence, the sphere with center D and radius DP contains P, A, B, and C and is the only sphere which contains those points.
5. By Exercise 2 any sphere contains four noncoplanar points. The argument of Exercise 4 shows that there is just one point which is the center of a sphere which contains these points.
6. Suppose that A, B, and C are three collinear points. It follows that the perpendicular bisectors of $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are two parallel planes. Any sphere containing A, B, and C must have for its center a point of the intersection of these two planes. Since the intersection is $\emptyset$, there is no such sphere.

*

Note that since a circle contains three noncollinear points, a circle determines its plane uniquely. As in the case of a sphere, a circle determines its center uniquely [but, this is yet to be proved]. Since a circle has one and only one center it follows at once that its [numerical] radius is unique. [As in the case of spheres, we shall speak of a radius of a given circle when we wish to describe an interval one of whose end points is the center of the given circle, the other being a point of the circle.]

Part B

1. Given three noncollinear points—say, P, Q, and R—we know that there is exactly one plane which contains them. Tell how to locate the circles which contain P, Q, and R. How many such circles are there?
2. Show that no circle is a subset of a line.
3. Given a circle, is there a line which is a subset of it? Explain your answer.
4. Prove Theorem 17-2.
5. Tell how to locate the spheres which contain a given circle. How many such spheres are there? Is there a largest one? A smallest one?
6. Can a circle have more than one center? Explain your answer.
7. Suppose that π is a plane which intersects a sphere $\mathscr{X}$ with center C and radius $r > 0$. Let D be the foot of the perpendicular from C to π, and let P be any point in $\pi \cap \mathscr{X}$.



   (a) What can you say about CP?
   (b) Suppose that $P = D$. Show that no other point of π is a point of $\mathscr{X}$. [*Hint:* Let $Q \in \pi$ and assume that $Q \neq D$. What kind of triangle is $\triangle CDQ$ and what is $\overline{CQ}$?]
   (c) Suppose that $P \neq D$. Show that $\pi \cap \mathscr{X}$ is the circle of π with center D and radius DP.
8. Given the sphere $\mathscr{X}$ described in Exercise 7, are there planes whose intersection with $\mathscr{X}$ is empty? If so, describe such planes; if not, tell how you know.

*

One consequence of Theorem 17-2 is that there is a unique circle which contains the vertices of a given triangle. Such a circle is said to *circumscribe* the triangle [or to be a *circumcircle* of the triangle]. In this connection we have the following:

Corollary Any triangle has a unique circumscribed circle.

[Does each quadrilateral have a unique circumscribed circle? How about each *plane* quadrilateral? Describe at least one class of plane quadrilaterals which have, and at least one class of plane quadrilaterals which do not have, circumscribed circles.]

The corollary above and your answer to Exercise 1(c) of Part A should suggest that any circle is the intersection of its plane with each of many spheres. From another point of view we have:

Answers for Part B

1. The center of any circle containing P, Q, and R must belong to $\overleftrightarrow{PQR}$ and to the perpendicular bisectors of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{QR}$. Since {P, Q, R} is noncollinear, the two perpendicular bisectors intersect in a line, and this line — being perpendicular to $\overleftrightarrow{PQR}$ — contains exactly one point of $\overleftrightarrow{PQR}$. So, there is one and only one point — say, S — which can be a center of a circle containing P, Q, and R. And the circle of $\overleftrightarrow{PQR}$ with center S and radius SP is the only such circle.
2. [The argument is similar to that for Exercise 2 of Part A. Here, let $(\vec{a}, \vec{b})$ be a basis for the bidirection of the plane π of the circle and be such that $\|\vec{a}\| = r = \|\vec{b}\|$. If the circle is a subset of ℓ then $C + \vec{a}$, $C + \vec{b}$, and $C - \vec{a}$ belong to ℓ and, so, $(\vec{b} - \vec{a}, -\vec{a}2)$ is linearly dependent. It is easy to show directly that the linear independence of $(\vec{a}, \vec{b})$ makes this impossible.]
3. As in the case of a sphere, the distance between any two points of a circle of radius r is at most 2r. But any line contains points whose distance apart is greater than 2r.
4. [This proof has been given in answer for Exercise 1.]
5. Any point on the line perpendicular to the plane of a circle and containing its center is the center of a sphere containing the given circle. [See Exercise 1.] There is no largest such sphere, but the smallest is the one whose center and radius are those of the given circle.
6. By Exercise 2 any circle contains three noncollinear points. So, the argument for Exercise 1 shows that any circle has only one center.
7. (a) CP = r
   (b) When P = D, CD = CP = r. If $Q \in \pi$ and $Q \neq D$ then CDQ is a right triangle with hypotenuse $\overline{CQ}$. Since the hypotenuse of a right triangle is longer than either leg it follows that CQ > r and, so, that Q is not a point of $\mathscr{X}$.
   (c) When $P \neq D$ the points of $\pi \cap \mathscr{X}$ are just those points of π whose distance from D is $\sqrt{r^2 - CD^2}$. [Since D is the point of π nearest to C, CD < CP = r.]
8. Yes.; These are just those planes whose distance from C is greater than r [or, for which CD > r].

Only plane quadrilaterals can have circumscribed circles since only plane quadrilaterals have coplanar vertices. Rectangles have circumscribed circles since their diagonals bisect each other and are of the same length. Parallelograms which are not rectangles do not have circumscribed circles because the perpendicular bisectors of two opposite sides of such a parallelogram are two parallel planes.

Theorem 17-3 The intersection of a plane π and the sphere with center C and radius r is either the empty set, or a set consisting of a single point, or a circle. Specifically, if F is the foot of the perpendicular from C to π and d is the distance between C and π then this intersection is $\emptyset$ if $d > r$, is $\{F\}$ if $d = r$, and is the circle of π with center F and radius $\sqrt{r^2 - d^2}$ if $d < r$.

Part C

1. Prove Theorem 17-3. [*Hint*: Show that A belongs to the intersection if and only if $A \in \pi$ and $\|A - F\|^2 = r^2 - d^2$.]
2. State a theorem like Theorem 17-3 concerning the intersection of a coplanar line and circle.
3. Prove the theorem you stated in Exercise 2.
4. If the word 'coplanar' is deleted from the theorem you stated in Exercise 2, is the resulting statement a theorem? Explain.
5. Discuss the various intersections which might result between a line and a sphere.
6. Consider two spheres with centers C_1 and C_2 and radii r_1 and r_2. Give conditions on C_1C_2 in terms of 'r_1' and 'r_2' for which
   (a) the spheres do not intersect;
   (b) the spheres have exactly one point in common; and
   (c) the spheres have more than one point in common.
   [*Hint*: You will have to consider $r_1 + r_2$ and $|r_1 - r_2|$ to get all of the conditions.]
7. Give some instances which illustrate each of the conditions you stated in Exercise 6.
8. Given that two spheres have more than one point in common, what is their intersection?
9. Can two circles have more than one point in common? More than two points in common? Explain your answers.

*

Several of the results of Part C are summarized in the following theorems:

Theorem 17-4 The intersection of coplanar line l and circle with center C and radius r is either the empty set, or a set consisting of a single point, or a set consisting of two points. Specifically, if F is the foot of the perpendicular from C to l and $d = CF$ then the intersection is $\emptyset$ if $d > r$, is $\{F\}$ if $d = r$, and consists of the two points of l whose distance from F is $\sqrt{r^2 - d^2}$ if $d < r$.

Answers for Part C

1. Suppose that $F \neq C$. In this case $A \in \pi$ if and only if AFC is a right triangle with hypotenuse $\overline{CA}$ — that is, if and only if $CA^2 = FA^2 + d^2$. $A \in \mathcal{K}$ if and only if $CA^2 = r^2$. Hence, $A \in \pi \cap \mathcal{K}$ if and only if $r^2 = FA^2 + d^2$ and $CA^2 = FA^2 + d^2$ — that is, if and only if $\|A - F\|^2 = r^2 - d^2$ and $A \in \pi$. Suppose, next, that $F = C$. In this case $d = 0$ and $A \in \pi \cap \mathcal{K}$ if and only if $A \in \pi$ and $FA^2 = r^2 = r^2 - d^2$. So, in any case, $A \in \pi \cap \mathcal{K}$ if and only if $A \in \pi$ and $\|A - F\|^2 = r^2 - d^2$. In case $d > r$ there is no such point A; in case $d = r$ the only such point is F; in case $d < r$ the points in question are those of the circle of π with center F and radius $\sqrt{r^2 - d^2}$.
2. [The theorem should approximate Theorem 17-4 on page 319.]
3. Proof of Theorem 17-4: Let $\mathcal{K}$ be the circle of σ with center C and radius r, let ℓ be a line in σ, and let F be the foot of the perpendicular from C to ℓ. As in Exercise 1 [with 'ℓ' for 'π'], $A \in \ell \cap \mathcal{K}$ if and only if $A \in \ell$ and $\|A - F\|^2 = r^2 - d^2$. In case $d > r$ there is no such point A; in case $d = r$ the only such point is F; in case $d < r$ A is one or the other of the two points of ℓ at the distance $\sqrt{r^2 - d^2}$ from F.
4. No. A line whose distance d from C is at most r and which is not coplanar with $\mathcal{K}$ may contain no point of $\mathcal{K}$. [Consider, for example, lines parallel to the plane of $\mathcal{K}$ through the point at a distance $d \le r$ "above" the center of $\mathcal{K}$.] Also, a line whose distance from C is less than r may contain just one point of $\mathcal{K}$.
5. Since no sphere contains three collinear points, the intersection of a line and a sphere must be empty or consist of one or of two points. It is easy to see that each of these three possibilities may occur.
6. If two spheres with centers C_1 and C_2 and radii r_1 and r_2 have a point P in common then, by the triangle inequality, $|r_1 - r_2| \le C_1C_2 \le r_1 + r_2$. [In Theorem 14-1 let $\vec{a} = P - C_1$ and $\vec{b} = C_2 - P$ so that $\|\vec{a}\| = r_1$, $\|\vec{b}\| = r_2$, and $C_2 - C_1 = \vec{a} + \vec{b}$.] On the other hand, if $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ then there is, on each side of $\overleftrightarrow{C_1C_2}$ a point P such that $C_1P = r_1$ and $C_2P = r_2$ — that is, a point P common to the two spheres. [For this, see Theorem 16-3.] Also, if $C_1C_2 = r_1 + r_2$ then, by Theorem 14-3, any common point must belong to $\overline{C_1C_2}$ and, as is easily seen, the only such point is $C_1 + (C_2 - C_1)[r_1/(r_1 + r_2)]$. Finally, suppose that $C_1C_2 = |r_1 - r_2|$. In this case, since we are dealing with two spheres $r_1 \neq r_2$. [For, if $r_1 = r_2$ then $C_1 = C_2$.] If $r_1 \neq r_2$, the spheres have only $C_1 + (C_2 - C_1)[r_1/(r_1 - r_2)]$ in common.

   As a consequence of the preceding we see that

   (a) the spheres do not intersect if and only if $C_1C_2 < |r_1 - r_2|$ or $C_1C_2 > r_1 + r_2$;

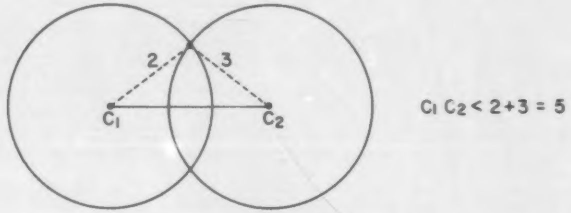
   (b) the spheres have exactly one point in common if and only if $C_1C_2 = |r_1 - r_2|$ or $C_1C_2 = r_1 + r_2$; and

   (c) the spheres have more than one point in common if and only if $|r_1 - r_2| < C_1C_2 < r_1 + r_2$.

   [The preceding argument holds equally well for two coplanar circles. Merely replace 'spheres' by 'circles'.]

Answers for Part C [cont.]

7. [The instances which the students generate can be verified in the appropriate parts of the answers for Exercise 6. It is likely that many students will not have the conditions involving '$r_1 - r_2$'. Making sketches on the chalkboard can illustrate the locations of points of intersection and can serve to reveal those conditions involving '$r_1 - r_2$'. For example, given $r_1 = 2$ and $r_2 = 3$, it is very natural to picture C_1 and C_2 as follows to illustrate intersecting spheres (or, coplanar circles):



Now, if you imagine "sliding" the smaller sphere towards the larger one, you can see that when C_1 gets within 1 (that is, within 3 - 2) of C_2 the spheres (or, circles) will not have points in common, for the smaller sphere will be "inside" the larger one. This seems to be a nice example to illustrate on an overhead projector.]

8. Two spheres which intersect in more than one point have a circle for their intersection. [This can be proved by making use of isometries. Suppose that P and Q are two points of the intersection. Since $PC_1 = QC_1$ and $PC_2 = QC_2$ there is an isometry f which leaves C_1 and C_2 fixed and maps P on Q (Theorem 14-29). Since f leaves each point of $\overleftrightarrow{C_1C_2}$ fixed and preserves perpendicularity the foot F of the perpendicular from P to $\overleftrightarrow{C_1C_2}$ is also the foot of the perpendicular from Q to $\overleftrightarrow{C_1C_2}$. Since f preserves distances, $FP = FQ$. It follows that the intersection of the spheres is the circle, of the plane through F perpendicular to $\overleftrightarrow{C_1C_2}$, whose center is F and whose radius is FP. (That any point of this circle belongs to both spheres follows easily from the Pythagorean theorem.)]

9. Two circles may have two points in common. They cannot have three since three points of a circle are noncollinear and are contained in no other circle.

Answers for Part D

1. Let $\mathcal{K}$ be a sphere with center C and radius r and let f be any isometry. Since f preserves distances it is clear that f maps each point $P \in \mathcal{K}$ on a point f(P) which belongs to the sphere $\mathcal{K}_1$ with center f(C) and radius r. If, now, Q is any point of $\mathcal{K}_1$, $f^{-1}(Q) \in \mathcal{K}$ [for the same reason that $f(P) \in \mathcal{K}_1$] and $f(f^{-1}(Q)) = Q$. So, f maps $\mathcal{K}$ onto $\mathcal{K}_1$.

2. Let $\mathcal{K}$ be a circle of the plane π with center C and radius r and let f be any isometry. As in Exercise 1, f maps $\mathcal{K}$ into a circle $\mathcal{K}_1$ of the plane $f(\pi)$ with center f(C) and radius r. As before, since f^{-1} is an isometry, f^{-1} maps $\mathcal{K}_1$ into $\mathcal{K}$. So, f maps $\mathcal{K}$ onto $\mathcal{K}_1$.

3. It should be clear from Exercises 1 and 2 that congruent spheres or circles have the same radius. Suppose, now, that $\mathcal{K}_1$ and $\mathcal{K}_2$ are spheres with centers C_1 and C_2 and the common radius r. Let f be any isometry — say, $C_2 - C_1$ — which maps C_1 on C_2. By Exercise 1, f maps $\mathcal{K}_1$ onto a sphere with center C_2 and radius r. But, $\mathcal{K}_2$ is the only such sphere. Hence, f maps $\mathcal{K}_1$ onto $\mathcal{K}_2$ and, since f is an isometry it follows, by definition that $\mathcal{K}_1$ and $\mathcal{K}_2$ are congruent.

   Suppose, on the other hand, that $\mathcal{K}_1$ and $\mathcal{K}_2$ are circles of π_1 and π_2 with centers C_1 and C_2 and common radius r. Let f be any isometry which maps π_1 onto π_2 and C_1 on C_2. [For example, f may be the resultant of an isometry g which maps π_1 onto π_2 followed by the translation $C_2 - g(C_1)$.] By Exercise 2, f maps $\mathcal{K}_1$ onto a circle of π_2 with center C_2 and radius r. But, $\mathcal{K}_2$ is the only such circle.

4. (a) Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be spheres with centers C_1 and C_2 and radii r_1 and r_2, respectively. Let $\mathcal{K}'$ be the sphere with center C_1 and radius r_2. The uniform stretching g about C_1 with stretching factor r_2/r_1 maps $\mathcal{K}_1$ onto $\mathcal{K}'$. By Exercise 3 there is an isometry f which maps $\mathcal{K}'$ onto $\mathcal{K}_2$. Since the similitude $f \circ g$ maps $\mathcal{K}_1$ onto $\mathcal{K}_2$ it follows that $\mathcal{K}_1$ and $\mathcal{K}_2$ are similar. [The proof of the corresponding result for circles is very similar to this.]

   (b) r_1/r_2

5. (a) Suppose that P and Q are on the same side of ℓ. It follows, by definition, that P and Q are coplanar with ℓ and $\overline{PQ} \cap \ell = \emptyset$. Let f be any isometry. Since f maps planes onto planes and lines onto lines, f(P) and f(Q) are coplanar with the line $f(\ell)$. Since f maps $\overline{PQ}$ onto $\overline{f(P)f(Q)}$ and is one-to-one $\overline{f(P)f(Q)} \cap f(\ell) = \emptyset$. So, by definition, f(P) and f(Q) are on the same side of $f(\ell)$.

   (b) Suppose that P and Q are on opposite sides of ℓ. It follows, by definition, that neither P nor Q belongs to ℓ and that $\overline{PQ} \cap \ell \neq \emptyset$. Arguing as in part (a) it follows that neither f(P) nor f(Q) belongs to the line $f(\ell)$ and that $\overline{f(P)f(Q)} \cap \ell \neq \emptyset$. So, by definition, f(P) and f(Q) are on opposite sides of $f(\ell)$.

6. (a) $\sqrt{21}$ and 3, respectively. (b) $\sqrt{21}/3$

Theorem 17-5 Two spheres [or: coplanar circles] with centers C_1 and C_2 and radii r_1 and r_2 intersect if and only if

$$|r_1 - r_2| \le \|C_1 - C_2\| \le r_1 + r_2.$$

Part D

1. Show than any isometry maps a sphere onto a sphere whose center is the image of the center of the given sphere. [*Hint*: Show, first, that any isometry maps a sphere *into* a sphere. Then, use the fact that the inverse of an isometry is an isometry.]
2. Repeat Exercise 1 with 'circle' in place of 'sphere'.
3. Prove:

Theorem 17-6 Two spheres [or: circles] are congruent if and only if they have the same radius.

4. (a) Prove this corollary to Theorem 17-6.

Corollary Any two spheres [or: circles] are similar in the ratio of their radii.

(b) Given that spheres $\mathscr{K}_1$ and $\mathscr{K}_2$ have radii r_1 and r_2, respectively, what is the ratio of similitude of $\mathscr{K}_1$ to $\mathscr{K}_2$?

5. (a) Show that any isometry maps two points which are on the same side of a line on two points which are on the same side of the image of the line.
   (b) Repeat (a) with 'opposite sides' in place of 'the same side'.
6. Suppose that two parallel planes—say, π_1 and π_2—intersect a sphere with center C and radius 5 and that the distances from C to π_1 and π_2 are 2 and 4, respectively.
   (a) Compute the radii of the circles of intersection of π_1 and π_2 with the sphere.
   (b) What is the ratio of similitude of the larger to the smaller of these circles?
7. Given $\triangle ABC$, consider $\triangle MNP$ whose vertices M, N, and P are the midpoints of the sides of $\triangle ABC$. Suppose that $\mathscr{K}_1$ and $\mathscr{K}_2$ are the circumscribing circles of $\triangle ABC$ and $\triangle MNP$, respectively. What is the ratio of similitude of $\mathscr{K}_1$ to $\mathscr{K}_2$?

Part E

In these exercises, all coordinates are given with respect to an orthonormal coordinate system with origin O.

1. In each of the following, you are given the radius of a sphere with center O. Give equations which describe the spheres.
   (a) 9 (b) 3 (c) 4 (d) 16 (e) 6

Answers for Part D [cont.]

7. 2/1 [$\triangle ABC$ is similar to $\triangle MNP$, the ratio of similitude being 2. Students should guess that distances which — like the circumradius — are defined for all triangles have the same ratio for similar triangles as do the sides of these triangles. Using the formula in Exercise 1 of Part D on page 334, students can, at that time, check this in the case of the circumradius. The similar property of the inradius is established by the results in both Exercise 2 and Exercise 7 of Part C on page 329.]

Answers for Part E

[Preliminary remark: X is on the sphere with center C and radius r if and only if $\|X - C\| = r$. Given that X and C have coordinates (x_1, x_2, x_3) and (c_1, c_2, c_3), respectively, X is on the sphere with center C and radius r if and only if

$$(x_1 - c_1)^2 + (x_2 - c_2)^2 + (x_3 - c_3)^2 = r^2.$$

In the answers which follow, we express the results in terms of (x, y, z)-coordinates instead of (x_1, x_2, x_3)-coordinates. It is reasonable to do at least Exercises 1 and 2 in class, while assigning Exercises 3 and 4 as part of a homework lesson.]

1. Each of the equations may begin '$x^2 + y^2 + z^2 =$'. The appropriate endings are:

   (a) 81 (b) 9 (c) 16 (d) 256 (e) 36

2. Suppose that A has coordinates (3, 0, 0).
   (a) Write an equation which describes the sphere with center A and radius 9.
   (b) Do the spheres described in (a) and Exercise 1(a) intersect? Explain your answer.
   (c) Which of the spheres described in Exercises 1(b) - 1(e) intersect the sphere described in (a), and which do not? Explain your answers.
3. Suppose that B has coordinates (3, 4, 5).
   (a) Write an equation which describes the sphere with center B and radius 6.
   (b) What are the coordinates of the points in which the sphere described in (a) intersects the first coordinate axis? The second coordinate axis? The third coordinate axis?
   (c) Do the spheres described in (a) and Exercise 2(a) intersect? Explain your answer.
4. Suppose that C has coordinates (5, −2, 1).
   (a) Write an equation which describes the sphere with center C and radius 6.
   (b) Let $\vec{a}$ be the translation from O to C. That is, let $\vec{a} = C - O$. Show that each point of the sphere described in Exercise 1(e) maps on a point of the sphere described in (a).
   (c) Is there a point of the sphere described in (a) which is the image under $\vec{a}$ of two points of the sphere described in Exercise 1(e)? Explain your answer.
   (d) Describe a translation which maps the sphere described in (a) onto the sphere described in Exercise 3(a).
5. Consider the equation:

$$x^2 + 6x + y^2 - 4y + z^2 + 8z = 7$$

By "completing the square" we see that this equation is equivalent to:

$$(x + 3)^2 + (y - 2)^2 + (z + 4)^2 = 36 \qquad \text{[Explain.]}$$

Thus, the given equation is one of a sphere whose center has coordinates (−3, 2, −4) and whose radius is 6.

Each of the following is an equation of a sphere. In each case, find the radius and the coordinates of the center of the sphere.

(a) $x^2 + 4x + y^2 - 6y + z^2 + 16z = 23$
(b) $x^2 + y^2 + 6y + z^2 + 8z = 25$
(c) $x^2 + 3x + y^2 + 3y + z^2 - 3z = \frac{9}{4}$
(d) $x^2 + 2x + y^2 - 2y + z^2 = 98$
(e) $x^2 + 6x + y^2 + z^2 - 6z = 18$
(f) $2x - x^2 + 4y - y^2 - z^2 = -45$

Answers for Part E [cont.]

2. (a) $(x - 3)^2 + y^2 + z^2 = 81$ [or: $x^2 - 6x + y^2 + z^2 = 72$]
   (b) Yes. $\|A - O\| = 3$ and, so, $9 - 9 = 0 \le 3 = \|A - C\| \le 9 + 9 = 18$. Thus, by Theorem 17-5, the given spheres intersect.
   (c) The sphere described in 1(e) intersects the sphere described in (a) because $|9 - 6| \le \|A - O\| \le 9 + 6$; the spheres described in 1(b) - 1(d) do not intersect the sphere described in (a) because, in each case, the inequation of Theorem 17-5 is not satisfied.
3. (a) $(x - 3)^2 + (y - 4)^2 + (z - 5)^2 = 36$ [or: $x^2 - 6x + y^2 - 8y + z^2 - 10z + 14 = 0$; encourage the first answer.]
   (b) The given sphere does not intersect the first coordinate axis because there are no real numbers which satisfy the equation '$(x - 3)^2 + (0 - 4)^2 + (0 - 5)^2 = 36$'; it intersects the second coordinate axis in the points with coordinates $(0, 4 + \sqrt{2}, 0)$ and $(0, 4 - \sqrt{2}, 0)$ because the latter are the only coordinate triples which satisfy the system of equations: $(x - 3)^2 + (y - 4)^2 + (z - 5)^2 = 36$, $x = 0$, and $z = 0$; it intersects the third coordinate axis in the points with coordinates $(0, 0, 5 + \sqrt{11})$ and $(0, 0, 5 - \sqrt{11})$ because the latter are the only coordinate triples which satisfy the system of equations: $(x - 3)^2 + (y - 4)^2 + (z - 5)^2 = 36$, $x = 0$, and $y = 0$.
   (c) The spheres intersect. Their radii are 9 and 6 and the distance between their centers is $\sqrt{41}$. Clearly, $|9 - 6| \le \sqrt{41} \le 9 + 6$. [For example, $6 < \sqrt{41} < 7$; while $3 < 6$ and $7 < 15$.]
4. (a) $(x - 5)^2 + (y + 2)^2 + (z - 1)^2 = 36$
   (b) $\vec{a}$ maps the center of the sphere of Exercise 1(e) on the center of the sphere of Exercise 4(a). Let P belong to the sphere of 1(e) and assume that P has coordinates (p_1, p_2, p_3). Then, $(p_1)^2 + (p_2)^2 + (p_3)^2 = 36$ and $P + \vec{a}$ has coordinates $(p_1 + 5, p_2 - 2, p_3 + 1)$. Note, then, that $P + \vec{a}$ belongs to the sphere of 4(a) because $(p_1 + 5 - 5)^2 + (p_2 - 2 + 2)^2 + (p_3 + 1 - 1)^2 = (p_1)^2 + (p_2)^2 + (p_3)^2 = 36$. Hence, $\vec{a}$ maps each point of the sphere described in 1(e) on a point of the sphere described in 4(a).
   (c) No, because any translation is a one-to-one mapping.
   (d) The translation whose components are (−2, 6, 4), for it maps C on B and, so, maps the sphere described in (a) onto the sphere with center B and radius 6, which is the sphere described in 3(a).
5. (a) 10; (−2, 3, −8) (b) $5\sqrt{2}$; (0, −3, −4)
   (c) 3; (−3/2, −3/2, 3/2) (d) 10; (−1, 1, 0)
   (e) 6; (−3, 0, 3) (f) $5\sqrt{2}$; (1, 2, 0)

6. Which of the spheres described in Exercise 5 are congruent? Which are similar?
7. Which of the following equations describe spheres? For each which does, give the sphere's radius.
   (a) $x^2 + y^2 + z^2 - 2x + 4y - 6z = 10$
   (b) $x^2 + y^2 + z^2 - 2x + 4y - 6z = -14$
   (c) $x^2 + y^2 + z^2 - 2x + 4y - 6z = 18$
   (d) $x^2 + 2y^2 + z^2 - 2x + 8y - 6z = 22$

17.02 Arcs and Chords

An interval whose end points belong to a circle is called *a chord* of the circle. A chord of a circle which contains the center of the circle is *a diameter* of the circle. Any line—say, *l*—which contains two points of a circle separates the circle into two arcs which are on opposite sides of *l*. And, the chord whose endpoints are those points of *l* is said to *subtend* each of the arcs. In case the chord is a diameter, the subtended arcs are called *semicircles*; in case the chord is not a diameter, the arc on the side opposite the center of the circle is *a minor arc* and the arc on the same side as the center is *a major arc*.

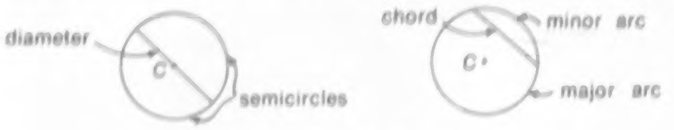


Fig. 17–3

By Exercise 2 of Part D, above, we have that any isometry maps a circle onto a circle whose radius is that of the given circle and whose center is the image of the center of the given circle. And, by Exercise 5 of Part D, we have that any isometry maps two points which are on the same side of the line on two points which are on the same side of the image of that line. Thus, we have the following:

> Theorem 17–7 Any isometry maps a circle onto a congruent circle, mapping minor arcs onto minor arcs, major arcs onto major arcs, and semicircles onto semicircles, mapping centers on centers and endpoints of arcs on endpoints of the image arcs.

Note that if we have an isometry which maps a minor arc of a circle onto a minor arc of a second circle then, by definition, these minor

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Answers for Part E [cont.]

6. The spheres dcscribed in (a) and (d) and those described in (b) and (f) are congruent; any two spheres are similar.

7. The equations in (a) and (c) describe spheres with radii $2\sqrt{6}$ and $4\sqrt{2}$, respectively. The equation in (b) is satisfied only by (1, −2, 3) and, so, describes the set consisting of the single point which has these coordinates. [A set consisting of a single point is sometimes — but not here — called a point circle.] The equation in part (d) is equivalent to '$(x - 1)^2 + 2(y + 2)^2 + (z - 3)^2 = 40$'. The set described would look like a sphere if we pictured it in such a way that the unit of length on the y-axis was $\sqrt{2}$ times that on the x- and z-axes. Using the same unit for graphing on all three axes the set looks like a sphere which has been compressed in the direction of the y-axis. The set in question is called an ellipsoid — in particular, it is an oblate spheroid.

#

Note that arcs, like intervals, do not contain their end points. We shall not need a word — analogous to 'segment' — to refer to arcs together with their end points.

The proof of the part of Theorem 17-7 dealing with minor arcs can be expanded as follows: Let $\mathcal{K}$ be any circle, $\widehat{AB}$ any minor arc of $\mathcal{K}$, and f any isometry. Since f maps $\mathcal{K}$ onto a circle $\mathcal{K}'$ and maps the center of $\mathcal{K}$ on that of $\mathcal{K}'$, it follows that f maps all points of $\mathcal{K}$ on the opposite side of $\overleftrightarrow{AB}$ from its center into the set of points of $\mathcal{K}'$ on the opposite side of $\overleftrightarrow{f(A)f(B)}$ from its center. So, f maps the arc $\widehat{AB}$ of $\mathcal{K}$ into the arc $\widehat{f(A)f(B)}$ of $\mathcal{K}'$. Since, similarly, the isometry f^{-1} maps $\widehat{f(A)f(B)}$ into $\widehat{AB}$ it follows that f maps $\widehat{AB}$ onto $\widehat{f(A)f(B)}$.

The same argument applies equally well to major arcs, and a slight modification applies to semicircles. [In the case of semicircle $\widehat{ABC}$, f maps all points of $\mathcal{K}$ on the same side of $\overleftrightarrow{AC}$ as B into the set of all points of $\mathcal{K}'$ on the same side of $\overleftrightarrow{f(A)f(C)}$ as f(B).]

Answers to questions in text: Yes. By definition two sets are congruent if and only if one is the image of the other under an isometry.

Given two points A and B, the center of any circle containing these points belongs to the perpendicular bisector of $\overline{AB}$; and, given any point of this plane, there is one and only one circle which has this point as center and contains A and B.

If A, B, and K are three noncollinear points then there is exactly one circle which contains them, and K belongs to exactly one of the two arcs of this circle whose end points are A and B.

arcs are congruent. [Is this also true of two major arcs? Of two semicircles?] What we wish to discover are some simple conditions on chords and arcs which will enable us to tell when two given minor arcs, or two chords, are congruent. Before getting at this problem, we will discuss some matters of notation.

Since an interval is determined by its endpoints, we have been able to describe intervals by indicating only their endpoints—as in, say, '$\overline{AB}$'. We cannot proceed in this way in the case of arcs because any two points are the endpoints of many circular arcs. [Given two points *A* and *B*, describe the set of all circles for which *A* and *B* are the endpoints of arcs of these circles.] In fact, given any point *K* which is not collinear with points *A* and *B*, there is exactly one arc with endpoints *A* and *B* which contains *K*. [Explain.] This suggests that we describe an arc by indicating its endpoints together with another point of the arc. For example, we can refer to the arc pictured below as 'arc *AKB*' or as 'arc *BKA*' or, for short, as '$\widehat{AKB}$' or '$\widehat{BKA}$'.

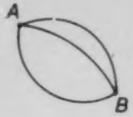


Fig. 17–4

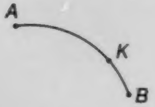


Fig. 17–5

When we are dealing with a given circle containing *A* and *B* and $\overline{AB}$ is not a diameter we shall use '$\widehat{AB}$' to refer to the minor arc of that circle with endpoints *A* and *B*. And we shall, in this case, sometimes use 'major $\widehat{AB}$' to refer to the major arc with endpoints *A* and *B*.

Exercises

Part A

Given a circle of π with center *C* and radius *r*, let $\overline{AB}$ be a chord of the circle and let *F* be the foot of the perpendicular from *C* to $\overleftrightarrow{AB}$.

1. (a) Show that $AB \le 2r$.
   (b) Give a necessary and sufficient condition that $AB = 2r$.
   (c) What is the length of a diameter of the given circle? What does your result in (b) suggest about two diameters of the given circle?
2. (a) Show that *F* is the midpoint of $\overline{AB}$.
   (b) Show that $CF < r$.
3. What must *CF* be in order that $AB = r$?

* * *

Suggestions for the exercises of section 17.02:
(i) Part A may be used for class examples and supervised work.
(ii) Part B may be used as homework.

Answers for Part A

1. (a) By the triangle inequality $AB \le BC + CA$. Since $BC = r = CA$ it follows that $AB \le 2r$.

   (b) Again by the triangle inequality, $AB = 2r$ if and only if $C - B$ and $A - C$ have the same sense. This is the case if and only if $C \in \overline{AB}$.

   (c) $2r$; Any two diameters of a circle are congruent. [Note, also, that it follows from (b) that a chord is a diameter of a circle if and only if its measure is twice the radius of the circle.]

2. (a) If $\overline{AB}$ is a diameter, $C = F$ so that F is the midpoint of $\overline{AB}$. If $\overline{AB}$ is not a diameter, ABC is an isosceles triangle with base $\overline{AB}$ and $\overline{CF}$ is its altitude from C. Thus, F is the midpoint of $\overline{AB}$ in this case, too.

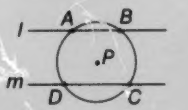
   (b) In case $C = F$, $CF = 0 < r$. In case $C \neq F$, $CF < CB = r$. So, in any case, $CF < r$.

3. In any case, $CF^2 + (\frac{1}{2}AB)^2 = CB^2$. So, $CF^2 + \frac{1}{4}AB^2 = r^2$. Thus, $AB = r$ if and only if $CF^2 = 3r^2/4$ and the latter is the case if and only if $CF = r\sqrt{3}/2$.

4. Assume that D is the point of the circle such that $F \in \overline{CD}$.
   - (a) Show that $ADBC$ is a kite unless $\overline{AB}$ is a diameter.
   - (b) How far must F be from C in order that $ADBC$ be a parallelogram? Is there something special about this parallelogram?
   - (c) How long must $\overline{AB}$ be in order that $ADBC$ be a square?
5. Assume that $\overline{AB}$ is a diameter, and let D be any point on one of the semicircles subtended by $\overline{AB}$. Show each of the following.
   - (a) $\triangle ADB$ is a right triangle.
   - (b) $\triangle ACD$ and $\triangle BCD$ have the same area-measure.
   - (c) The area-measure of $\triangle ADB$ is not greater than r^2.

Part B

1. Let $\overline{AB}$ and $\overline{CD}$ be two chords of the same circle of π with center P. Show that
   - (a) $AB = CD$ if and only if P is equidistant from $\overline{AB}$ and $\overline{CD}$ – that is, if and only if $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{CD})$ – and
   - (b) $AB < CD$ if and only if $d(P, \overleftrightarrow{AB}) > d(P, \overleftrightarrow{CD})$.
2. Let $\overline{AB}$ and $\overline{CD}$ be two diameters of the same circle of π with center P.
   - (a) Show that $ACBD$ is a rectangle.
   - (b) Give a necessary and sufficient condition that $ACBD$ be a square.
3. Suppose that $ABCD$ is a parallelogram. Give a necessary and sufficient condition that A, B, C, and D be contained in a circle.
4. Suppose that $l \parallel m$ and that each of l and m intersects the circle of π with center P in two points, as shown in the picture at the right.



   - (a) Show that $\overline{AD}$ and $\overline{BC}$ are congruent. [*Hint*: Consider the reflection in the perpendicular bisector of $\overline{AB}$.]
   - (b) Show that $\overline{AC}$ and $\overline{BD}$ are congruent.
   - (c) What kind of figure is $ABCD$? [Assume that, as in the figure, A and D are on the same side of $\overleftrightarrow{BC}$.]
   - (d) What can you say about the minor arcs, $\widehat{AD}$ and $\widehat{BC}$, subtended by $\overline{AD}$ and $\overline{BC}$? About $\widehat{BD}$ and $\widehat{AC}$? About major $\widehat{AD}$ and major $\widehat{BC}$? About major $\widehat{BD}$ and major $\widehat{AC}$?
   - (e) What can you say about $\angle BAC$ and $\angle ABD$? About $\angle ACD$ and $\angle BDC$? About $\angle BAC$ and $\angle ACD$?
5. Given that $l \parallel m$, that l intersects a coplanar circle in one point – say, A – and that m intersects the circle in two points – say, C and D.
   - (a) Show that $\overline{AC}$ and $\overline{AD}$ are congruent.
   - (b) What can you say about $\widehat{AC}$ and $\widehat{AD}$? About major $\widehat{AC}$ and major $\widehat{AD}$?

Answers for Part A [cont.]

4. (a) Unless $\overline{AB}$ is a diameter, $\{A, C, B\}$ is noncollinear. In any case $\{A, D, B\}$ is noncollinear. Clearly, unless $\overline{AB}$ is a diameter, $\widehat{ADB}$ is a minor arc and, so, neither $\{D, B, C\}$ nor $\{D, A, C\}$ is collinear. Hence, ADBC is a quadrilateral. Since F is the midpoint of $\overline{AB}$ and $\overline{CD} \perp \overline{AB}$, $\overline{CD}$ is contained in the perpendicular bisector of $\overline{AB}$. So [Theorem 16-22], ADBC is a kite.

   (b) $r/2$; The parallelogram is a rhombus. [Since F is the midpoint of $\overline{AB}$ the diagonals of ADBC bisect each other if and only if F is the midpoint of $\overline{CD}$. In this case, the parallelogram is a rhombus because its diagonals are perpendicular.]

   (c) ADBC cannot be a square. [ADBC could be a square only if it were a rhombus and the latter is the case if and only if $CF = r/2$. But, in this case $AF = r\sqrt{3}/2$ and, so, the diagonals of ADBC are not congruent.]

5. (a) Since $\{A, B, D\}$ is noncollinear, ABD is a triangle. Since its side $\overline{AB}$ has measure twice that of the median $\overline{DC}$ to that side it follows [by Theorem 14-11] that $\triangle ABD$ is a right triangle with hypotenuse $\overline{AB}$.

   (b) $\triangle ACD$ and $\triangle BCD$ have congruent bases $\overline{AC}$ and $\overline{BC}$ and have the same altitude from D. Hence, $\triangle ACD$ and $\triangle BCD$ have the same area-measure.

   (c) The measure of the altitude from D is at most r and the measure of $\overline{AB}$ is 2r. So, the area-measure of $\triangle ADB$ is at most $r(2r)/2$ — that is, it is at most r^2.

Answers for Part B

1. The measure of a chord which is at the distance d from the center of a circle of radius r is $2\sqrt{r^2 - d^2}$. Hence, (a), chords of a given circle have the same measure if and only if they are equidistant from the center and, (b), shorter chords are farther from the center and chords farther from the center are shorter.

2. (a) Since its four vertices belong to a circle, ACBD is a quadrilateral. Since its diagonals bisect each other, quadrilateral ACBD is a parallelogram. Since its diagonals are congruent, parallelogram ACBD is a rectangle [Corollary to Theorem 16-23].

   (b) That $\overline{AB} \perp \overline{CD}$ is a necessary and sufficient condition that ACBD be a square. [Another is that $AC = r\sqrt{2}$.]

3. That parallelogram ABCD is a rectangle is a necessary and sufficient condition that A, B, C, and D are contained in a circle [for short, that {A, B, C, D} is concyclic]. Another is that AC = BD.]

Answers for Part B [cont.]

4. (a) The perpendicular bisector of $\overline{AB}$ and that of $\overline{CD}$ both contain P and are parallel because $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. So, the perpendicular bisector of $\overline{AB}$ is the perpendicular bisector of $\overline{CD}$. It follows that the reflection in this plane maps A on B and D on C and, so, maps $\overline{AD}$ onto $\overline{BC}$. Since a plane reflection is an isometry it follows that $\overline{AD} \cong \overline{BC}$. [Note, for part (d), that this reflection leaves P fixed and, so, maps the circle onto itself.]

   (b) Since the reflection discussed in part (a) maps A on B and C on D it follows that it maps $\overline{AC}$ onto $\overline{BD}$. So, as before, $\overline{AC} \cong \overline{BD}$.

   (c) ABCD is an isodiagonal trapezoid and so, by Theorem 16-23 is either an isosceles trapezoid or a rectangle.

   (d) Congruent [in each case] because an isometry which maps the circle onto itself and maps $\overline{AD}$ onto $\overline{BC}$ and $\overline{AC}$ onto $\overline{BD}$ maps the minor or major arc subtended by $\overline{AD}$ or $\overline{AC}$ onto the corresponding arc subtended by $\overline{BC}$ or $\overline{BD}$, respectively.

   (e) $\angle BAC \cong \angle ABD$ and $\angle ACD \cong \angle BDC$ because the isometry of part (a) maps each of either pair of angles onto the other. $\angle BAC$ and $\angle ACD$ are either congruent or supplementary according as B and D are on opposite sides of $\overleftrightarrow{AC}$ or on the same side of $\overleftrightarrow{AC}$. [Other pairs of congruent angles are $(\angle BAD, \angle ABC)$ and $(\angle BCD, \angle ADC)$.]

5. By Theorem 17-4 A is the foot of the perpendicular from P to ℓ. Since $m \parallel \ell$ it follows that $\overline{PA} \perp m$ and, so, is contained in the perpendicular bisector of $\overline{CD}$. The reflection f in this latter plane leaves A fixed and interchanges C and D. Hence,

   (a) $\overline{AC} \cong \overline{AD}$ because f maps $\overline{AC}$ onto $\overline{AD}$, and

   (b) $\widehat{AC} \cong \widehat{AD}$ because f maps the circle onto itself and, so, in mapping $\overline{AC}$ onto $\overline{AD}$, maps $\widehat{AC}$ onto $\widehat{AD}$. The same statement holds for major $\widehat{AC}$ and major $\widehat{AD}$.

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6. (a) It is perpendicular to each of the given parallel lines. Also, it is a diameter of the circle.

   (b) Each of the arcs is a semicircle [and, so the two are congruent].

*

The fact — on which the proof of Theorem 17-9 depends — that an arc determines its end points can be proved but, like that of the similar Theorem 16-22(a), is tedious. The same applies to Theorem 17-10. We shall leave them unproved even in this commentary.

The explanations asked for in the statement of the proof of Theorem 17-9 go as follows:

There is an isometry which maps A on C, B on D and P_1 on P_2 because AC = BD, $AP_1 = CP_2$, and $BP_1 = DP_2$. [See Theorem 14-29 or the lemma on page 190.] This isometry maps $\mathcal{K}_1$ onto $\mathcal{K}_2$ because it maps $\mathcal{K}_1$ onto a circle of $\overline{CP_2D}$ with center at P_2 and radius that of $\mathcal{K}_1$, and $\mathcal{K}_2$ is the only such circle.

In the proof of the only if-part of Theorem 17-9, the isometry f maps any three noncollinear points of $\widehat{AB}$ on three noncollinear points of $\widehat{CD}$ and, since it maps $\mathcal{K}_1$ onto some circle, must map $\mathcal{K}_1$ onto the circle determined by the latter three points. This is, of course, $\mathcal{K}_2$.

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6. Suppose that each of two parallel lines intersects a coplanar circle in exactly one point.
   (a) What can you say about the chord of the circle whose endpoints are the points common to the circle and the given parallel lines?
   (b) What can you say about the arcs subtended by the chord described in (a)?

*

The result in Exercise 1, above, is summarized in:

> **Theorem 17-8** Given two noncongruent chords of the same circle, the shorter of the chords is farther from the center of the circle.

The following theorem—whose proof we shall leave incomplete—states a useful relation between chords and arcs:

> **Theorem 17-9** Minor [or: Major] arcs of congruent circles are congruent if and only if the chords which subtend them are congruent.

The proof of the if-part is not difficult. For that of the only if-part we need to know that, as we have shown for intervals in Theorem 7-22(a), no arc has two pairs of endpoints. It is this that we shall not take time to prove. Suppose, then that $\mathscr{K}_1$ and $\mathscr{K}_2$ are congruent circles with centers P_1 and P_2 To prove the if-part of the theorem suppose that $\overline{AB}$ and $\overline{CD}$ are congruent chords which are not diameters. It follows that there is an isometry—say, f—which maps A on C, B on D, and P_1 on P_2 [Explain.] and, so, maps $\mathscr{K}_1$ onto $\mathscr{K}_2$ [Why?]. By Theorem 17-7 f maps $\widehat{AB}$ onto $\widehat{CD}$ and major $\widehat{AB}$ onto major $\widehat{CD}$. To prove the only if-part of Theorem 17-9 suppose that $\widehat{AB}$ and $\widehat{CD}$ are congruent minor arcs of the congruent circles $\mathscr{K}_1$ and $\mathscr{K}_2$ and that f is an isometry which maps $\widehat{AB}$ onto $\widehat{CD}$. It follows that f maps $\mathscr{K}_1$ onto $\mathscr{K}_2$. [Why?] Since f also maps $\widehat{AB}$ onto $\widehat{f(A)f(B)}$, and since an arc determines its endpoints, it follows that $\{f(A), f(B)\} = \{C, D\}$. Hence, $\overline{AB} \cong \overline{CD}$. [The same argument holds if we concern ourselves with major $\widehat{AB}$ and major $\widehat{CD}$.

Here is another useful theorem for which we shall give no proof. It is, at least, intuitively obvious.

> **Theorem 17-10** An arc $\widehat{ABC}$ is a union $\widehat{AB} \cup \{B\} \cup \widehat{BC}$ where $\widehat{AB}$ and $\widehat{BC}$ have no common point and each may be either a minor arc, a semicircle, or a major arc.

17.03 Tangents and Secants

We have seen that the intersection of a coplanar line and circle is either the empty set, or a set consisting of a single point, or a set consisting of two points. Given that a coplanar line and circle have exactly one point in common, the line is said to be *a tangent line* [or: a tangent] of the circle. The common point is sometimes called *the point of tangency*. If a coplanar line and circle have two points in common, the line is called *a secant line* [or: a secant] of the circle. For convenience, we give the following definition:

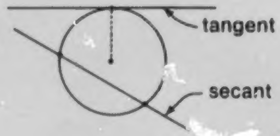


Fig. 17–6

> **Definition 17–3** Given a coplanar line l and circle $\mathscr{K}$,
> (a) l is a tangent of $\mathscr{K}$ if and only if $l \cap \mathscr{K}$ consists of exactly one point, and
> (b) l is a secant of $\mathscr{K}$ if and only if $l \cap \mathscr{K}$ consists of two points.

[Discuss the importance of the word 'coplanar' to part (a) of the definition. Is the word 'coplanar' equally important to part (b)?] Given that a line is a tangent of a circle we may say that the line is tangent to the circle or that the circle is tangent to the line.

The following theorem is a consequence of Definition 17–3 and Theorem 17–4:

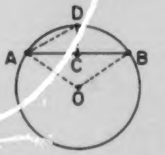
> **Theorem 17–11** A coplanar line is tangent to a circle at a given point of the circle if and only if the line contains the point and is perpendicular to the radius at that point.

Exercises

Part A

1. Suppose that l is the tangent at T to a circle with center P and radius 5. Let S be a point of l such that $ST = 12$, and let R be the point of the circle contained in $\overline{SP}$. Compute the following.
   (a) SR (b) $\cos \angle SPT$ (c) $\sin \angle SPT$
   (d) RT (e) area of $\triangle SRT$ (f) $d(T, \overleftrightarrow{SP})$

Sample Quiz



Given that C is the midpoint of $\overline{AB}$, that D is the midpoint of $\widehat{AB}$, that O is the center of the circle, and that $AB = 36$ and $CD = 9$, as shown in the picture at the right. Compute the following.

1. Radius of the circle
2. Measure of chord $\overline{AD}$
3. $\cos \angle AOB$
4. Area of $\triangle AOB$

Key to Sample Quiz

1. 45/2
2. $9\sqrt{5}$
3. −7/25
4. 243

*

There are many lines which have only a given point P of a circle $\mathscr{K}$ in common with $\mathscr{K}$. Only one of these is coplanar with $\mathscr{K}$. [It is, by Theorem 17-4, the line through P, and coplanar with $\mathscr{K}$, which is perpendicular to the radius of $\mathscr{K}$ to P.] On the other hand, any line which contains two points of $\mathscr{K}$ is coplanar with $\mathscr{K}$. So, 'coplanar' is not needed in the definition of 'secant'.

* * *

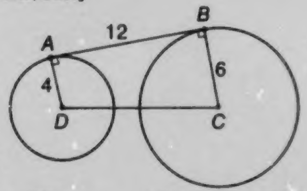
Suggestions for the exercises of section 17.03:

(i) Use Part A for class demonstration.
(ii) Parts B, C, and D are appropriate for homework, but will probably require more than one such assignment.

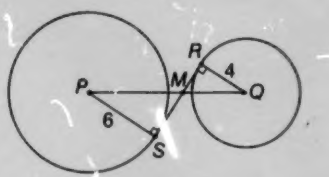
Answers for Part A

1. (a) 8 [$SR = SP - RP = 13 - 5$] (b) 5/13 (c) 12/13
   (d) $20/\sqrt{13}$ (e) 240/13 (f) 60/13

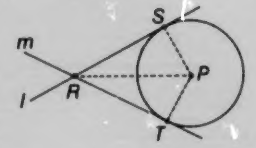
2. Given the information from Exercise 1, let Q be such that $\overline{RQ}$ is a diameter of the given circle.
   (a) Find cos $\angle STR$ and cos $\angle TQS$. What can you say about $\angle STR$ and $\angle TQS$? About $\triangle SRT$ and $\triangle STQ$?
   (b) What can you say about $\angle TRQ$ and $\angle TQR$?
   (c) Show that ST is the mean proportional between SQ and SR. [That is, show that $SQ/ST = ST/SR$.]



3. Suppose that $\overleftrightarrow{AB}$ is tangent at A and B, respectively, to coplanar circles with centers D and C, and that $DA = 4$, $BC = 6$, and $AB = 12$, as shown in the picture at the right. Compute the following.
   (a) DC (b) AC (c) BD (d) area of $ABCD$
   [*Hint for (a)*: Consider the line through D parallel to $\overleftrightarrow{AB}$.]



4. Suppose that $\overleftrightarrow{RS}$ is tangent at R and S, respectively, to coplanar circles with centers Q and P, that $\overline{RS}$ and $\overline{PQ}$ intersect in the point M, and that $PS = 6$, $RS = 10$, and $RQ = 4$, as shown in the picture at the right. Complete each of the following sentences.
   (a) $PRQS$ is a ____ (b) PM/MQ = ____
   (c) PQ = ____ (d) The area of $\triangle PMR$ is ____.



5. Suppose that l and m are two lines which contain R and are tangent at S and T, respectively, to a circle with center P, as shown in the picture at the right.
   (a) Show that $RS = RT$.
   (b) Show that $\overline{ST} \perp \overline{RP}$.
   (c) Given that $RT = 8$ and $PT = 5$ compute ST, cos $\angle SPT$, cos $\angle SRT$, and $d(S, \overleftrightarrow{RP})$.
   (d) Find RS and RP given that $\triangle RST$ is equilateral and $PT = 10$.

Part B

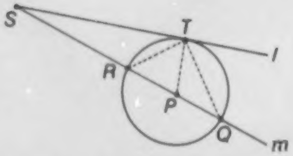
1. Suppose that l and m are tangents to a circle at A and B, respectively.
   (a) Given that l and m are parallel, what can you say about $\overline{AB}$?
   (b) Given that $\overline{AB}$ is a diameter of the circle, what can you say about l and m?
   (c) Given that $\overline{AB}$ is not a diameter, what can you say about l and m?

Answers for Part A [cont.]

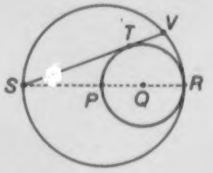
2. (a) $3/\sqrt{13}$, $3/\sqrt{13}$; $\angle STR \cong \angle TQS$; $\triangle SRT$ and $\triangle STQ$ are similar.
   (b) $\angle TRQ$ and $\angle TQR$ are complementary.
   (c) Since, by a.a., $SRT \longleftrightarrow STQ$ is a similarity, $SQ/ST = ST/SR$. [So, $ST = \sqrt{SQ \cdot SR}$.]
3. (a) $2\sqrt{37}$ (b) $6\sqrt{5}$ (c) $4\sqrt{10}$ (d) 60
4. (a) trapezoid (b) 3/2 (c) $10\sqrt{2}$ (d) 12
   [In part (d) the area-measure of $\triangle PMR$ is that of $\triangle PSR$ minus that of $\triangle PSM$.]
5. (a) Since $RS^2 = RP^2 - SP^2 = RP^2 - TP^2 = RT^2$, $RS = RT$.
   (b) By part (a), RSPT is a kite. So, $\overline{RP}$ is contained in the perpendicular bisector of $\overline{ST}$.
   (c) $80/\sqrt{89}$, $-39/89$, $39/89$, $40/\sqrt{89}$ [See Corollary to Th. 15-13.]
   (d) $10\sqrt{3}$, 20 [$(ST/2)RP = 10 \cdot RT$]

Answers for Part B

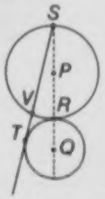
1. (a) $\overline{AB}$ is a diameter of the circle and is perpendicular to both ℓ and m.
   (b) $\ell \parallel m$ and both ℓ and m are perpendicular to $\overline{AB}$.
   (c) $\ell \nparallel m$



2. Assume that l is a tangent at T to a circle with center P, that m is a secant of the circle which intersects the circle at R and Q, passes through P, and intersects l at S, as shown in the picture at the right. Prove the following.
   (a) $\angle STR$ and $\angle TRP$ are complementary angles.
   (b) $\angle TQS$ is congruent to $\angle STR$. [*Hint*: What is a complement of $\angle TQR$?]
   (c) $\triangle SRT$ is similar to $\triangle STQ$.
   (d) $SR \cdot SQ = ST^2$



3. The two circles shown at the right are said to be *internally tangent*. Suppose that P and Q are the centers of the circles, that $\overline{PR}$ is a diameter of the smaller circle, that $\overline{SR}$ is a diameter of the larger circle and that $\overleftrightarrow{SV}$ is tangent at T to the smaller circle.
   (a) Show that $\overline{QT} \parallel \overline{RV}$.
   (b) Given that $SP = 6$, find ST, TV, VR, and $d(T, \overleftrightarrow{SR})$.



4. The two circles shown at the right are said to be *externally tangent*. Assume that P and Q are the centers of the circles, that $\overline{SR}$ is a diameter of the larger circle, that $\overleftrightarrow{SR}$ contains Q, and that $\overleftrightarrow{SV}$ is tangent at T to the smaller circle.
   (a) Show that $\overline{RV} \parallel \overline{QT}$.
   (b) Given that $SP = 6$ and $QR = 2$, find ST, TV, VR, and $d(T, \overleftrightarrow{SR})$.

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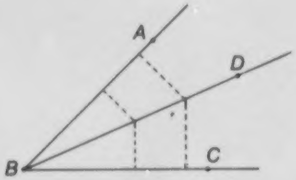

Fig. 17–7

Consider an angle—say, $\angle ABC$. We know that if $\overrightarrow{BD}$ is the bisector of $\angle ABC$ then each point of $\overrightarrow{BD}$ is equidistant from the sides of $\angle ABC$. One consequence of this fact is that each point of $\overrightarrow{BD}$ is the center of a circle which is tangent to the sides of $\angle ABC$. [Explain.] Another is that the point of intersection of the bisectors of the angles of a triangle is the center of a circle which is tangent to the sides of the triangle. [How many such circles are there for a given triangle? How many cir-

Answers for Part B [cont.]

2. (a) Since ∠PTS is a right angle, ∠STR and ∠RTP are complementary. Since PR = PT, ∠RTP = ∠TRP. Hence, ∠STR and ∠TRP are complementary.
   (b) Since RTQ is a right triangle with right angle at T, ∠TQR and ∠TRP are complementary. So, by part (a), ∠TQR and ∠STR are congruent. But, ∠TQR = ∠TQS.
   (c) By part (b) and the a.a. similarity theorem the matching SRT ⟷ STQ is a similarity.
   (d) By part (c), SR/ST = ST/SQ.
3. (a) Since $\overline{QT}$ and $\overline{RV}$ are both perpendicular to $\overline{SV}$ and are coplanar it follows that they are parallel.
   (b) $6\sqrt{2}$, $2\sqrt{2}$, 4, $2\sqrt{2}$
4. (a) Since $\overline{RV}$ and $\overline{QT}$ are both perpendicular to $\overline{ST}$ and are coplanar it follows that they are parallel.
   (b) $8\sqrt{3}$, $8\sqrt{3}/7$, $12/7$, $8\sqrt{3}/7$

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Given a point P of the angle bisector of ∠ABC, the feet of the perpendiculars from P to the sides of ∠ABC are equidistant from P. So, the sides of ∠ABC are tangents at those feet to the circle whose center is P and whose radius is the distance from P to either side [Th. 17-4].

Note that, by Theorem 15-20, the bisectors of the angles of a triangle are concurrent.

There is just one circle which is tangent to the sides of a given triangle. There are, however three other circles [called escribed circles] which are tangent to the lines containing the sides of the triangle. Each has its center at the intersection of the bisectors of two exterior angles.

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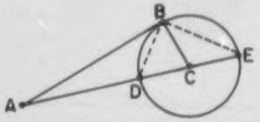
Sample Quiz



Given that $\overleftrightarrow{AB}$ is tangent to the circle with center C, that AB = 8 and that the radius of the circle is 6, determine each of the following.

1. AD 2. AE 3. BD 4. BE

The measure of the common chord of two intersecting circles is 48, and the radii of the circles are 26 and 25.

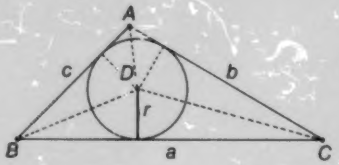
5. Compute the measure of the segment joining the centers of the circles. [Give all possible answers.]
6. Compute the area-measure of the quadrilateral whose vertices are the centers of the given circles and the points of intersection of the circles. [Give all possible answers, assuming that the common chord is a diagonal of the quadrilateral.]

Key to Sample Quiz

1. 4 2. 16 3. $12/\sqrt{5}$ 4. $24/\sqrt{5}$
5. 17 [when centers are on opposite sides of chord]; 3 [when centers are on same side of chord]
6. 408 [when distance between centers is 17]; 72 [when distance between centers is 3]

cles can you find which are tangent to the *lines* containing the sides of a given triangle?] In the next exercises, we get at the problem of computing the radius of the circle whose center is the point of intersection of the angle bisectors of a triangle and which is tangent to the sides of the triangle. This circle is sometimes called *the incircle* of the triangle.

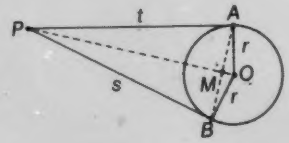
Part C



Suppose that D is the point of intersection of the angle bisectors of $\triangle ABC$ and that r is the radius of the incircle of $\triangle ABC$, as shown in the picture at the right. Let K be the area-measure of $\triangle ABC$ and let $s = \frac{1}{2}(a + b + c)$.

1. Show that $K = rs$. [*Hint*: Consider the triangles, $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$.]
2. Show that $r = K/s$.
3. Find r in each of the following. [*Note*: You will make use of these results in Part D on page 334. So, save your work.]
   (a) $a = 8, b = 6, c = 10$ (b) $a = 8 = b, c = 12$
   (c) $a = b = c = 5$ (d) $a = 15, b = 8, c = 17$
   (e) $a = 6, b = 8, \cos \angle C = \sqrt{3}/2$ (f) $a = 6, b = 8, \cos \angle C = -\sqrt{3}/2$
4. We already have established that $K = rs$ and that $K = \frac{1}{2}ab \sin \angle C$. Show that $r = \dfrac{ab \sin \angle C}{a + b + c}$. State similar results which involve 'sin $\angle A$' and 'sin $\angle B$'.
5. Suppose that $\triangle ABC$ is equilateral, and that $AB = a$. Find r and K in terms of 'a'.
6. Suppose that $\triangle ABC$ is isosceles with base $\overline{AB}$, that $AB = c$ and $BC = a$. Find r and K in terms of 'a' and 'c'.
7. We have seen that $K = \sqrt{s(s - a)(s - b)(s - c)}$. Show that $r = \sqrt{\dfrac{(s - a)(s - b)(s - c)}{s}}$.

Part D



Suppose that $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are tangents at A and B, respectively, to the circle with center O and radius r, that $PA = t$ and $PB = s$, and that M is the point of intersection of $\overline{AB}$ and $\overrightarrow{PO}$, as shown in the picture at the right.

1. Show that
   (a) $OP = \sqrt{r^2 + t^2}$, (b) $t = s$, and (c) M is the midpoint of $\overline{AB}$.

Answers for Part C

1. The sum of the area-measures of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$ is $(ar)/2 + (br)/2 + (cr)/2$ — in other words, it is rs. But, this sum is also the area-measure K of $\triangle ABC$. Hence, $K = rs$.
2. This follows at once from the result of Exercise 1 since $s \neq 0$.
3. (a) 2 [See Exercise 7.] (b) $6/\sqrt{7}$
   (c) $5/(2\sqrt{3})$ (d) 3
   (e) $12/(7 + \sqrt{25 - 12\sqrt{3}})$ (f) $12/(7 + \sqrt{25 + 12\sqrt{3}})$
4. $r = K/s = [\frac{1}{2}ab \sin\angle C]/\frac{1}{2}(a + b + c) = ab \sin\angle C/(a + b + c)$. Similarly, $r = bc \sin\angle A/(a + b + c)$ and $r = ca \sin\angle B/(a + b + c)$.
5. $K = a^2\sqrt{3}/4$, $r = a/(2\sqrt{3})$
6. $K = c\sqrt{4a^2 - c^2}/4$, $r = c\sqrt{4a^2 - c^2}/(4a + 2c)$
7. $r = K/s = \sqrt{s(s - a)(s - b)(s - c)}/s = \sqrt{(s - a)(s - b)(s - c)/s}$ [since $s > 0$].

Answers for Part D

1. (a), (b) By the Pythagorean theorem $OP = \sqrt{r^2 + t^2} = \sqrt{r^2 + s^2}$. Hence, [(b)] $s = t$.
   (c) The altitudes, $\overline{AM}$ and $\overline{BM}$, of the congruent triangles, $\triangle PAO$ and $\triangle PBO$, are congruent. Hence, M is the midpoint of $\overline{AB}$.

2. Express each of the following in terms of 'r' and 't'.
   (a) PM (b) MO (c) AM
   (d) AB (e) $\cos \angle APB$ (f) $\cos \angle AOB$
3. (a) What do the results in Exercises 2(e) and 2(f) tell you about $\angle APB$ and $\angle AOB$?
   (b) Under what conditions on 'r' and 't' is $\angle APB$ acute? Right? Obtuse?
4. Given that $r = 8$ and $t = 15$, find PO, AB, and $\cos \angle AOB$, and tell whether $\angle APB$ is acute, right, or obtuse.
5. Given that $OP = 10$ and $t = 5$, find AB, PB, and $\cos \angle APB$.
6. Given that $r = 8$ and $\cos \angle APB = -\frac{1}{2}$, find AB, OP, and t.
7. Show that $t = r$ if and only if $\angle APB$ is a right angle.
8. Show that $AB = r$ if and only if $r = t\sqrt{3}$.
9. Show that $AB = r$ if and only if $\angle APO \cong \angle AOB$.

17.04 Angles and Circles

An angle which is coplanar with a given circle and whose vertex is the center of the circle is called *a central angle* [of the circle]. It follows, then, that a central angle intersects its circle in two points, and these two points are the endpoints of a chord and of a minor arc. This chord and its related minor arc are said to *subtend* the central angle, and the central angle is said to *intercept* the minor arc.

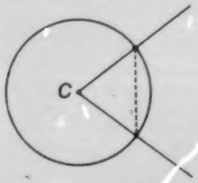


Fig. 17–8

Exercises

Part A

Given a circle of π with center C and radius r, assume that $\overline{AB}$ and $\overline{PQ}$ are chords of the circle and that neither $\overline{AB}$ nor $\overline{PQ}$ are diameters of the circle.

1. (a) Express $\cos \angle ACB$ and $\cos \angle PCQ$ in terms of 'r', 'AB', and 'PQ'.
   (b) Find $\cos \angle BAC$ in terms of 'r' and 'AB'.
   (c) Find $\cos \angle PQC$ in terms of 'r' and 'PQ'.
2. Suppose that $AB = PQ$.
   (a) Show that the central angles $\angle ACB$ and $\angle PCQ$ are congruent.
   (b) Show that $\angle BAC$ is congruent to $\angle PQC$.

Answers for Part D [cont.]

2. (a) $t^2/\sqrt{r^2 + t^2}$ (b) $r^2/\sqrt{r^2 + t^2}$ (c) $tr/\sqrt{r^2 + t^2}$
   (d) $2tr/\sqrt{r^2 + t^2}$ (e) $(t^2 - r^2)/(t^2 + r^2)$ (f) $(r^2 - t^2)/(r^2 + t^2)$
3. (a) $\angle APB$ and $\angle ACB$ are supplementary.
   (b) $r < t$; $r = t$; $r > t$
4. 17, 240/17, −161/289, acute
5. $5\sqrt{3}$, $5\sqrt{3}$, 1/2
6. 8, $16/\sqrt{3}$, $8/\sqrt{3}$
7. By Exercise 2(e), $t = r$ if and only if $\cos \angle APB = 0$ — that is, if and only if $\angle APB$ is a right angle.
8. By Exercise 2(d), $AB = r$ if and only if $2t = \sqrt{r^2 + t^2}$ — that is, if and only if $r = t\sqrt{3}$.
9. Since $\cos \angle APO = t/\sqrt{r^2 + t^2}$ it follows by Exercise 2(f) that $\angle APO \cong \angle AOB$ if and only if $t/\sqrt{r^2 + t^2} = (r^2 - t^2)/(r^2 + t^2)$. Multiplying on both sides with '$\sqrt{r^2 + t^2}$', simplifying, and squaring on both sides, we see that this equation is satisfied if and only if $r = t\sqrt{3}$. But, by Exercise 8 this is the case if and only if $AB = r$.

*

Since two points, one from each side of an angle, and the vertex of the angle are not collinear it follows that the points in which a central angle intersects a circle and the center of the circle are not collinear. Hence, the points in which a central angle intersects a circle are not end points of a diameter.

* * *

Suggestions for the exercises of section 17.04:

(i) Part A and the discussion that follows should be presented by the teacher.
(ii) The experiments in Part B should be conducted individually, and then discussed by the group.
(iii) Part C may be used for homework.
(iv) Part D can be used as a class demonstration.
(v) Part E may be used for homework.
(vi) Following a careful discussion of pages 336-338, Part F may be used as a supervised exercise.
(vii) Part G may be assigned for homework, but should be subsequently discussed in class. This class discussion should include number line graphs of the exercises of Part G as well as similar exercises for other subsets of R. Students are quicker to grasp the ideas of upper bounds, least upper bounds, greatest members, etc., when the discussions are illustrated with number line graphs.

Answers for Part A

1. (a) $\cos \angle ACB = 1 - AB^2/(2r^2)$, $\cos \angle PCQ = 1 - PQ^2/(2r^2)$
   (b) $\cos \angle BAC = AB/(2r)$ (c) $\cos \angle PQC = PQ/(2r)$
2. (a) Since $AB = PQ$ it follows from Exercise 1(a) that $\cos \angle ACB = \cos \angle PCQ$. So, $\angle ACB \cong \angle PCQ$.
   (b) Since $AB = PQ$ it follows from Exercises 1(b) and (c) that $\cos \angle BAC = \cos \angle PQC$. So, $\angle BAC \cong \angle PQC$.

Answers for Part A [cont.]

3. (a) Given that $\angle ACB \cong \angle PCQ$ it follows that these angles have the same cosine and, so, by Exercise 1(a), that $AB = PQ$. Since $AB = PQ$, $\overline{AB} \cong \overline{PQ}$. By Theorem 17-9 it follows that, since $\overline{AB} \cong \overline{PQ}$, $\widehat{AB} \cong \widehat{PQ}$.

   (b) Yes. This follows by parts (b) and (c) of Exercise 1.

4. Given that $AB < PQ$ it follows by Exercise 1(a) that $\cos \angle ACB > \cos \angle PCQ$ and, so, that $\angle ACB$ is smaller than $\angle PCQ$.

5. (a) $7/8$, $17/25$, $2/5$, $25\sqrt{15}/4$, $8\sqrt{21}$

   (b) 16, 12, $-7/25$, $4/5$, $7/25$, $3/5$

   (c) $1/2$, $1/2$, $r\sqrt{3}/2$

*

In Figure 17-9 the inscribed angles are $\angle DCA$, $\angle ACB$, and $\angle BCD$. The intercepted arcs are $\widehat{DA}$, $\widehat{AB}$, and $\widehat{BD}$, respectively.

Students have already used Theorem 14-11 to prove that any angle inscribed in a semicircle is a right angle.

3. (a) Given that $\angle ACB$ is congruent to $\angle PCQ$, what can you say about $\overline{AB}$ and $\overline{PQ}$? About $\widehat{AB}$ and $\widehat{PQ}$? Explain your answers.
   (b) Given that $\angle BAC$ is congruent to $\angle PQC$, is it the case that $\overline{AB}$ and $\overline{PQ}$ are congruent? Explain your answer.

4. Given that $AB < PQ$, which of $\angle ACB$ and $\angle PCQ$ is the larger? Explain your answer.

5. In each of the following, make use of the given information to do the indicated computations.
   (a) $r = 10$, $AB = 5$, and $PQ = 8$. Find $\cos \angle ACB$, $\cos \angle PCQ$, $\cos \angle PQC$, area-measure of $\triangle ABC$, and area-measure of $\triangle PQC$.
   (b) $r = 10$, $d(C, \overleftrightarrow{AB}) = 6$, and $d(C, \overleftrightarrow{PQ}) = 8$. Find AB, PQ, $\cos \angle ACB$, $\cos \angle ABC$, $\cos \angle PCQ$, and $\cos \angle PQC$.
   (c) $r = AB = PQ$. Find $\cos \angle ACB$, $\cos \angle QPC$, and $d(C, \overleftrightarrow{AB})$ in terms of 'r'.

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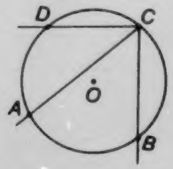
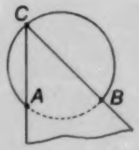


Fig. 17–9

An angle is said to be *inscribed* in a circle if and only if its vertex and a point of each of its sides belong to the circle. The points of the circle which are on the sides of the inscribed angle are the endpoints of two arcs of the circle. One of these arcs is contained in the interior of the inscribed angle, and is said to be *the intercepted arc*. The other arc contains the vertex of the angle and the angle is said to be *inscribed in* this arc. [In Fig. 17–9, there are three inscribed angles pictured with vertex C. Find them and describe their intercepted arcs.]

Consider any angle inscribed in a semicircle. It is not difficult to show that such an inscribed angle is a right angle. And, this being so implies that any two angles inscribed in the same semicircle—or, intercepting the same semicircle—are congruent. It is natural to ask whether this is the case for two angles inscribed in the same major [or, the same minor] arc. For example, it is natural to ask whether the inscribed angles $\angle ACB$ and $\angle ADB$, in Fig. 17–9, are congruent. In the exercises which follow, we shall investigate this question.

Part B

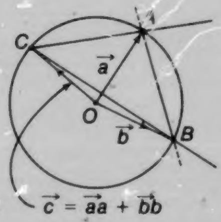


1. Make a tracing of $\angle ACB$ and cut out a wedge-shaped piece of paper as suggested in the picture. Mark the point *A* on one edge of your wedge.
2. Place your wedge on the figure so that the edge marked '*A*' just covers the point marked '*A*' on the figure and so that the vertex of your wedge is on the circle but not at *C*. [*Note*: The mark for *A* on your wedge need not coincide with the mark for *A* on the circle.] What do you notice about the other edge of your wedge?
3. Place your wedge on the figure so that the edge marked '*A*' just covers the point marked '*A*' on the figure and so that the other edge covers the point marked '*B*'. What do you notice about the vertex of your wedge?
4. Make a conjecture about angles which are inscribed in the same arc.

*

The exercises you have just completed should suggest that two angles which are inscribed in the same arc are congruent. Of course, in order to *prove* that this is the case, we must either find an isometry which maps one of the inscribed angles onto the other or we must show that the cosine-value of the first is the cosine-value of the second. We look into the latter possibility in the next exercises.

Part C



Suppose that $\{O, A, B\}$ is noncollinear and that $OA = OB = r$. Then, *A* and *B* are on a circle with center *O* and radius *r*. Let $\vec{a} = A - O$ and $\vec{b} = B - O$, as shown in the picture at the right. Also, let $C = O + \vec{c}$, where $\vec{c} = \vec{a}a + \vec{b}b$, for some *a* and *b*.

1. Show each of the following.
   (a) $C \in \overleftrightarrow{AB}$ if and only if $a + b = 1$.
   (b) *C* is a point of the circle with center *O* and radius *r* if and only if $a^2 + b^2 + 2ab \cos \angle AOB = 1$. [*Hint*: The former is the case if and only if $\vec{c} \cdot \vec{c} = r^2$.]
2. Since $\{O, A, B\}$ is noncollinear it follows from Exercise 1(a) that $a + b \neq 1$. What do you think is the case with respect to *O* and *C* if $a + b > 1$? If $a + b < 1$?
3. (a) To check your answers for Exercise 2, take *A* as origin and $(\vec{a}, \vec{b} - \vec{a})$ as basis for $[\overline{OAB}]$ and compute the position vectors

Answers for Part B

2. The other edge of the wedge should just cover the point marked 'B'.
3. The vertex of the wedge should just cover a point of the circle.
4. Exercises 2 and 3 suggest that angles inscribed in the same arc are congruent.

*

Answers for Part C

1. (a) $C \in \overleftrightarrow{AB}$ if and only if for some t, $\vec{c} = \vec{a} + (\vec{b} - \vec{a})t$. Since $\vec{c} = \vec{a}a + \vec{b}b$ and $(\vec{a}, \vec{b})$ is linearly independent it follows that $C \in \overleftrightarrow{AB}$ if and only if there is a number t such that $a = 1 - t$ and $b = t$. This last is the case if and only if $a + b = 1$.

   (b) C belongs to the circle $[\text{of } \overline{OAB}]$ with center O and radius r if and only if $||\vec{c}|| = r$. Since $\vec{c} = \vec{a}a + \vec{b}b$, this is the case if and only if $||\vec{a}||^2a^2 + ||\vec{b}||^2b^2 + 2ab(\vec{a} \cdot \vec{b}) = r^2$. Since $||\vec{a}|| = r = ||\vec{b}||$ and $\vec{a} \cdot \vec{b} = ||\vec{a}|| \, ||\vec{b}|| \cos \angle AOB$, this condition reduces to '$a^2 + b^2 + 2ab \cos \angle AOB = 1$'. [Note that this last equation can be thought of as an equation of the circle with respect to the coordinate system for $\overline{OAB}$ with origin O and basis $(\vec{a}, \vec{b})$. In case $\vec{a} \perp \vec{b}$ the equation reduces to the more familiar '$a^2 + b^2 = 1$'.]

2. Apparently $a + b > 1$ if and only if O and C are on opposite sides of $\overleftrightarrow{AB}$; and $a + b < 1$ if and only if O and C are on the same side of $\overleftrightarrow{AB}$. [This guess is verified in Exercise 3.]

3. (a) $O - A = -\vec{a} = \vec{a} \cdot -1 + (\vec{b} - \vec{a})0$;

   $C - A = \vec{c} - \vec{a} = (\vec{a}a + \vec{b}b) - \vec{a} = \vec{a}(a - 1) + \vec{b}b = \vec{a}(a + b - 1) + (\vec{b} - \vec{a})b$

   [The last step in determining C - A is by inspection. We know that C - A is some unique linear combination of $\vec{a}$ and $\vec{b} - \vec{a}$ and juggle to find it. One can proceed more systematically: We wish to find numbers c and d so that $\vec{a}(a - 1) + \vec{b}b = \vec{a}c + (\vec{b} - \vec{a})d$ — that is, so that $\vec{a}(a - 1) + \vec{b}b = \vec{a}(c - d) + \vec{b}d$. Since $(\vec{a}, \vec{b})$ is linearly independent it is necessary and sufficient that $a - 1 = c - d$ and $b = d$. Solving for 'c' and 'd' gives the desired result.]

Answers for Part C [cont.]

3. (b) Since $\vec{b} - \vec{a} \in [\overleftrightarrow{AB}]$ it follows that O and C are on the same side of $\overleftrightarrow{AB}$ if and only if -1 and $a + b - 1$ are both positive or both negative and are on opposite sides of $\overleftrightarrow{AB}$ if and only if one of -1 and $a + b - 1$ is positive and the other negative. Since $\underline{-1}$ is negative it follows that O and C are on the same side of $\overleftrightarrow{AB}$ if and only if $a + b - 1 < 0$ and are on opposite sides if and only if $a + b - 1 > 0$. [The criterion in Part C on pages 211 and 212 may be used with 'A' for 'V', 'B' for 'A', and 'O' for 'C'.]

4. $\angle ACB$ intercepts a major arc if and only if C and O are on opposite sides of $\overleftrightarrow{AB}$ — that is, if and only if $a + b - 1 > 0$. [The intercepted arc is the arc with end points A and B which does not contain C.] Similarly, $\angle ACB$ intercepts a minor arc if and only if $a + b - 1 < 0$.

5. (a) $\vec{w} = \vec{c}/r = (\vec{a}a + \vec{b}b)/r = (\vec{a}/r)a + (\vec{b}/r)b = \vec{u}a + \vec{v}b$.

   (b) Since $C \notin \overleftrightarrow{AB}$ it follows by Exercise 1(a) that $a + b \neq 1$.

   (c) Since $\|\vec{a}\| = r = \|\vec{b}\|$ it follows that $\vec{u}$ and $\vec{v}$ are unit vectors in the senses of $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively. Hence, by definition, $\cos \angle AOB = \vec{u} \cdot \vec{v}$.

   (d) This follows from Theorem 15-6 since $\vec{u} - \vec{w}$ and $\vec{v} - \vec{w}$ belong to the senses of $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

6. (a) [Square on both sides of the equation in Exercise 5(a), remembering that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are unit vectors.]

   (b) [Add with '1' on both sides of the equation in part (a).]

   (c), (d) [These follow at once from the equation in Exercise 5(a).]

   (e) [Use the equations in parts (c) and (d), remembering that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are unit vectors.]

7. (a) [Compute '$(\vec{u} - \vec{w}) \cdot (\vec{u} - \vec{w})$', remembering that $\vec{u}$ and $\vec{w}$ are unit vectors.]

   (b) [Compute '$\vec{u} \cdot \vec{w}$' from the equation in Exercise 5(a) and substitute in the expression in part (a).]

   (c) [Substitute from Exercise 6(a) into the expression in part (b).]

   (d) [Factor the difference of squares in the numerator of the expression in part (c).]

8. Interchange 'a' and 'b' to obtain '$\|\vec{v} - \vec{w}\|^2 = (1 - b - a)[1 + (b - a)]/b$'.

9. (a) [This follows at once from Exercises 7(d) and 8 and Exercise 6(b).]

   (b) $\|\vec{u} - \vec{w}\|\,\|\vec{v} - \vec{w}\| = |1 - a - b|\sqrt{2(1 + \vec{u} \cdot \vec{v})}$

10. By Exercises 5(d), 6(e), and 9(b),

$$\begin{aligned}\cos \angle ACB &= \frac{(1 - a - b)(1 + \vec{u} \cdot \vec{v})}{|1 - a - b|\sqrt{2(1 + \vec{u} \cdot \vec{v})}}\\ &= \operatorname{sgn}(1 - a - b)\sqrt{\frac{1 + \vec{u} \cdot \vec{v}}{2}}\end{aligned}$$

11. $\cos \angle ACB = \sqrt{(1 + \vec{u} \cdot \vec{v})/2}$ [minor arc]

    $= -\sqrt{(1 + \vec{u} \cdot \vec{v})/2}$ [major arc]

$O - A$ and $C - A$ of O and C with respect to this origin and basis.

(b) In answer to part (a) you should have expressions for $O - A$ and $C - A$ as linear combinations of $\vec{a}$ and $\vec{b} - \vec{a}$. Determine, from these, conditions on 'a' and 'b' in order that O and C are on opposite sides, or on the same side of $\overleftrightarrow{AB}$.

4. For what condition on 'a' and 'b' is it the case that $\angle ACB$ intercepts a major arc? A minor arc?

5. Let C be any point of the given circle other than A and B, and let $\vec{u} = \vec{a}/r$, $\vec{v} = \vec{b}/r$, and $\vec{w} = \vec{c}/r$. Establish each of the following:

(a) $\vec{w} = \vec{u}a + \vec{v}b$ (b) $a + b \neq 1$

(c) $\cos \angle AOB = \vec{u} \cdot \vec{v}$ (d) $\cos \angle ACB = \dfrac{(\vec{u} - \vec{w}) \cdot (\vec{v} - \vec{w})}{\|\vec{u} - \vec{w}\|\,\|\vec{v} - \vec{w}\|}$

6. We are seeking a link between $\cos \angle AOB$ and $\cos \angle ACB$. To begin with let's express both of these in terms of 'a' and 'b'. Establish each of the following:

(a) $\vec{u} \cdot \vec{v} = \dfrac{1 - a^2 - b^2}{2ab}$ (b) $1 + \vec{u} \cdot \vec{v} = \dfrac{1 - (a - b)^2}{2ab}$

(c) $\vec{u} - \vec{w} = \vec{u}(1 - a) - \vec{v}b$ (d) $\vec{v} - \vec{w} = -\vec{u}a + \vec{v}(1 - b)$

(e) $(\vec{u} - \vec{w}) \cdot (\vec{v} - \vec{w}) = (1 - a - b)(1 + \vec{u} \cdot \vec{v})$

7. Show that $\|\vec{u} - \vec{w}\|^2$ is equal to each of the following:

(a) $2[1 - (\vec{u} \cdot \vec{w})]$ (b) $2[1 - a - (\vec{u} \cdot \vec{v})b]$

(c) $\dfrac{b^2 - (a - 1)^2}{a}$ (d) $\dfrac{(1 - a - b)[-1 + (a - b)]}{a}$

8. Obtain an expression for $\|\vec{v} - \vec{w}\|^2$ like that in Exercise 7(d). [*Hint*: Is there an easier way than that of repeating the work in Exercise 7?]

9. (a) Show that

$$\begin{aligned}(\|\vec{u} - \vec{w}\|\,\|\vec{v} - \vec{w}\|)^2 &= (1 - a - b)^2\left[\frac{1 - (a - b)^2}{ab}\right]\\ &= (1 - a - b)^2 \cdot 2(1 + \vec{u} \cdot \vec{v}).\end{aligned}$$

[*Hint*: Use results from Exercises 6, 7, and 8.]

(b) Express '$\|\vec{u} - \vec{w}\|\,\|\vec{v} - \vec{w}\|$' in terms of '$a$', '$b$', and '$\vec{u} \cdot \vec{v}$'. [*Hint*: What is $\sqrt{(1 - a - b)^2}$?]

10. Express '$\cos \angle ACB$' in terms of 'a', 'b', and '$\cos \angle AOB$'. [*Hint*: Use the results of Exercises 5(d), 6(e), and 9(b). Recall the function sgn discussed on pages 313–314.]

11. Simplify the result obtained in Exercise 10 for the case in which $\angle ACB$ intercepts minor arc $\widehat{AB}$ and for the case in which $\angle ACB$ intercepts major arc $\widehat{AB}$. [*Hint*: Interpret your answer for Exercise 4 in terms of 'sgn'.]

*

Let's collect the result of Exercise 10:

$$\cos \angle ACB = \operatorname{sgn}(1 - a - b)\sqrt{\frac{1 + \cos \angle AOB}{2}}$$

and the result of Exercise 4:

$\angle ACB$ intercepts a major arc if and only if $\operatorname{sgn}(1 - a - b) = -1$ and $\angle ACB$ intercepts a minor arc if and only if $\operatorname{sgn}(1 - a - b) = 1$; and the result of Exercise 5 of Part C on page 333.

$\sqrt{\frac{1 + \cos \angle AOB}{2}}$ is the cosine of an angle half as large as $\angle AOB$.

Together, these give us information about the size of an inscribed angle and of the corresponding central angle. Note that an inscribed angle which intercepts a major [minor] arc is inscribed in a minor [major] arc.

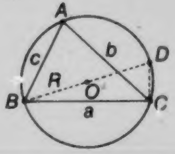
Theorem 17–12 An angle inscribed in a major arc is half as large as its corresponding central angle; an angle inscribed in a minor arc is the supplement of an angle half as large as its corresponding central angle.

Corollary 1 Any two inscribed angles which intercept the same arc are congruent.

*

Part D

Consider the circle which circumscribes $\triangle ABC$. Given that R is the radius of this circle and r is the radius of the incircle of $\triangle ABC$, it is reasonable to search for a relation between R and r.



1. (a) Show that $R = \dfrac{a}{2 \sin \angle A}$. [Treat three cases, according as A and O are on opposite sides of $\overleftrightarrow{BC}$, $\overline{BC}$ is a diameter, and A and O are on the same side of $\overleftrightarrow{BC}$. It should be helpful to consider the diameter $\overline{BD}$.]

   (b) Show that $rR = abc/(4s)$. [*Hint*: Use results from Part C on page 329.]

Answers for Part D

1. (a) Consider $\angle BDC$ where $\overline{BD}$ is a diameter. Since $\angle C$ is a right angle, $\sin \angle BDC = a/(2R)$.

   In case, as in the figure, A and O are on the same side of $\overleftrightarrow{BC}$, so are A and D. In this case, $\angle A$ and $\angle D$ are inscribed in the same arc and, so, $\angle A \cong \angle D$ and $\sin \angle A = a/(2R)$. [Congruent angles have the same sine.]

   In case A and O are on opposite sides of $\overleftrightarrow{BC}$, so are A and D. In this case, $\angle A$ and $\angle D$ are inscribed in "supplementary arcs" and, so, $\angle A$ and $\angle D$ are supplementary and $\sin \angle A = a/(2R)$. [Supplementary angles have the same sine.]

   In case $\overline{BC}$ is a diameter, $\triangle CAB$ is a right triangle and, so, $\sin \angle A = a/(2R)$.

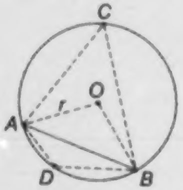
   Consequently, in any case, $R = a/(2 \sin \angle A)$.

   [Of course, R is also $b/(2 \sin \angle B)$ and $c/(2 \sin \angle C)$. So, we have a new proof for the law of sines.]

   (b) By Exercise 4 of Part C on page 329, $r = bc \sin \angle A/(a + b + c)$. So, using the result of part (a), $rR = abc/[2(a + b + c)] = abc/(4s)$.

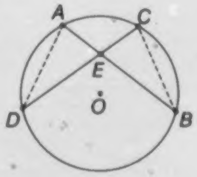
2. Find R in each of the following.
   (a) $a = 8, b = 6, c = 10$ (b) $a = 8 = b, c = 12$
   (c) $a = b = c = 5$ (d) $a = 15, b = 8, c = 17$
   (e) $a = 6, b = 8, \cos \angle C = \sqrt{3}/2$ (f) $a = 6, b = 8, \cos \angle C = -\sqrt{3}/2$
3. In Exercise 3 of Part C on page 329, you found r from the same information you used in Exercise 2 to find R. Verify the formula in Exercise 1(b) for the values you found.
4. Suppose that $\triangle ABC$ is equilateral, and that $AB = a$. Find r, R, and K in terms of 'a'. What is $r : R$?
5. Suppose that $\triangle ABC$ is isosceles with base $\overline{BC}$, that $AB = c$ and $BC = a$. Find r, R, and K in terms of 'a' and 'c'. What is $r : R$?

Part E



1. Given a chord $\overline{AB}$ of a circle with center O and radius r, consider the inscribed angles $\angle ACB$ and $\angle ADB$, as shown in the picture at the right.
   (a) If $r = 5$ and $AB = 8$, find $\cos \angle AOB$, $\cos \angle ACB$, and $\cos \angle ADB$.
   (b) if $r = 8$ and $AB = 8$, find $\cos \angle AOB$ $\cos \angle ACB$, and $\cos \angle ADB$.
   (c) if $r = 10$, $AB = 8$, and $AC = 6$, find $\cos \angle ACB$ and $\cos \angle ABC$.
   (d) Do you have enough information in (c) to find $\cos \angle BAC$? If so, do it.
   (e) If $AB = 5$, $AC = 6$, and $BC = 7$, find r, $\cos \angle ADB$, and $d(O, \overleftrightarrow{AB})$.
2. Prove this corollary to Theorem 17–12:

> Corollary 2 If $ABCD$ is a convex quadrilateral inscribed in a circle, each two opposite angles of $ABCD$ are supplementary.



3. Suppose that $\overline{AB}$ and $\overline{CD}$ are two chords of a circle which intersect in the point E.
   (a) Show that $\triangle AED$ is similar to $\triangle CEB$.
   (b) Show that $AE/CE = DE/BE = AD/CB$.
   (c) Show that $AE \cdot BE = DE \cdot CE$.
   (d) Do you think that $\triangle AEC$ is similar to $\triangle DEB$? Justify your answer.

Answers for Part D [cont.]

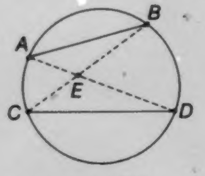
2. (a) 5 (b) $16/\sqrt{7}$
   (c) $5/\sqrt{3}$ (d) $17/2$
   (e) $2\sqrt{25 - 12\sqrt{3}}$ (f) $2\sqrt{25 + 12\sqrt{3}}$
3. [The formula is verified in each case.]
4. $r = a/(2\sqrt{3})$, $R = a/\sqrt{3}$, $K = a^2\sqrt{3}/4$; $r : R = 1/2$
5. $r = a\sqrt{4c^2 - a^2}/[2(a + 2c)]$, $R = c^2/\sqrt{4c^2 - a^2}$, $K = a\sqrt{4c^2 - a^2}/4$; $r : R = a(2c - a)/(2c^2)$

Answers for Part E

1. (a) $-7/25$, $3/5$, $-3/5$
   (b) $1/2$, $\sqrt{3}/2$, $-\sqrt{3}/2$
   (c) $\sqrt{21}/5$, $\sqrt{91}/10$
   (d) By Exercise 6 of Part D on page 324, $\cos \angle BAC = \cos \angle BAO \cos \angle OAC - \sin \angle BAO \sin \angle OAC$. Computing the cosines of $\angle BAO$ and $\angle OAC$ and, from them, the sines of these angles, leads to:
   $$\cos \angle BAC = (6 - \sqrt{21 \cdot 91})/50$$
   (e) $35/(4\sqrt{6})$, $-5/7$, $25/4\sqrt{6}$
2. Suppose that ABCD is a convex quadrilateral inscribed in a circle. Since ABCD is convex, $\overline{AC}$ and $\overline{BD}$ intersect and, so, A and C are on opposite sides of $\overleftrightarrow{BD}$. It follows that A and C are inscribed in different arcs with end points B and D. Hence, by Theorem 17-12, $\angle A$ and $\angle C$ are supplementary. Similarly, $\angle B$ and $\angle D$ are supplementary.
3. (a) The matching $ADE \longleftrightarrow CBE$ is a similarity because vertical angles $\angle DEA$ and $\angle BEC$ are congruent and $\angle EAD$ and $\angle ECB$ are congruent since they are inscribed in the same arc.
   (b) From the similarity established in part (a) $AE/CE = DE/BE = AD/CB$.
   (c) This results immediately from part (b).
   (d) Yes. $AEC \longleftrightarrow DEB$ is a similarity for reasons like those in the answer for part (a).

4. Suppose that the chord $\overline{AB}$ shown in the picture for Exercise 3 has measure 8, that $AE = 2$, and that the radius of the circle is 12. Use the results of Exercise 3 to do the following.
   (a) If $CE = 3$, find ED.
   (b) If $CD = 7$ and $AD = 3$, find CE, DE, and CB.
   (c) If $ED = 4$ and $CB = 6$, find CD and AD.
   (d) If $\overline{CD}$ is a diameter of the circle, find CE and ED.

5. Let $\overline{AB}$ and $\overline{CD}$ be two chords of a circle which have no points in common. Assume that they are as pictured at the right, so that $\overline{AD}$ and $\overline{BC}$ intersect in a point —say, E. By Exercise 3, we know that $AE \cdot ED = CE \cdot EB$.



   (a) Give a necessary and sufficient condition that $\overline{AB}$ be parallel to $\overline{CD}$.
   (b) Give a necessary and sufficient condition that $\overline{AC}$ be parallel to $\overline{BD}$.
   (c) Is it possible to have both $\overline{AB} \parallel \overline{CD}$ and $\overline{AC} \parallel \overline{BD}$? If so, what can you say about E? About $ABDC$?

6. Suppose that the lines containing the chords $\overline{AB}$ and $\overline{CD}$ described in Exercise 4 intersect in the point P. Show that
   (a) $\triangle PAD$ is similar to $\triangle PCB$,
   (b) $PA \cdot PB = PC \cdot PD$, and
   (c) $\overline{AC} \parallel \overline{BD}$ if and only if $\triangle PAC$ is isosceles with base $\overline{AC}$.

7. Look at the picture of inscribed quadrilateral $ABCD$ in Exercise 5. Its sides $\overline{AD}$ and $\overline{BC}$ intersect in the point E. Show that each two opposite angles of $ABCD$ are congruent.

*

Two of the results in the exercises in Part E are summarized in the following theorems. That of Exercise 3(c) is:

> **Theorem 17-13** If two chords of a circle intersect, the point of intersection divides each chord into segments such that the product of the measures of the segments of one chord is the product of the measures of the segments of the other.

That of Exercise 6(c) is:

> **Theorem 17-14** If two secants of a circle intersect at a point in the exterior of the circle, the product of the distances between the exterior point and the points of intersection on one of the secants is the product of the distances between the exterior point and the points of intersection on the other secant.

Answers for Part E [cont.]

4. (a) 4
   (b) There are two possibilities: (1) CE = 4, DE = 3, and CB = 6; (2) CE = 3, DE = 4, and CB = 9/2. The alternate solutions arise from solving the system of equations 'CE + ED = 7 and CE · ED = 12'.
   (c) 7, 4
   (d) There are two possibilities: (1) $CE = 12 + 2\sqrt{33}$ and $ED = 12 - 2\sqrt{33}$; (2) $CE = 12 - 2\sqrt{33}$ and $ED = 12 + 2\sqrt{33}$. They arise from solving the system of equations 'CE + ED = 24 and CE · ED = 12'.

5. (a) That E divides each of the segments from A to D and from B to C in the same ratio. Alternately, that $\overline{AE} : \overline{ED} = \overline{BE} : \overline{EC}$, or that (E - B):(C - E) = (E - A):(D - E).
   (b) That (E - A):(D - E) = (E - C):(B - E).
   (c) Yes.; E is the center of the circle.; ABDC is a rectangle.

6. (a) In $\triangle PAD$ and $\triangle PCB$, $\angle P = \angle P$ and $\angle B \cong \angle D$. Hence, by a.a., PAD $\longleftrightarrow$ PCB is a similarity.
   (b) By part (a), PA/PC = PD/PB.
   (c) Suppose that $\triangle PAC$ is isosceles with base $\overline{AC}$. Then PA = PC so that, by part (b) PD = PB. So, A and C divide the segments from P to B and from P to D, respectively, in the same ratio. Hence, $\overline{AC} \parallel \overline{BD}$.

   Suppose, next, that $\overline{AC} \parallel \overline{BD}$. Then, $\overset{\frown}{AB} \cong \overset{\frown}{CD}$ so that AB = CD. Now, PA/AB = PC/CD so that PA = PC. Hence, $\triangle PAC$ is isosceles with base $\overline{AC}$.

7. In quadrilateral ABCD, $\angle A$ and $\angle C$ are both inscribed in $\overset{\frown}{BACD}$ and, so, are congruent. Similarly, $\angle B \cong \angle D$. Hence, each two opposite angles of ABCD are congruent.

Using the result of Exercise 2(d) of Part B on page 328 we obtain:

> **Corollary** If a secant and a tangent of a circle intersect at a point exterior to the circle then the product of the distances between the exterior point and the points of intersection on the secant is the square of the distance between the exterior point and the point of tangency.

Given an inscribed angle, $\angle ACB$, of a circle with center O and radius r, we have established that if $\overline{AB}$ is not a diameter,

$$(*)\ \cos\angle ACB = \operatorname{sgn}(1-a-b)\sqrt{\frac{1+\cos\angle AOB}{2}},$$

where $(A-O)/r = \vec{u}, (B-O)/r = \vec{v}$, $C - O/r = \vec{u}a + \vec{v}b = \vec{w}$, and $\cos\angle AOB = \vec{u}\cdot\vec{v}$. And, if $\overline{AB}$ is a diameter we know that $\angle ACB$ is a right angle and since, in this case, $\vec{u}\cdot\vec{v} = -1$, this is what the formula (*) would lead us to expect. If we imagine that B and C remain fixed, while A moves along the circle toward the point B' diametrically opposite to B, then an angle $\angle AOM$ half the size of $\angle AOB$ becomes more nearly a right angle, while $\angle ACB$ becomes more nearly inscribed in a semicircular arc.

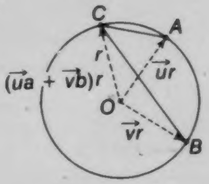


Fig. 17–10

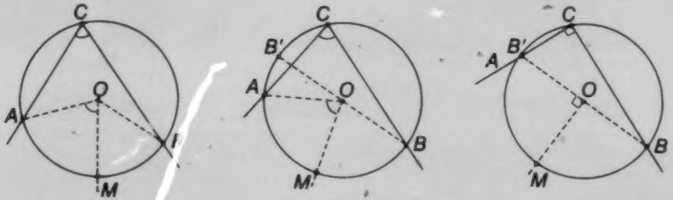


Fig. 17–11

The success of this "limiting argument" suggests that we consider what happens when A and B remain fixed while C moves along the

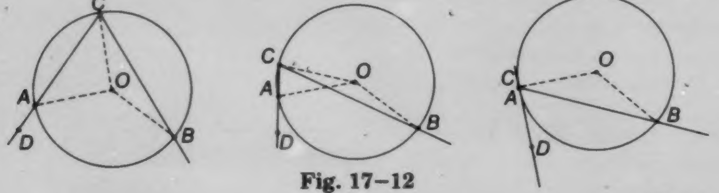


Fig. 17–12

circle toward A. If C remains on the same side of $\overleftrightarrow{AB}$, we can be sure that the size of $\angle ACB$ does not change, regardless of how close C comes to A. [Explain. Also, explain the change that occurs in the size of $\angle ACB$ if C moves to the opposite side of $\overleftrightarrow{AB}$.] Now, as C approaches A, $\overrightarrow{CA}$ appears to approach a limiting position $\overrightarrow{CD}$ [with $C = A$], and we may feel confident that $\angle DCB$ is the same size as each of the angles $\angle ACB$. We can find this limiting position of $\overrightarrow{CA}$ by considering the angle $\angle ACO$, and calculating its cosine value. Using the same notation as above [and, as in Part C on page 332], we have that

$$\begin{aligned}\cos \angle ACO &= \frac{(\overrightarrow{u} - \overrightarrow{w}) \cdot -\overrightarrow{w}}{\|\overrightarrow{u} - \overrightarrow{w}\|\,\|-\overrightarrow{w}\|} && \text{[Why?]}\\ &= \frac{\overrightarrow{w} \cdot \overrightarrow{w} - \overrightarrow{u} \cdot \overrightarrow{w}}{\|\overrightarrow{u} - \overrightarrow{w}\|\,\|\overrightarrow{w}\|} && \text{[Why?]}\\ &= \frac{1 - \overrightarrow{u} \cdot (\overrightarrow{u}a + \overrightarrow{v}b)}{\|\overrightarrow{u} - \overrightarrow{w}\|} && \text{[Why?]}\\ &= \frac{1 - a - (\overrightarrow{u} \cdot \overrightarrow{v})b}{\|\overrightarrow{u} - \overrightarrow{w}\|} && \text{[Why?]}\\ &= \frac{\|\overrightarrow{u} - \overrightarrow{w}\|^2/2}{\|\overrightarrow{u} - \overrightarrow{w}\|} && \text{[Why?]}\\ &= \|\overrightarrow{u} - \overrightarrow{w}\|/2\end{aligned}$$

Now, $A - C = \overrightarrow{a} - \overrightarrow{c} = (\overrightarrow{u} - \overrightarrow{w})r$ and, so, $d(A, C) = \|\overrightarrow{u} - \overrightarrow{w}\|r$. So,

$$(**) \qquad \cos \angle ACO = d(A, C)/(2r).$$

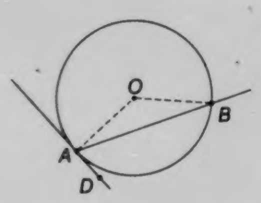
From (**), we see that as C approaches A, $\cos \angle ACO$ approaches 0. [Explain.] Thus, in the limiting case, $\overleftrightarrow{AD} \perp \overleftrightarrow{OA}$. That is, in the limiting case, $\overleftrightarrow{AD}$ is tangent to the circle at A. [Why?]

The preceding discussion suggests that the angle "between" a tangent and a chord is related to its corresponding central angle in the same way that an inscribed angle is. In the next exercises, a proof that this is the case is outlined.

Part F

Suppose that $\overleftrightarrow{AB}$ is a secant and that $\overleftrightarrow{AD}$ is a tangent of a circle with center O, and that A is the point of tangency, as shown in the picture at the right.

1. Show that if $\overleftrightarrow{AB}$ contains O, then $\angle BAD$ is a right angle.



As long as C remains on the same side of $\overleftrightarrow{AB}$ ∠ACB remains inscribed in the same arc and, hence, does not change in size. If C crosses $\overleftrightarrow{AB}$ then the arc in which ∠ACB is inscribed is the supplement of the former arc and ∠ACB is supplementary to the former angle.

Answers for the 'Why?'s in the computation of the cosine of ∠ACO:

$\vec{u} - \vec{w}$ is in the sense of $\overrightarrow{CA}$ and $-\vec{w}$ is in the sense of $\overrightarrow{CO}$.

Dot product algebra, and $||-\vec{w}|| = ||\vec{w}||$.

$\vec{w}$ is a unit vector and $\vec{w} = \vec{u}a + \vec{v}b$.

$\vec{u}$ is a unit vector.

See Exercise 7(b) of Part C on page 333.

As C approaches A, d(A, C) approaches 0 and so does d(A, C)/(2r).

The tangent to the circle at A is the perpendicular through A, in the plane of the circle, to $\overline{OA}$.

Answers for Part F

1. Since $\overleftrightarrow{AD}$ is tangent to the circle at A, ∠OAD is a right angle. And, in case $\overleftrightarrow{AB}$ contains O, ∠OAD = ∠BAD.

Answers for Part F [cont.]

2. (a) If ℓ intersected the circle in the single point B it would be tangent to the circle at B and, since it is parallel to the tangent at A, A and B would be end points of a diameter.

   (b) Since $\overleftrightarrow{OA} \perp \overleftrightarrow{AD}$ it follows that $\overleftrightarrow{OA} \perp \overleftrightarrow{BC}$ and, so, that $\overleftrightarrow{OA}$ intersects $\overline{BC}$. Assuming that O is exterior to ∠BAD it follows that C is also exterior to ∠BAD and, so, that C and D are on opposite sides of $\overleftrightarrow{AB}$. Hence, in this case, ∠CBA and ∠BAD are congruent as alternate interior angles.

   (c) As in part (b), $\overleftrightarrow{OA}$ intersects $\overline{BC}$. Assuming that O is interior to ∠BAD it follows that C is also interior to ∠BAD and, so, that C and D are on the same side of $\overleftrightarrow{AB}$. Hence, in this case, ∠CBA and ∠BAD are supplementary as consecutive interior angles.

   (d) Let F be the point of intersection of $\overrightarrow{AO}$ and $\overline{BC}$. Then $\overline{AF}$ is both the median and the altitude from A of △ABC. Hence △ABC is isosceles with base $\overline{BC}$ and, consequently, AB = AC. Also, since △ABC is isosceles with base $\overline{BC}$, ∠ACB ≅ ∠CBA.

   (e) In case O is exterior to ∠BAD, ∠BAD ≅ ∠ACB; in case O is interior to ∠BAD, ∠BAD and ∠ACB are supplementary. [Note that, in any case, C and O are on the same side of $\overleftrightarrow{AB}$ and, so, that ∠ACB is, in any case, half as large as ∠AOB.]

The exercises of Part G are exploratory for the content of section 17.05.

Answers for Part G

1.

(a)	(i) Yes.	(ii) Yes.	(iii) No.	(iv) No.
(b)	(i) No.	(ii) No.	(iii) No.	(iv) No.
(c)	(i) Yes.	(ii) Yes.	(iii) No.	(iv) No.
(d)	(i) Yes.	(ii) Yes.	(iii) Yes.	(iv) No.
(e)	(i) Yes.	(ii) Yes.	(iii) Yes.	(iv) No.
(f)	(i) Yes.	(ii) Yes.	(iii) No.	(iv) No.

[Students should see that answers for (i) and (ii) will always be the same, that a 'No.' answer for (i) implies a 'No.' answer for (iii)—so, a 'Yes.' answer for (iii) implies a 'Yes.' answer for (i)—and that the answer for (iv) is always 'No.'.]

2. The sets in (a), (c), (d), (e), and (f) have upper bounds; the sets in (d) and (e) have greatest members.; The sets of upper bounds for the sets in Exercise 1 are

   (a) $\{x: x \geq 0\}$ (b) $\emptyset$ (c) $\{x: x \geq 1\}$
   (d) $\{x: x \geq 0\}$ (e) $\{x: x \geq 2\}$ (f) $\mathbb{R}$

2. Suppose that $\overline{AB}$ is not a diameter. Let l be the line through B which is parallel to $\overleftrightarrow{AD}$.
   (a) Show that l intersects the given circle in two points.
   (b) Let C be the second point in which l intersects the circle, and assume that O is exterior to $\angle BAD$. Show that $\angle CBA \cong \angle BAD$.
   (c) Assume now that O is interior to $\angle BAD$. What can you say about $\angle CBA$ and $\angle BAD$?
   (d) Show that A is equidistant from B and C. What can you say about $\angle ACB$ and $\angle CBA$?
   (e) What can you say about $\angle BAD$ and inscribed angle $\angle ACB$?
3. Find the cosine and sine of *acute* $\angle BAD$ in each of the following:
   (a) $r = 5$, $AB = 8$ (b) $r = 10$, $AB = 10$
   (c) $\cos \angle AOB = \frac{3}{5}$ (d) $r = 10$, $\cos \angle OAB = \frac{1}{4}$
   (e) $AB = 10$, $\cos \angle OAB = \frac{1}{4}$ (f) $AB = 10$, $\sin \angle OAB = \frac{1}{4}$
4. Find $d(B, \overleftrightarrow{AD})$ in each part of Exercise 3.

*

The theorem whose proof is outlined in Part F is:

> **Theorem 17-15** If a secant and tangent of a circle intersect in a point of the circle, the angle between the secant and the tangent is either congruent to, or the supplement of, an angle half as large as its corresponding central angle according as the center of the circle is exterior or interior to the former angle.

Part G

1. In each of the following you are given a set S of real numbers. For each set, answer the following questions:
   (i) Is there a number which is greater than or equal to each member of S?
   (ii) Is there more than one number z such that, for each $x \in S$, $x \leq z$?
   (iii) Is there a number z in S such that $\forall_x\, [x \in S \longrightarrow x \leq z]$?
   (iv) Is there more than one number $z \in S$ such that for each $x \in S$, $x \leq z$?
   (a) $\{x: x < 0\}$ (b) $\{x: x \geq 1\}$ (c) $\{x: 0 < x < 1\}$
   (d) $\{x: x \leq 0\}$ (e) $\{x: x < 1 \text{ or } x = 2\}$ (f) $\emptyset$
2. We shall say that b is *an upper bound* of a set S of real numbers if and only if $\forall_x\, [x \in S \longrightarrow x \leq b]$. A number b is *a greatest member* of S if it is an upper bound of S and belongs to S. Which of the sets in Exercise 1 have upper bounds? Which of the sets in Exercise 1 have greatest members? For each set of Exercise 1 describe its set of upper bounds.

3. Suppose that b is an upper bound of a set S of real numbers. Are there any other numbers which you are certain are upper bounds of S? Explain.

4. (a) Can a set have just one upper bound?
   (b) Can a set have more than one greatest member? [Justify your answers for (a) and (b).]

5. (a) Is there a set which has no upper bounds?
   (b) Is there a set which has every real number as an upper bound?

6. If you have answered Exercise 5 correctly you see that the set of all upper bounds of a given set of real numbers may be $\varnothing$ or may be $\mathscr{R}$. These possibilities occur if the given set has no upper bound or is the empty set. Given a nonempty set of real numbers which has an upper bound, the set of upper bounds of the given set is a set of a special kind. Guess what kind.

7. Give definitions, in analogy with those in Exercise 2, of 'lower bound' and 'least member'.

8. How many sets can you find each of which has upper bounds but whose set of upper bounds does not have a least member?

9. If a set has a greatest member, does the set have a least upper bound? Explain.

17.05 An Important Property of Real Numbers

In the background work at the end of Chapter 15 we have seen that each nonempty set of integers which has an upper bound has a greatest member. In the preceding exercises of Part G we have seen that other nonempty sets of real numbers which have upper bounds need not have greatest members. [The set of negative real numbers is an example. Name some others.] Nevertheless, the exercises of Part G should have suggested to you a property of sets of arbitrary real numbers which is something like the property of sets of integers which we discovered in Chapter 15. This property—called *the least upper bound property*—may be stated as the final part of our Postulate 5:

> 5_{13} Each nonempty subset of $\mathscr{R}$ which has an upper bound has a least upper bound.

[For example, although the set of all negative numbers does not have a greatest member, the set of all its upper bounds—$\{x: x \geq 0\}$—does have a least member.] You may have found yourself agreeing with 5_{13}

Answers for Part G [cont.]

3. Any number greater than an upper bound of S is also an upper bound of S. This is because the relation of being less than or equal to is transitive.

4. (a) No. For, given any upper bound there is a number greater than it and, by Exercise 3, this number is also an upper bound of the given set.
   (b) No. For if a and b are greatest members of S then, since $a \in S$ and b is an upper bound of S, $a \leq b$. Similarly, $b \leq a$. Hence, $a = b$.

5. (a) Yes, there are many such sets.
   (b) $\varnothing$.

6. The set of all upper bounds of a nonempty set is a positively sensed ray.

7. A number b is a lower bound of a set S if and only if it is less than or equal to each member of S. A number b is a least member of S if and only if it is a lower bound of S and belongs to S. [After an argument like that in answer to Exercise 4(b) we are justified in speaking of the least member of a set which has a least member.]

8. Only one, the set $\varnothing$.

9. Yes. Suppose that b is a greatest member of S. Since $b \in S$, no number less than b is an upper bound of S. So, since b is an upper bound of S it is the least member of the set of upper bounds of S.

*

Postulate 5_{13} is a postulate and so, if it is to be accepted, must be accepted on intuitive evidence or on authority. We have tried to furnish a basis for intuitive evidence in the preceding exercises of Part G. What Postulate 5_{13} says is, essentially, that there are no "gaps" in the ordered system of real numbers. As a foil you may point out that the system consisting of only the rational real numbers satisfies all the earlier parts of Postulate 5, but fails to satisfy Postulate 5_{13}, One proof of this involves showing that there is no rational number whose square is 2, but that there are rational numbers as close as one wishes on either side of $\sqrt{2}$. From this it follows that the set of all rational numbers less than $\sqrt{2}$ has no least rational upper bound.

when you answered Exercise 6 of Part G. [The set of all upper bounds of a set of real numbers is either ∅ or $\mathscr{R}$ or a positively-sensed ray of $\mathscr{R}$.] Or, you may have come to the conclusion that 5_{13} is correct while you were working on Exercise 8. [∅ is the only set which has an upper bound but has no least upper bound.] In Section 17.06 we shall find that Postulate 5_{13} is essential when we define the measures of circular arcs.

Up to this point our postulates for $\mathscr{R}$ have been Postulates $5_0 - 5_{12}$ on page 501. As we remarked in Chapter 4 these postulates—which we combined into a single Postulate 5′—can be condensed into the statement that $\mathscr{R}$ is an ordered field. Similarly Postulates $5_0 - 5_{13}$ can be condensed into:

‖ 5 $\mathscr{R}$ is a complete ordered field.

The "completeness" of $\mathscr{R}$ which is expressed in 5_{13} amounts, roughly, to the fact that there are as many real numbers as we have believed that there are. For example, it is possible, using 5_{13} to prove that every nonnegative real number has a square root:

$$a \geq 0 \longrightarrow \exists_x\, x^2 = a$$

We shall not go through the details of proving this; but you may recall that we assumed this property of $\mathscr{R}$ when we introduced square roots in the introduction to this volume.

There is another property of real numbers which we assumed earlier in this course and used in proving that each nonempty set of integers which has an upper bound has a greatest member. Our assumption was that the set Nn of nonnegative integers has no upper bound:

$$\text{(C)} \qquad \exists_x (x \in Nn \text{ and } x > a)$$

[Note that sentence (C) is an assertion about all values of 'a'.] As an example of the use of Postulate 5_{13} we shall show that (C) is now a theorem. To do so, assume that, contrary to (C), Nn has an upper bound. Since $Nn \neq \varnothing$ it follows from our assumption and 5_{13} that Nn has a least upper bound—say, b. Then, each member of Nn is less than or equal to b and, since $b - 1$ is not an upper bound of Nn, there is a member of Nn—say, c—such that $c \leq b - 1$. It follows that $c + 1 \leq b$—but, this is impossible since $c + 1 \in Nn$ and b is an upper bound of Nn. So there is no such number as b, and Nn does not have an upper bound.

To show that a nonnegative number a has a nonnegative square root we may proceed by showing that $\{x\colon x \geq 0 \text{ and } x^2 \leq a\}$ has a least upper bound and that this least upper bound is nonnegative and has a as its square.

Since, for $a \geq 0$, $0^2 \leq a$ it follows that $\{x\colon x \geq 0 \text{ and } x^2 \leq a\}$ is nonempty. Moreover, there is a number — say, c — such that $c > 0$ and $c^2 > a$. For, in case $a \leq 1$ we may take c to be any number greater than 1, and in case $a > 1$ we may take c to be a. It follows that this number c is an upper bound of the set under consideration and, so, that the set has a least upper bound — say, u. Since the set contains 0, $u \geq 0$. It remains to be proved that $u^2 = a$.

To show that $u^2 = a$ we begin by choosing

$$b = \frac{cu + a}{u + c}.$$

Clearly, $b \geq 0$ [$b = 0$ if and only if $u = a = 0$]. Also,

$$b - u = \frac{cu + a}{u + c} - u = \frac{a - u^2}{u + c}, \text{ and}$$

$$\begin{aligned} a - b^2 = a - \left(\frac{cu + a}{u + c}\right)^2 &= \frac{a(u + c)^2 - (cu + a)^2}{(u + c)^2} \\ &= \frac{au^2 + ac^2 - c^2u^2 - a^2}{(u + c)^2} \\ &= \frac{c^2(a - u^2) - a(a - u^2)}{(u + c)^2} = \frac{(c^2 - a)(a - u^2)}{(u + c)^2} \end{aligned}$$

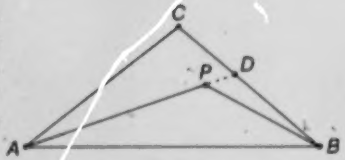
Suppose, now, that $u^2 < a$. It follows that $b - u > 0$ and [since $c^2 > a$] that $a - b^2 > 0$. So, $b^2 < a$ and $b > u$. This is impossible since it makes b a member of $\{x\colon x \geq 0 \text{ and } x^2 \leq a\}$ which is greater than the least upper bound of this set. So, $u^2 \not< a$.

Suppose, on the other hand, that $u^2 > a$. It follows that $b - u < 0$ and $a - b^2 < 0$. So, $b < u$ and $a < b^2$. Since $\sqrt{b^2} = b$ it follows that each member of $\{x\colon x \geq 0 \text{ and } x^2 \leq a\}$ is less than b and, so, that b is an upper bound of this set and, indeed, one which is less than the least upper bound of the set. This last being impossible, we conclude that $u^2 \not> a$.

Finally, since $u^2 \not< a$ and $u^2 \not> a$ it follows, as we wished to show, that $u^2 = a$.

Exercises

Given $\triangle ABC$, let P be any point interior to the triangle, as shown in the picture at the right. [A point is interior to a triangle if and only if it is interior to each angle of the triangle.]



1. Show that $AC + CB > AP + PB$. [*Hint*: Make use of the facts that $AC + CD > AP + PD$ and $PD + DB > PB$, where D is as shown in the picture.]
2. What does the result in Exercise 1 tell you about $AC + CB$ with respect to the set of all numbers $AP + PB$, where P is chosen interior to $\triangle ABC$?
3. Show that, given any real number $d > 0$, there is a point P interior to $\triangle ABC$ such that $AP + PB > (AC + BC) - d$. [*Hint*: Show that any point P which is interior to $\triangle ABC$ and close enough to C will work. How close is "close enough"?]
4. What does the result in Exercise 3 tell you about any number less than $AC + CB$ with respect to the set of numbers described in Exercise 2? What can you say about $AC + CB$ in consequence of this and Exercise 2?
5. Consider the set of all numbers $AQ + QR + RP$, where $Q \in \overline{AP}$ and and $R \in \overline{PB}$. [Draw a picture.] Show that $AP + PB$ is an upper bound for this set.
6. Is $AC + CB$ an upper bound for the set described in Exercise 3? Explain.

17.06 Measures of Circular Arcs

We have found it convenient to define the measure of a segment – say, $\overleftrightarrow{PQ}$ – to be $\|Q - P\|$. This measure gives us some idea of the length – or the size – of $\overleftrightarrow{PQ}$. Intuitively, the measure of $\overleftrightarrow{PQ}$ is $\|Q - P\|$ times the measure of a unit segment $\overleftrightarrow{P(P + \vec{u})}$, where $\vec{u}$ is a vector of norm 1. If, as suggested in Part I on pages 165 and 166, we chose a different scalar product [but one which gives the same notion of orthogonality] then we would have a different notion of unit vector, and all distances would be multiplied by some number [on page 165, by the number /c]. In any case, ratios of segment measures are unaffected, and the measure of a segment $\overleftrightarrow{PQ}$ is the ratio of $Q - P$ to $\vec{u}$ where $\vec{u}$ is a unit vector in the direction of $Q - P$.

Given several segments which have at most endpoints in common – as, for example, the segments containing the sides of a triangle or

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Answers for Exercises

1. Since P is interior to $\angle A$, $\overrightarrow{AP}$ intersects $\overline{BC}$ at a point — say, D. Since P is interior to $\angle C$, and $\angle C = \angle ACD$, $P \in \overrightarrow{AD}$. By the triangle inequality, $AC + CD > AD = AP + PD$. Similarly, $PD + DB > PB$. So,

$$\begin{aligned} AC + CB &= AC + CD + DB \\ &> AP + PD + DB \\ &> AP + PB. \end{aligned}$$

2. $AC + CB$ is an upper bound of all sums $AP + PB$, for P interior to $\triangle ABC$.
3. Choose P interior to $\triangle ABC$ and at a distance at most $d/2$ from C. It follows from the triangle inequality that $AC < AP + d/2$ and that $CB < PB + d/2$. So, $AC + CB < (AP + PB) + d$ and, consequently, $AP + PB > (AC + CB) - d$.
4. No number less than $AC + CB$ is an upper bound for the set described in Exercise 2.; Hence, $AC + CB$ is the least upper bound of the set of all sums $AP + PB$, P interior to $\triangle ABC$.
5. Since, by the triangle inequality, $QR < QP + PR$ it follows that $AQ + QR + RB < (AQ + QP) + (PR + RB) < AP + PB$. So, $AP + PB$ is an upper bound for all sums $AQ + QR + RB$ where $Q \in \overline{AP}$ and $R \in \overline{PB}$.
6. Yes. $AC + CB > AP + PB$ and $AP + PB$ is, itself, an upper bound of the set in question.

quadrilateral, or those whose union is the "polygonal line" shown in Fig. 17–13—it is customary to define the measure of their union as the sum of their measures. [In the case of a closed figure, such as a triangle or quadrilateral, this measure is called *the perimeter* of the figure.] This way of assigning measures to unions of segments is reasonable on intuitive grounds. For example, if you imagine laying the given segments end to end and measuring the resulting "straight" figure, you would expect it to have the same measure as the given figure. And as a segment, the measure of the straight figure is the sum of the measures of the segments which compose it.

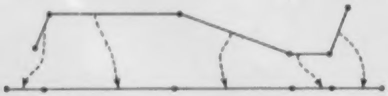


Fig. 17–13

Consider, now, a circular arc. A reasonable question to ask is:

How do we proceed to measure this arc?

Intuitively, we can take hold of the ends of the arc, pull it "straight", and measure the resulting segment. Then, by *defining* the measure of the arc to be the measure of this segment, we would have found the measure of the arc. Of course, we do have the problem of having to straighten an arc in order to measure it, so such a definition is not very useful. Taking a hint from the above-mentioned procedure used to define the measure of a polygonal line, we choose some successive points on the arc from one end of the arc to the other and join them, in order, by segments. Defining the measure of the inscribed polygonal line to be the sum of the measures of the segments of this polygonal line, it is intuitively clear that, no matter how the measure of an arc is defined, the measure of any such inscribed polygonal line is an approximation to the measure of the arc. Furthermore, given any inscribed polygonal line, a longer one can be inscribed in the same arc in the manner suggested by the dashed lines in Fig. 17–14, and this

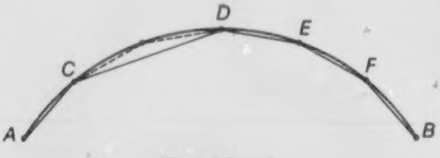


Fig. 17–14

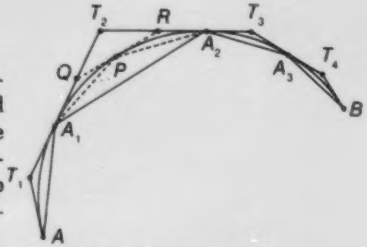
longer polygonal line is a better approximation to the measure of the arc. This suggests that the measure of an arc must be an upper bound for the set of all measures of polygonal lines inscribed in the arc.

It also seems reasonable that we can find inscribed polygonal lines which approximate an arc as closely as we wish. [Think of a polygonal line made up of a million segments, all of about the same length, inscribed in the arc of Fig. 17 – 14.] And, intuitively, we should obtain in this way inscribed polygonal lines whose measures approximate that of the arc as closely as we wish. So, it would seem natural to define the measure of an arc to be the *least* upper bound of the measures of its inscribed polygonal lines. So, if we can establish that the set of all such measures has a least upper bound, this number is the chief candidate for the title of measure of the arc. Our job, then, is to determine whether the set of all measures of polygonal lines inscribed in an arc has an upper bound. For, if we can manage to do this we will know, by Postulate 5_{13}, that the given set of measures has a least upper bound, and we may define this to be the measure of the arc.

Exercises

Part A

Consider arc $\widehat{AB}$ [minor, semicircular, or major] and its inscribed polygonal line $AA_1A_2A_3B$. Note the corresponding circumscribed polygonal line $AT_1T_2T_3T_4B$ made up of segments of tangents to the circle at the vertices of $AA_1A_2A_3B$.



1. Show that $AA_1A_2A_3B$ is shorter than $AT_1T_2T_3T_4B$.
2. Given that $P \in \widehat{A_1A_2}$, as shown in the figure, show that the polygonal line $AA_1PA_2A_3B$ is longer than $AA_1A_2A_3B$.
3. Show that the polygonal line $AT_1QRT_3T_4B$ is shorter than $AT_1T_2T_3T_4B$.
4. What can you say about the lengths of $AA_1PA_2A_3B$ and $AT_1QRT_3T_4B$?

*

The preceding exercises illustrate the fact that, corresponding with any polygonal line p inscribed in $\widehat{AB}$, there is a polygonal line c circumscribed about $\widehat{AB}$ and made up of segments of the tangents to the arc at the vertices of p. Evidently, given corresponding inscribed and circumscribed polygonal lines p and c, p is shorter than c.

The exercises also illustrate the fact that, given an inscribed polygonal line p the "insertion of a new vertex" P results in a longer in-

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Suggestions for the exercises of section 17.06:

(i) Part A and the discussion which follows should be directed by the teacher.

(ii) Parts B and C may be assigned as homework.

(iii) Part D may be used for class discussion.

Answers for Part A

1. By the triangle inequality,

$$AA_1 < AT_1 + T_1A_1,$$
$$A_1A_2 < A_1T_2 + T_2A_2,$$
$$A_2A_3 < A_2T_3 + T_3A_3, \text{ and}$$
$$A_3B < A_3T_4 + T_4B.$$

On adding, noting that $T_1A_1 + A_1T_2 = T_1T_2$, etc., one finds that $AA_1A_2A_3B$ is shorter than $AT_1T_2T_3T_4B$.

2. The measure of $AA_1PA_2A_3B$ is greater than that of $AA_1A_2A_3B$ by $A_1P + PA_2 - A_1A_2$.
3. The measure of $AT_1QRT_3T_4B$ is less than that of $AT_1T_2T_3T_4B$ by $QT_2 + T_2R - QR$.
4. The former is shorter than the latter.

scribed polygonal line p′. Also, if c and c' are the circumscribed polygonal lines corresponding with p and p' then c' is shorter than c [and, of course, p' is shorter than c']. Similar results would be obtained if we inserted any finite number of new vertices in p. [Explain.]

Suppose, now, that p_1 is any polygonal line inscribed in $\widehat{AB}$ and that c_2 is any polygonal line circumscribed about $\widehat{AB}$. We wish to show that p_1 is shorter than c_2 and, so, that the measure of c_2 is an upper bound for the set of measures of polygonal lines inscribed in $\widehat{AB}$. To do so, let p be the inscribed polygonal line whose vertices are those of p_1 together with the points of tangency of c_2. Let c be the corresponding circumscribed polygonal line. [Its points of tangency are those of c_2 together with the vertices of p_1.] It follows from the remarks made in the preceding paragraph that p_1 is shorter than p, that p is shorter than c, and that c is shorter than c_2. So, as we wished to show, p_1 is shorter than c_2.

The result just obtained shows that the set of measures of polygonal lines inscribed in $\widehat{AB}$ has an upper bound. Hence, by Postulate 5_{13}, this set of measures has a least upper bound. Consequently, we are justified in adopting:

> **Definition 17-4** The measure of a circular arc is the least upper bound of the set of measures of polygonal lines inscribed in the arc.

We have previously seen that this definition is intuitively satisfactory.

For 'measure of $\widehat{AB}$' we shall often write '$m(\widehat{AB})$'.

There is another result illustrated by the exercises of Part A. We saw there that the result of inserting a new vertex P in a polygonal line p inscribed in $\widehat{AB}$ is a new polygonal line p' which is inscribed in $\widehat{AB}$ and whose measure is greater than that of p. It follows that the set of measures of just those inscribed polygonal lines which have a given point P of $\widehat{AB}$ as a vertex has the same least upper bound as does the set of measures of all inscribed polygonal arcs. [Explain.] From this we obtain:

> **Theorem 17-16** If $\widehat{AP}$ and $\widehat{PB}$ are arcs [minor, semicircular, or major] which have no point in common then

$$m(\widehat{APB}) = m(\widehat{AP}) + m(\widehat{PB}).$$

[Explain.]

Suppose that the polygonal line p′ results from inserting a finite number of new vertices in the polygonal line p. We may think of p′ as the end result of a sequence of steps each of which involves inserting one new vertex in a previously obtained polygonal line. Since each step results in an increase in length p′ will be longer than p.

The least upper bound, $m(\widehat{APB})$, of measures of polygonal lines inscribed in $\widehat{APB}$ is certainly an upper bound of the measures of those polygonal lines which have P as one vertex. On the other hand, given any number $a < m(\widehat{APB})$, there is a polygonal line p inscribed in $\widehat{APB}$ whose measure is greater than a. Inserting P as a new vertex in this polygonal line gives a polygonal line, with P as a vertex, whose measure is greater than that of p and, so, is greater than a. Hence, no number less than $m(\widehat{APB})$ is an upper bound of the measures of those inscribed polygonal lines which have P as a vertex. Consequently, the least upper bound of the measures of these latter polygonal lines is the same as that $[m(\widehat{APB})]$ of the measures of all polygonal lines inscribed in $\widehat{APB}$.

Since $m(\widehat{AP}) + m(\widehat{PB})$ is an upper bound for the measures of polygonal lines which are inscribed in $\widehat{AB}$ and have P as a vertex it follows from the preceding argument that $m(\widehat{AP}) + m(\widehat{PB}) \geq m(\widehat{APB})$. But, if inequality held we could combine properly chosen polygonal lines p_1 and p_2 inscribed in $\widehat{AP}$ and in $\widehat{PB}$ to obtain a polygonal line inscribed in $\widehat{APB}$ whose measure would be greater than $m(\widehat{APB})$. [All we would need to do would be to choose p_1 and p_2 so that their measures differed from $m(\widehat{AP})$ and $m(\widehat{PB})$, respectively, by less than half of $m(\widehat{AP}) + m(\widehat{PB}) - m(\widehat{APB})$.] This completes the proof of Theorem 17-16.

The preceding procedures can be extended easily to give a definition of the measure [or: the circumference] of a circle $\mathscr{K}$. Instead of inscribed polygonal lines, we use inscribed simple closed polygons. And, essentially as a corollary of Theorem 17 - 16 we would have:

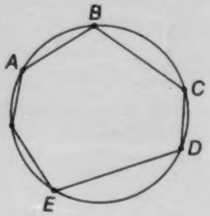


Fig. 17-15

Corollary The circumference of a circle is the sum of the measures of any two arcs which have the same endpoints.

We know by the corollary to Theorem 17 - 6 that if $\mathscr{K}_1$ and $\mathscr{K}_2$ are circles whose diameters have measures d_1 and d_2, respectively [for short, whose diameters are d_1 and d_2], then $\mathscr{K}_1$ is similar to $\mathscr{K}_2$ and the ratio of similitude of $\mathscr{K}_1$ to $\mathscr{K}_2$ is d_1/d_2. [Explain.] Any similitude which maps $\mathscr{K}_1$ onto $\mathscr{K}_2$ maps any polygonal line inscribed in $\mathscr{K}_1$ onto a similar polygonal line inscribed in $\mathscr{K}_2$, and the inverse of any such similitude maps any polygonal line inscribed in $\mathscr{K}_2$ onto a polygonal line inscribed in $\mathscr{K}_1$. [Explain.] So, there is a one-to-one correspondence between polygonal lines inscribed in $\mathscr{K}_1$ and polygonal lines inscribed in $\mathscr{K}_2$ and the ratio of the perimeters of two such lines is the ratio of similitude d_1/d_2. [Explain.] It follows that, if c_1 and c_2 are the circumferences [or, measures] of $\mathscr{K}_1$ and $\mathscr{K}_2$,

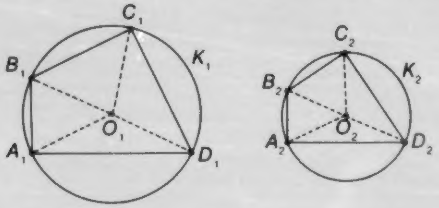


Fig. 17-16

$$(*) \qquad c_1/c_2 = d_1/d_2.$$

We have seen that the ratio of similarity of $\mathscr{K}_1$ to $\mathscr{K}_2$ is r_1/r_2, and this is, also, d_1/d_2.

A similitude which maps $\mathscr{K}_1$ onto $\mathscr{K}_2$ maps the vertices of a polygon inscribed in $\mathscr{K}_1$ on the vertices of a polygonal line inscribed in $\mathscr{K}_2$, and a similitude which maps the end points of a segment on those of another maps the first segment onto the second. Since such a similitude maps the center of $\mathscr{K}_1$ on that of $\mathscr{K}_2$ it follows by the a.a. similarity theorem that the ratio of the measures of the sides of the first polygon to those of the second will be d_1/d_2.

A procedure for computing approximations to π by finding the measures of polygonal arcs inscribed in a semicircle is outlined in the Teacher's Edition of High School Mathematics, Course 2. See T 420.

So, we have

Theorem 17–17 The ratio of the circumference of one circle to that of a second is the ratio of the diameter of the first to that of the second.

It follows from (*) that

$$c_1/d_1 = c_2/d_2$$

and, so, that the ratio of circumference to diameter is the same for all circles. This ratio is always denoted by 'π', and it can be shown that the number π is approximately 3.141592653. Another rational number which is a fairly good approximation to π is $\frac{22}{7}$. [One way of obtaining approximations to π is to compute, by using the Pythagorean Theorem, the lengths of polygonal lines inscribed in a semicircle of diameter 2, choosing polygonal lines with 2, 4, 8, and, in general, 2^n sides.] We have, then, as a corollary to Theorem 17–18:

Corollary If a circle has diameter d and circumference c then $c = \pi d$.

Part B

1. In each of the following, compute the circumference of the circle whose diameter is given.
   (a) 12 (b) 15/2 (c) $9/\pi$ (d) $13/(4\pi)$ (e) $\pi/4$
2. Find decimal approximations, to the nearest tenth, for the circumferences computed in Exercise 1.
3. Give a formula for the circumference, c, of a circle in terms of its radius, r. [*Hint*: What is a relation between r and d?]
4. In each of the following, find the radius of a circle whose circumference is given.
   (a) 4π (b) $\pi/4$ (c) $\pi\sqrt{15}/6$ (d) 8 (e) $8/\pi$

Part C

1. What is the length of a semicircular arc of a circle whose radius is 13?
2. A square whose sides measure 7 is inscribed in a circle.
   (a) What is the diameter of the circle?
   (b) Choose any one of the four minor arcs subtended by the sides of the square. Describe isometries which map this minor arc onto the remaining three arcs.
   (c) Compute the length of a minor arc subtended by a side of the square.

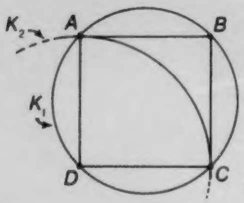
Answers for Part B

1. (a) 12π (b) $15\pi/2$ (c) 9 (d) 13/4 (e) $\pi^2/4$
2. (a) 37.7 (b) 23.6 (c) 9 (d) 3.3 (e) 2.5
3. $c = 2\pi r$
4. (a) 2 (b) 1/8 (c) $\sqrt{15}/12$ (d) $4/\pi$ (e) $4/\pi^2$

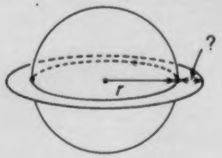
Answers for Part C

1. 13π
2. (a) $7\sqrt{2}$
   (b) Two such isometries are reflections in the planes which are perpendicular to the plane of the circle and each of which contains a diameter of the circle. The third is the reflection in the plane which is perpendicular to the plane of the circle and contains the midpoints of the two sides of the square which are adjacent to the one which subtends the given minor arc.
   (c) It seems reasonable to assume [and see the corollary to Theorem 17-18] that all four minor arcs referred to in part (b) have the same measure. This being so it follows by the corollaries to Theorems 17-16 and 17-17 that each minor arc has measure $7\pi\sqrt{2}/4$.

3. Consider the square $ABCD$ inscribed in a circle, $\mathscr{K}_1$, of radius 4, as shown at the right. Let $\mathscr{K}_2$ be the circle with center D and radius $\overline{AD}$.
   (a) What is the ratio of the circumferences of $\mathscr{K}_1$ to $\mathscr{K}_2$?
   (b) Which is the larger of $m(\widehat{AC})$ and $m(\widehat{ABC})$, where $\widehat{AC}$ is a minor arc of $\mathscr{K}_2$?
   (c) Given $\widehat{AB}$ of $\mathscr{K}_1$ and $\widehat{AC}$ of $\mathscr{K}_2$, which is the longer arc? What is the ratio of $m(\widehat{AB})$ to $m(\widehat{AC})$?



4. Suppose that a circular plate has a radius of 6 feet and that a circular piece of rope just fits around the edge of the plate.
   (a) How long is the rope?
   (b) Given that the length of the rope is increased by 6 feet, what is the radius of a circular plate around which this new rope will just fit?
5. Suppose that a circular rope just fits around the equator of a globe of radius r. Given that the length of the rope is increased by 6 feet, how far above the surface of the globe will the circular rope be, given that the center of the circle containing the lengthened rope is the center of the globe.



6. Consider isosceles triangle, $\triangle ABC$, with base $\overline{AC}$. Let $AB = BC = a$ and $AC = b$, and draw semicircles with diameters $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.
   (a) What is the measure of the semicircle whose diameter is $\overline{AC}$?
   (b) What is the sum of the measures of the semicircles whose diameters are $\overline{AB}$ and $\overline{BC}$?
   (c) Show that the sum in (b) is greater than the measure in (a).

*

Although any two circles are similar it is not the case that any two arcs are similar. [Explain.] The same argument which led to Theorem 17–17 establishes:

Theorem 17–18 If two arcs are similar then the ratio of their measures is the ratio of their radii.

Answers for Part C [cont.]

3. (a) $1/\sqrt{2}$
   (b) $m(\widehat{ABC}) > m(\widehat{AC})$ [$m(\widehat{ABC}) = 4\pi$, $m(\widehat{AC}) = 2\pi\sqrt{2}$]
   (c) $\widehat{AC}$ is longer than $\widehat{AB}$.; $1/\sqrt{2}$ [$m(\widehat{AC}) = 2\pi\sqrt{2}$, $m(\widehat{AB}) = 2\pi$]
4. (a) 12π feet
   (b) $6 + 3/\pi$ feet
5. $3/\pi$ feet [$(c + 6)/(2\pi) - r = 6/(2\pi)$ since $c/(2\pi) = r$.]
6. (a) $\pi b/2$
   (b) πa
   (c) $\pi a > \pi b/2$ because, by the triangle inequality, $2a > b$.

*

Two arcs of the same circle, one of which is a subset of the other, are not similar.

TC 349 (1)

Answers for Part D

1. (a) $\pi/2$ (b) $\pi/2$ (c) π (d) $\pi/2$
2. (a) $\pi/3$
   (b) $\pi/6$; $\pi/3$ [$m(\widehat{UP'}) = m(\widehat{UP})$ because the isometry f maps $\widehat{UP}$ onto $\widehat{UP'}$.]
   (c) In $\triangle PCP'$, $\angle P' \cong \angle P$. Since $\overleftrightarrow{PP'} \parallel \overleftrightarrow{VV'}$ [both being perpendicular to $\overleftrightarrow{UU'}$] and since P' and V are on opposite sides of $\overleftrightarrow{CP}$ it follows that $\angle P \cong \angle VCP$. Since $m(\widehat{PUP'}) = \pi/3 = m(\widehat{PV})$ it follows that $\widehat{PUP'} \cong \widehat{PV}$. So, $\overline{PP'} \cong \overline{PV}$ and, hence, $\angle P'CP \cong \angle VCP$. Since, as we have now shown, $\triangle PCP'$ is equiangular it follows that $\triangle PCP'$ is equilateral. [Since $\overleftrightarrow{PP'} \parallel \overleftrightarrow{VV'}$, an alternative way to show that $\angle P \cong \angle VCP$ is to show that these angles are not supplementary. This can be done by showing that both are acute. $\angle VCP$ is acute because it is smaller than $\angle VCU$. $\angle P$ is acute because it is inscribed in a major arc and, so, is half as large as the corresponding central angle.]

Corollary Congruent arcs have the same measure.

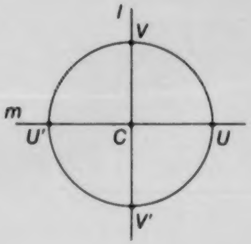
The converse of Theorem 17–18 is also a theorem but its proof is quite difficult. We shall accept it without proof:

Theorem 17–19 If the ratio of the measures of two arcs is the ratio of their radii then the arcs are similar.

Corollary Arcs which have the same radius and the same measure are congruent.

Part D

Consider a circle of π with center C and whose radius is 1. Let l and m be coplanar lines through C and such that $l \perp m$, as shown in the picture at the right.



1. What is the measure of
   (a) $\overset{\frown}{UV}$?
   (b) $\overset{\frown}{U'V'}$?
   (c) $\overset{\frown}{UVU'}$?
   (d) $\overset{\frown}{UV'}$?
2. Let P be the point of $\overset{\frown}{UV}$ such that $m(\overset{\frown}{UP}) = \pi/6$.
   (a) What is $m(\overset{\frown}{PV})$?
   (b) Given that f is the reflection in the perpendicular bisector of $\overleftrightarrow{VV'}$, let $P' = f(P)$. What is $m(\overset{\frown}{UP'})$? What is $m(\overset{\frown}{PUP'})$?
   (c) Show that $\triangle PCP'$ is equilateral, where P' is the point described in part (b).
3. Let g be the reflection in the perpendicular bisector of $\overleftrightarrow{UU'}$, and let $Q = g(P)$, where P is the point described in Exercise 2.
   (a) What is $m(\overset{\frown}{U'Q})$? $m(\overset{\frown}{UVQ})$? $m(\overset{\frown}{UV'Q})$?
   (b) Let $Q' = g(P')$. What is $m(\overset{\frown}{U'Q'})$ and $m(\overset{\frown}{QQ'})$?
   (c) Give an argument to show that $f(Q) = Q'$, where f is the reflection described in Exercise 2.
4. Consider the coordinate system for π with origin C and orthonormal basis $(U - C, V - C)$. Give the coordinates of each of the following points with respect to this coordinate system.
   (a) U (b) U' (c) V (d) V'
   (e) P (f) P' (g) Q (h) Q'
   (i) $\text{proj}_l(P)$ (j) $\text{proj}_m(P)$ (k) $\text{proj}_l(Q)$ (l) $\text{proj}_m(Q)$

Answers for Part D [cont.]

3. (a) $\pi/6$ [$= m(\overset{\frown}{UP})$ because the isometry g maps $\overset{\frown}{UP}$ onto $\overset{\frown}{U'Q}$.] $5\pi/6$; $7\pi/6$
   (b) $\pi/6$ [$= m(\overset{\frown}{UP'})$]; $\pi/3$
   (c) The isometry g maps $\overleftrightarrow{PP'}$ onto $\overleftrightarrow{QQ'}$ and maps $\overleftrightarrow{UU'}$ onto itself. So, since $\overleftrightarrow{PP'} \perp \overleftrightarrow{UU'}$, $\overleftrightarrow{QQ'} \perp \overleftrightarrow{UU'}$. Since $\overset{\frown}{QU'} \simeq \overset{\frown}{U'Q'}$ it follows that $\overline{QU'} \cong \overline{U'Q'}$ and so, by h.ℓ., that Q and Q' are equidistant from $\overleftrightarrow{UU'}$. So, $Q' = f(Q)$.
4. (a) (1, 0) (b) (−1, 0) (c) (0, 1) (d) (0, −1)
   (e) $(\sqrt{3}/2, 1/2)$ [By Exercise 2(c) and properties of equilateral triangles.]
   (f) $(\sqrt{3}/2, -1/2)$ (g) $(-\sqrt{3}/2, 1/2)$ (h) $(-\sqrt{3}/2, -1/2)$
   (i) $(0, 1/2)$ (j) $(\sqrt{3}/2, 0)$ (k) $(0, 1/2)$ (ℓ) $(-\sqrt{3}/2, 0)$

*

Sample Quiz

1. An equilateral triangle, $\triangle ABC$, whose sides have measure 6 is inscribed in a circle with center O and radius r.
   (a) What is the radius of the given circle? What is its circumference?
   (b) Compute $\cos \angle AOC$ and $\cos \angle ABC$.
   (c) Show that $\angle AOC$ and $\angle ABC$ are supplements.
   (d) What is the distance from the midpoint of $\overset{\frown}{AC}$ to the chord $\overline{AC}$?
2. Given that $\triangle PQR$ is a right triangle with hypotenuse $\overline{PR}$ and inscribed in a circle with center T and radius 10, assume that $PQ = 5$.
   (a) Compute QR and $\cos \angle QTR$.
   (b) If $\overline{PS}$ is the angle bisector from P, compute QS and PS.
   (c) Which of $\angle QPS$ and $\angle QRP$ is the larger?

Key to Sample Quiz

1. (a) $2\sqrt{3}$; $4\pi\sqrt{3}$ (b) $-1/2$; $1/2$
   (c) By (b), $\cos \angle AOC + \cos \angle ABC = 0$. So, $\angle AOC$ and $\angle ABC$ are supplements.
   (d) $\sqrt{3}$
2. (a) $5\sqrt{3}$; $-1/2$ (b) $5\sqrt{3}/3$; $10\sqrt{3}/3$
   (c) Neither is larger, for they are congruent. Note that $\cos \angle QPS = \sqrt{3}/2 = \cos \angle QRP$.

17.07 Measures of Angles

Consider $\angle AOB$ where, for convenience, $d(O, A) = d(O, B)$. Consider, also, the circles $\mathscr{K}$ and $\mathscr{K}'$ of $\overline{AOB}$ with the common center O

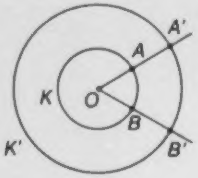


Fig. 17–17

and radii r and r', where $r = OA$ and $r' = OA'$. [A' and B' are the intersections of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with $\mathscr{K}'$.] There is a similitude [in fact, a uniform stretching about O] which leaves O fixed, maps A on A' and B on B', and maps $\widehat{AB}$ onto $\widehat{A'B'}$. The ratio of similitude is r/r'. It follows from Theorem 17–18 that

$$m(\widehat{AB})/m(\widehat{A'B'}) = r/r'$$

and, so, that

$$m(\widehat{AB})/r = m(\widehat{A'B'})/r'.$$

In other words, the ratio of the measure of the arc intercepted by a given central angle to the radius of that arc is the same for all circles having the given angle as a central angle. This ratio evidently tells us something about the size of the given angle and is called *the radian-measure of the given angle*:

> Definition 17–5 The radian-measure of an angle is the ratio of the measure of the arc intercepted by the angle, on any circle for which it is a central angle, to the radius of the arc.

We shall use '$m(\angle ABC)$' to refer to the radian-measure of $\angle ABC$. It is customary to call an angle whose radian-measure is k *an angle of k radians*. So a central angle which intercepts an arc whose measure is the radius of the circle is an angle of 1 radian. Sketch an angle of approximately 1 radian.

Using the corollaries to Theorems 17–18 and 17–19 we have an analog of Theorem 15–8:

> Theorem 17–20 Angles are congruent if and only if they have the same radian-measure.

An angle of 1 radian is nearly two-thirds of a right angle. More precisely, it is an angle of about 57.3°. [This last is to help you to evaluate the sketches asked for in the text following Definition 17-5. Students will be introduced to degree-measure later in this chapter.

Proof of Theorem 17-20: Let $\mathcal{K}$ and $\mathcal{K}'$ be circles of radius 1 in the planes of $\angle A$ and $\angle A'$ and with centers A and A', respectively. Let B and C be the points of intersection of $\angle A$ with $\mathcal{K}$ and let B' and C' be the points of intersection of $\angle A'$ with '.

Suppose, first, that $\angle A \cong \angle A'$. Since $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$ it follows that there is an isometry which maps A on A', B on B', and C on C'. This isometry will also map $\mathcal{K}$ onto $\mathcal{K}'$ and will map the arc $\widehat{BC}$ intercepted by $\angle A$ onto the arc $\widehat{B'C'}$ intercepted by $\angle A'$. It follows that $\widehat{BC} \cong \widehat{B'C'}$ and so, by the corollary to Theorem 17-18, that $m(\angle A) = m(\widehat{BC}) = m(\widehat{B'C'}) = m(\angle A')$.

Suppose, on the other hand, that $m(\angle A) = m(\angle A')$. It follows that $m(\widehat{BC}) = m(\widehat{B'C'})$ and so, by the corollary to Theorem 17-19, that $\widehat{BC} \cong \widehat{B'C'}$. So, $\overline{BC} \cong \overline{B'C'}$ and, since $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$, $ABC \longleftrightarrow A'B'C'$ is a congruence. In particular, $\angle A \cong \angle A'$.

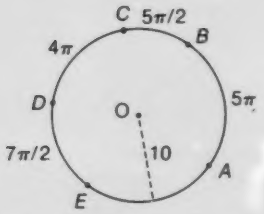
* * *

Suggestions for the exercises of section 17.07:

(i) Use Parts A and B for demonstration and practice exercises in class.
(ii) Parts C and D may be used for homework, but is a long assignment. You may wish to have students work together.
(iii) Part E and the discussion preceding it are good demonstration exercises for class.
(iv) Part F may be assigned as homework.

Exercises

Part A



Given a circle with center O and radius 10, suppose that A, B, C, D, and E are points of this circle and that the measures of some of the arcs of the circle are as given in the picture at the right.

1. (a) What is the circumference of the given circle?
   (b) What is $m(\widehat{AE})$?
2. Give the radian-measure of each of these angles.
   (a) $\angle AOB$ (b) $\angle BOC$ (c) $\angle AOC$ (d) $\angle COD$
   (e) $\angle EOC$ (f) $\angle AOD$ (g) $\angle AOE$ (h) $\angle DOE$
3. Consider the diameter of the circle one of whose endpoints is A. Let F be the other endpoint of this diameter. What is the measure of
   (a) $\angle COF$? (b) $\angle FOE$? (c) $\angle DOF$? (d) $\angle BOF$?
4. What is $\overrightarrow{OC}$ in relation to $\angle BOF$, where F is described in Exercise 3?
5. Suppose that G is a point of the circle such that $\angle DOG$ is a right angle.
   (a) How many such points G are there?
   (b) What is the measure of the arc intercepted by $\angle DOG$?
   (c) What is $m(\angle DOG)$?
   (d) For each of the points G you found in (a), determine $m(\angle EOG)$.

Part B

Given a circle, $\mathscr{K}$, with center O and radius r, suppose that P and Q are two points of $\mathscr{K}$ such that $\overline{PQ}$ is not a diameter.

1. Find the measure of the minor arc $\widehat{PQ}$ given the following information.
   (a) $r = 10$, $m(\angle POQ) = \pi/4$ (b) $r = 5$, $m(\angle POQ) = \pi/4$
   (c) $r = 2$, $m(\angle POQ) = \pi/6$ (d) $r = \frac{1}{2}$, $m(\angle POQ) = \pi/3$
   (e) $r = 4$, $m(\angle POQ) = 5\pi/6$ (f) $r = 6$, $m(\angle POQ) = 3\pi/4$
   (g) $r = \frac{1}{3}$, $m(\angle POQ) = \frac{5}{6}$ (h) $r = \frac{3}{4}$, $m(\angle POQ) = \frac{3}{4}$
2. Find r given the following information.
   (a) $m(\widehat{PQ}) = 2\pi$, $m(\angle POQ) = \pi/2$
   (b) $m(\widehat{PQ}) = \pi$, $m(\angle POQ) = 2\pi/3$
   (c) $m(\widehat{PQ}) = 3\pi/4$, $m(\angle POQ) = 3\pi/4$
   (d) $m(\widehat{PQ}) = \frac{11}{4}$, $m(\angle POQ) = \frac{11}{4}$
3. Let R and S be points of $\mathscr{K}$ such that $\overline{PR}$ and $\overline{QS}$ are diameters.
   (a) If $r = 5$ and $m(\widehat{PQ}) = 3\pi$, find $m(\widehat{RS})$, $m(\widehat{PS})$, and $m(\angle ROQ)$.
   (b) If $r = 6$ and $m(\widehat{PS}) = 3\pi$, show that $PQRS$ is a square.
   *(c) If $\triangle POQ$ is equilateral and $r = 10$, find $m(\widehat{PQ})$, $m(\widehat{QR})$, $m(\angle ROS)$, and $m(\angle POS)$. [*Hint*: See Part D on page 349.]

Answers for Part A

1. (a) 20π
   (b) 5π
2. (a) $\pi/2$ (b) $\pi/4$ (c) $3\pi/4$ (d) $2\pi/5$
   (e) $3\pi/4$ (f) $17\pi/20$ (g) $\pi/2$ (h) $7\pi/20$

*

Some questions which might be raised in connection with this exercise are:

(1) Which of the pairs of angles are congruent?
(2) Which of the angles are acute? Right? Obtuse?
(3) Is $\overleftrightarrow{OB} \cup \overleftrightarrow{OE}$ an angle? Explain.
(4) Which angle is "half" of $\angle AOB$? Is this angle half of any other of the angles mentioned in Exercise 2?

*

3. (a) $\pi/4$ (b) $\pi/2$ (c) $3\pi/20$ (d) $\pi/2$
4. $\overrightarrow{OC}$ is the angle bisector of $\angle BOF$, for C is interior to $\angle BOF$ and $\angle BOC \cong \angle COF$.
5. (a) Two. They are the points of intersection of the circle with the plane which contains O and is perpendicular to $\overleftrightarrow{DO}$.
   (b) 5π
   (c) $\pi/2$
   (d) $3\pi/20$ and $17\pi/20$. [The measures of the arcs from E to each of the points G are $3\pi/2$ and $17\pi/2$.]

Answers for Part B

1. By Definition 18-5, $m(\widehat{PQ}) = r \cdot m(\angle POQ)$.
   (a) $5\pi/2$ (b) $5\pi/4$ (c) $\pi/3$ (d) $\pi/6$
   (e) $10\pi/3$ (f) $9\pi/2$ (g) $5/18$ (h) $9/16$
2. By Definition 18-5, $r = m(\widehat{PQ})/m(\angle POQ)$.
   (a) 4 (b) 3/2 (c) 1 (d) 1
3. (a) $m(\widehat{RS}) = 3\pi$; $m(\widehat{PS}) = 2\pi$; $m(\angle ROQ) = 2\pi/5$.
   (b) Since $r = 6$ and $m(\widehat{PS}) = 3\pi$, $m(\angle POS) = \pi/2$. So, $\angle POS$ is a right angle, and the four arcs $\widehat{PQ}$, $\widehat{QR}$, $\widehat{RS}$, and $\widehat{PS}$ are congruent. Thus, PQRS is equilateral, which implies that PQRS is a rhombus. Since the diagonals of PQRS are congruent — for, they are diameters of the circle — PQRS is a square.
   (c) $m(\widehat{PQ}) = \frac{1}{2}m(\widehat{QR})$ and $m(\widehat{PQR}) = 10\pi$. Since $m(\widehat{PQ}) + m(\widehat{QR}) = 10\pi$, $m(\widehat{PQ}) = 10\pi/3$ and $m(\widehat{QR}) = 20\pi/3$. Now, $\angle ROS$ and $\angle POQ$ are congruent (vertical) angles, $m(\angle ROS) = m(\angle POQ) = (10\pi/3)/10 = \pi/3$. Similarly, $m(\angle POS) = m(\angle QOR) = 2\pi/3$.

*

Using Theorem 17-16 it is possible to prove:

> **Theorem 17-21** If D is interior to $\angle ABC$ then $m(\angle ABC) = m(\angle ABD) + m(\angle DBC)$.

and:

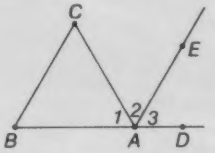
> **Theorem 17-22** Angles are supplementary if and only if the sum of their measures is π.

Here is a third theorem which is analogous to Theorem 15-7, but which we shall not prove:

> **Theorem 17-23** Given a number k such that $0 < k < \pi$, and given a half-line r, there is one and only one angle whose radian-measure is k, which has r as one of its sides, and whose other side is in a given side of the line which contains r.

Part C

1. What can you say about the measure of a right angle? An acute angle? An obtuse angle? Explain your answers.
2. (a) Given that $\angle A$ is congruent to $\angle B$, do $\angle A$ and $\angle B$ have the same measure? Justify your answer.
   (b) Suppose that $\angle A$ and $\angle B$ have the same measure. Are they congruent? Explain.
3. What is the sum of the measures
   (a) of two supplementary angles?
   (b) of two complementary angles?
4. Given a point D in the interior of $\angle ABC$.
   (a) Show that $m(\angle ABD) + m(\angle CBD) = m(\angle ABC)$.
   (b) If $\overrightarrow{BD}$ is the bisector of $\angle ABC$, what can you say about $m(\angle ABD)$ and $m(\angle ABC)$? About $m(\angle ABD)$ and $m(\angle CBD)$?
5. Show that the sum of the measures of the angles of a triangle is π. [*Hint*: Consider exterior angle, $\angle CAD$, of $\triangle ABC$ and let $\overrightarrow{AE}$ be the ray from A in the sense of $\overrightarrow{BC}$, as shown in the picture at the right. What is $m(\angle A_1) + m(\angle A_2) + m(\angle A_3)$?]



Theorem 17-21 follows from Theorem 17-16, Definition 17-5, and the fact that, if D is interior to $\angle ABC$ then the interiors of $\angle ABD$ and $\angle DBC$ have no points in common and contain, respectively, the arcs intercepted by these angles, on any circle for which they are central angles.

To prove Theorem 17-22, note that it is sufficient to consider the case of adjacent angles — say, $\angle ABD$ and $\angle DBC$. Those are supplementary if and only if A and C are end points of a diameter of a circle for which the angles are central angles. And this is the case if and only if the sum of the measures of the intercepted arcs is the measure of a semicircle.

Although Theorem 17-23 should seem reasonable on intuitive grounds, its proof requires the use of Postulate 5_{13} and of methods beyond the scope of this text.

Answers for Part C

1. The measure of a right angle is $\pi/2$. [A central angle which is a right angle intercepts an arc of its circle whose measure is one fourth the circumference of the circle.]

   The measure, k, of an acute angle is such that $0 < k < \pi/2$. [A central angle which is acute intercepts an arc of its circle whose measure is less than one fourth the circumference of the circle.]

   The measure, k, of an obtuse angle is such that $\pi/2 < k < \pi$. [A central angle which is obtuse intercepts an arc of its circle whose measure is greater than one fourth of and less than one half of the circumference of the circle.]

2. (a) Yes. Suppose that $\angle A$ is congruent to $\angle B$. Let f be an isometry which maps $\angle A$ onto $\angle B$, and let $\mathcal{K}$ be a circle of which $\angle A$ is a central angle. Then, $\angle B$ is a central angle of $f(\mathcal{K})$ and the arc of $f(\mathcal{K})$ which $\angle B$ intercepts is the image, under f, of the arc of $\mathcal{K}$ which $\angle A$ intercepts. So, the measures of these intercepted arcs are equal, as are the radii of $\mathcal{K}$ and $f(\mathcal{K})$. Hence, $\angle A$ and $\angle B$ have the same measure.

   (b) Yes. Suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are circles of radius r such that $\angle A$ and $\angle B$ are central angles of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Let $\overset{\frown}{P_1Q_1}$ and $\overset{\frown}{P_2Q_2}$ be the arcs of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, intercepted by $\angle A$ and $\angle B$. That $\angle A$ and $\angle B$ have the same measure implies that $\overset{\frown}{P_1Q_1}$ and $\overset{\frown}{P_2Q_2}$ have the same measure. Let f be any isometry which maps P_1 on P_2, Q_1 and Q_2 and the center of $\mathcal{K}_1$ on the center of $\mathcal{K}_2$. Then, f maps $\overset{\frown}{P_1Q_1}$ onto $\overset{\frown}{P_2Q_2}$ and, so, maps $\angle A$ onto $\angle B$. Hence, $\angle A$ is congruent to $\angle B$.

3. (a) π [The union of the arcs of a circle intercepted by supplementary adjacent central angles is a semicircular arc of the circle.]

   (b) $\pi/2$

Answers for Part C [cont.]

4. (a) Let $\mathcal{K}$ be any circle of which $\angle ABC$ is a central angle. Then, $\overrightarrow{BA}$, $\overrightarrow{BD}$, and $\overrightarrow{BC}$ intersect $\mathcal{K}$ in A_1, D_1, and C_1, respectively, such that the sum of the measures of $\widehat{A_1D_1}$ and $\widehat{D_1C_1}$ is the measure of $\widehat{A_1C_1}$. So, given that $\mathcal{K}$ has radius r, we have that

$$m(\widehat{A_1D_1})/r + m(\widehat{D_1C_1})/r = m(\widehat{A_1C_1})/r.$$

Thus, by definition,

$$m(\angle A_1BD_1) + m(\angle C_1BD_1) = m(\angle A_1BC_1).$$

Since $\angle A_1BD_1 = \angle ABD$, $\angle C_1BD_1 = \angle CBD$ and $\angle A_1BC_1 = \angle ABC$, we have that

$$m(\angle ABD) + m(\angle CBD) = m(\angle ABC).$$

(b) $m(\angle ABD) = \frac{1}{2}m(\angle ABC)$; They are equal.

5. Using the notion suggested in the figure, we note that $\angle A_2 \cong \angle C$ and $\angle A_3 \cong \angle B$. So, $m(\angle A_2) = m(\angle C)$ and $m(\angle A_3) = m(\angle B)$. Now, $\angle A_1$ and $\angle CAD$ are supplementary adjacent angles. So, $m(\angle A_1) + m(\angle CAD) = \pi$. Also, since E is interior to $\angle CAD$, we know by Exercise 4(a) that $m(\angle A_2) + m(\angle A_3) = m(\angle CAD)$. So, $m(\angle A_1) + m(\angle A_2) + m(\angle A_3) = \pi$. Hence, $m(\angle A) + m(\angle B) + m(\angle C) = \pi$. [Note that the preceding argument also shows that the measure of an exterior angle [$\angle CAD$] of a triangle is the sum of the measures of the opposite interior angles.

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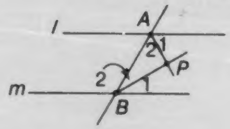
6. From the hypothesis, $m(\angle A_2) = m(\angle A_1)$ and $m(\angle B_2) = m(\angle B_1)$. Furthermore, $m(\angle A_1) + m(\angle A_2) + m(\angle B_1) + m(\angle B_2) = \pi$. So, $m(\angle A_2) + m(\angle B_2) = \pi/2$. Now, since $m(\angle A_2) + m(\angle B_2) + m(\angle APB) = \pi$, it follows that $m(\angle APB) = \pi/2$. Hence, $\angle APB$ is a right angle.

6. In the picture at the right, $l \parallel m$ and P is such that $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are bisectors of the interior angles at A and B. Show that $\angle APB$ is a right angle. [*Hint*: Compute $m(\angle APB)$.]



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Some of the results we can prove concerning the angles of triangles and convex quadrilaterals generalize to figures which have five or more intervals as sides. As examples of such figures, a *pentagon* is the union of five segments $\overset{\bullet\!-\!\bullet}{AB}$, $\overset{\bullet\!-\!\bullet}{BC}$, $\overset{\bullet\!-\!\bullet}{CD}$, $\overset{\bullet\!-\!\bullet}{DE}$, and $\overset{\bullet\!-\!\bullet}{EA}$ which are such that each of the sets $\{A, B, C\}$, $\{B, C, D\}$, $\{C, D, E\}$, $\{D, E, A\}$, and $\{E, A, B\}$ is noncollinear—in short, such that no three consecutive vertices of the pentagon are collinear. A pentagon will be said to be *convex* if and only if, for each three consecutive vertices X, Y, and Z, each of the remaining vertices is interior to $\angle XYZ$. [A similar definition of convexity for quadrilaterals turns out to be equivalent to the one we have previously adopted.]

A convex pentagon A nonconvex pentagon

Fig. 17–18

The notions of *a hexagon* and of *a convex hexagon* can be defined in a similar manner, but using six sides rather than five. More generally, an *n-gon* is the union of n segments, $\overset{\bullet\!-\!\bullet}{A_1A_2}$, $\overset{\bullet\!-\!\bullet}{A_2A_3}$, . . ., $\overset{\bullet\!-\!\bullet}{A_nA_1}$ which are such that no three consecutive vertices are collinear. A *convex n-gon* is an n-gon with the property that, for each three consecutive vertices X, Y, and Z, each of the remaining vertices is interior to $\angle XYZ$.

It can be proved that, for $n \geq 3$, the n-gon obtained by replacing two consecutive sides $\overline{AB}$ and $\overline{BC}$ of a convex $(n + 1)$-gon by the interval $\overline{AC}$ is also convex.

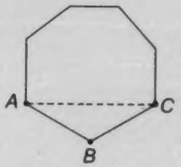
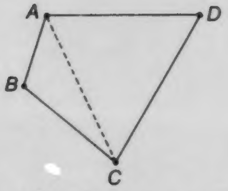


Fig. 17–19

Part D



1. Suppose that $ABCD$ is a convex quadrilateral. Show that the sum of the measures of the angles of $ABCD$ is 2π. [*Hint*: Consider the diagonal from A. Use the results of Exercises 4 and 5 of Part C.]
2. Find the sum of the measures of the angles of a convex pentagon. [*Hint*: Draw a convex pentagon—say, $ABCDE$—and draw the diagonals from one of its vertices.]
3. Find the sum of the measures of the angles of a convex hexagon.
4. What is the sum of the measures of the angles of a convex n-gon [$n \geq 3$]?

*5. Use mathematical induction to justify your answer for Exercise 4. [*Hint*: Use the fact that the result of "cutting off a corner" of a convex $(n + 1)$-gon is a convex n-gon.]

6. (a) Consider the exterior angles, one at each vertex, of a triangle. What is the sum of the measures of these exterior angles?
   (b) Answer the question in (a) for a convex quadrilateral.
   (c) Answer the question in (a) for a convex pentagon.
   (d) Answer the question in (a) for a convex n-gon.
7. A *regular* polygon is a convex polygon all of whose angles are congruent *and* all of whose sides are congruent.
   (a) What is a regular polygon of three sides? Of four sides?
   (b) What can you say about the exterior angles of a regular polygon?
   (c) In Exercise 6, you found that the sum of the measures of the exterior angles, one at each vertex, of a convex polygon is 2π. In a regular polygon, all of the exterior angles are congruent. Find the measure of an exterior angle of a regular polygon of n sides in terms of 'n'.
   (d) How can you use your result from (c) to find the measure of an angle of a regular polygon of n sides?
8. Find the measure of an exterior angle of a regular polygon of each of the following number of sides.
   (a) 3 (b) 4 (c) 5 (d) 6
   (e) 8 (f) 10 (g) 12 (h) 60
9. Find the measures of the angles of the regular polygons described in Exercise 8.
10. Find the number of sides in a regular polygon if the measure of an interior angle of the polygon is k times the measure of an exterior angle, where
    (a) $k = 2$; (b) $k = 1$; (c) $k = \frac{1}{2}$; (d) $k = 3$.

*

Answers for Part D

1. Suppose that ABCD is a convex quadrilateral. It follows that C is interior to $\angle DAB$ and that A is interior to $\angle BCD$. So, $m(\angle A) = m(\angle BAC) + m(\angle CAD)$ and $m(\angle C) = m(\angle BCA) + m(\angle ACD)$. By Exercise 5 of Part C, $m(\angle BAC) + m(\angle BCA) + m(\angle B) = \pi$ and $m(\angle CAD) + m(\angle ACD) + m(\angle D) = \pi$. Combining these results, $m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 2\pi$.
2. An argument similar to that given for Exercise 1, but using Exercise 1 in place of Exercise 5 of Part C, shows that the sum of the measures of the angles of a convex pentagon is 3π.
3. A repetition of the same kind of argument, using Exercise 2 in place of Exercise 1, shows that the sum of the measures of the angles of a regular hexagon is 4π.
4. It is easily guessed that the sum of the measures of the angles of a convex n-gon is $(n - 2)\pi$.
5. The sum of the measures of the angles of a convex 3-gon [a triangle] is $(3 - 2)\pi$, by Exercise 5 of Part C. Suppose, then, that the sum of the angles of a convex k-gon [$k \geq 3$] is $(k - 2)\pi$ and consider a convex $(k + 1)$-gon with vertices $A_1, A_2, \ldots, A_{k+1}$. Since the polygon $A_2 \ldots A_{k+1}$ is a convex k-gon, the sum of the measures of its angles is $(k - 2)\pi$. The sum of the measures of the angles of $\triangle A_1A_2A_{k+1}$ is π. Since A_2 is interior to $\angle A_1A_{k+1}A_k$ and A_{k+1} is interior to $\angle A_1A_2A_3$ it follows, as in Exercise 1 that the sum of the measures of the angles of $A_1A_2 \ldots A_{k+1}$ is $(k - 2)\pi + \pi$. But, this is $[(k + 1) - 2)\pi$. Hence, by induction, it follows that, for each $n \geq 3$, the sum of the measures of the angles of a convex n-gon is $(n - 2)\pi$.
6. (a) 2π [$= 3\pi - \pi$]
   (b) 2π [$= 4\pi - 2\pi$]
   (c) 2π [$= 5\pi - 3\pi$]
   (d) 2π [$= n\pi - (n - 2)\pi$]
7. (a) A regular polygon of three sides is an equilateral triangle; one of four sides is a square.
   (b) Any two exterior angles of a regular polygon are congruent [as supplements of congruent angles of the polygon].
   (c) $2\pi/n$
   (d) The measure of an angle of a regular n-gon is $\pi - (2\pi)/n$ or, in other words, $(n - 2)\pi/n$. [This, of course, checks with the result of Exercise 4.]
8. (a) $2\pi/3$ (b) $\pi/2$ (c) $2\pi/5$ (d) $\pi/3$
   (e) $\pi/4$ (f) $\pi/5$ (g) $\pi/6$ (h) $\pi/30$
9. (a) $\pi/3$ (b) $\pi/2$ (c) $3\pi/5$ (d) $2\pi/3$
   (e) $3\pi/4$ (f) $4\pi/5$ (g) $5\pi/6$ (h) $29\pi/30$
10. (a) 6 (b) 4 (c) 3 (d) 8

Some of the results of the exercises just completed are summarized in the following theorems:

Theorem 17–24 Given that $m(\angle ABC) = k$. Then,
(a) $\angle ABC$ is right if and only if $k = \pi/2$,
(b) $\angle ABC$ is acute if and only if $0 < k < \pi/2$, and
(c) $\angle ABC$ is obtuse if and only if $\pi/2 < k < \pi$.

Theorem 17–25 The measure of an exterior angle of a triangle is the sum of the measures of the opposite interior angles.

Theorem 17–26 The sum of the measures of the angles of a convex polygon of n sides is $(n - 2)\pi$.

Corollary 1 The sum of the measures of the angles
(a) of a triangle is π;
(b) of a convex quadrilateral is 2π.

Corollary 2 The sum of the measures of the exterior angles, one at each vertex, of a convex polygon is 2π.

Corollary 3 The measure of an angle of a regular polygon of n sides is $(n - 2)\pi/n$.

Earlier, we considered inscribed angles of a circle. We learned that an angle inscribed in a major arc is half as large as its corresponding central angle and that an angle inscribed in a minor arc is the supplement of an angle half as large as its corresponding central angle.

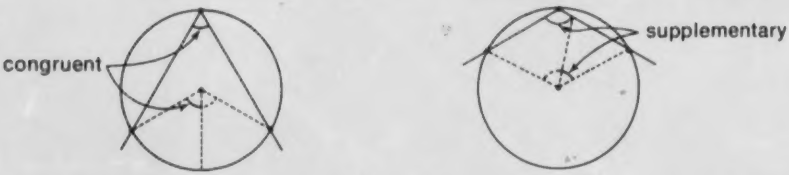


Fig. 17–20

Since congruent angles have the same measure, it follows immediately that the measure of an angle inscribed in a major arc of a circle of radius r is the quotient by r of one-half the measure of the intercepted arc. Furthermore, since the sum of the measures of supplementary angles is π, it follows that the measure of an angle inscribed in a minor arc is also the quotient by r of one-half the measure of the intercepted arc. To see that this is the case, let $\mathscr{K}_1$ be the minor arc and $\mathscr{K}_2$ the corresponding major arc of a circle, and let $m(\mathscr{K}_2) = p$. Then $m(\mathscr{K}_1) = 2\pi - p$. [Explain.] If $\angle A$ is inscribed in $\mathscr{K}_1$ then $\angle A$ is the supplement of half the central angle which intercepts $\mathscr{K}_1$. So,

$$m(\angle A) = [\pi - \tfrac{1}{2}m(\mathscr{K}_1)]/r = [\pi - \tfrac{1}{2}(2\pi - p)]/r = \tfrac{1}{2}p/r.$$

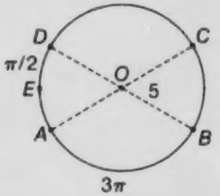
Finally, it is easy to establish that the measure of an angle inscribed in a semicircle is /r times one-half the measure of the intercepted arc. So, in any case, we have:

> **Theorem 17-27** The measure of an angle inscribed in a circle of radius r is half the measure of the arc intercepted by that angle divided by r.

*

Part E

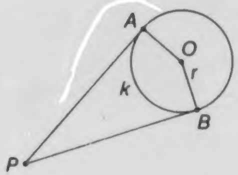
Given a circle with center O and radius 5, suppose that A, B, C, D, and E are points of the circle, that $\overline{AC}$ and $\overline{BD}$ are diameters, and that the measures of some of the arcs of the circle are as indicated in the picture at the right.



1. Give the measures of each of these inscribed angles.
   (a) $\angle AEB$ (b) $\angle ADB$ (c) $\angle ACB$ (d) $\angle ABE$
   (e) $\angle CDA$ (f) $\angle CDE$ (g) $\angle DEA$ (h) $\angle EAB$
2. What is the sum of the measures of the angles of quadrilateral $ABCD$? Of quadrilateral $AEDC$?
3. (a) Show that both $\angle DCA$ and $\angle DBA$ are supplements of $\angle DEA$.
   (b) Show that $\angle BAC$ is a supplement of $\angle DEA$.
4. Suppose that l and m are the tangents at A and B, respectively, and that $l \cap m = \{P\}$.
   (a) What is $m(\angle OAP)$ and $m(\angle OBP)$?
   (b) Show that $\angle AOB$ and $\angle APB$ are supplements.
   (c) What is $m(\angle APB)$?

Part F

1. Suppose that $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents at A and B to a circle with center O and radius r.
   (a) Explain why $\overline{AB}$ is not a diameter.
   (b) What kinds of angles are $\angle PAO$ and $\angle PBO$? What are their measures?
   (c) Show that $\angle P$ and $\angle O$ are supplements.
   (d) Given that $m(\widehat{AB}) = k$, find $m(\angle P)$ in terms of 'k'.



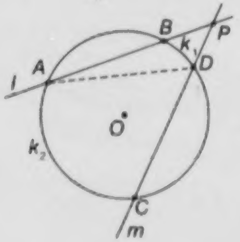
Answers for Part E

1. (a) $3\pi/10$ (b) $3\pi/10$ (c) $3\pi/10$ (d) $3\pi/20$
   (e) $\pi/2$ (f) $13\pi/20$ (g) $4\pi/5$ (h) $11\pi/20$
2. 2π; 2π
3. (a) $\angle DCA$ and $\angle DBA$ intercept $\widehat{DEA}$ and $\angle DEA$ intercepts $\widehat{DCA}$. So, $m(\angle DEA) + m(\angle DCA) = m(\angle DEA) + m(\angle DBA) = \pi$. Hence, both $\angle DCA$ and $\angle DBA$ are supplements of $\angle DEA$.
   (b) Since $m(\angle BAC) = \pi/5$ and $m(\angle DEA) = 4\pi/5$ and $\pi/5 + 4\pi/5 = \pi$, it follows that $\angle BAC$ and $\angle DEA$ are supplementary.
4. (a) $\pi/2$, $\pi/2$
   (b) It follows from part (a) and Corollary 1(b) to Theorem 17-26 that the sum of the measures of these angles is π. So, by Theorem 17-22, the angles are supplementary.
   (c) $2\pi/5$

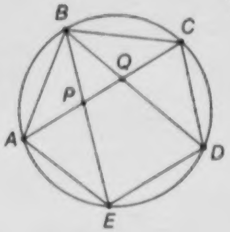
Answers for Part F

1. (a) If $\overline{AB}$ were a diameter then the two tangents would be perpendicular to $\overleftrightarrow{AB}$ and, so, would not intersect at P.
   (b) They are right angles.; $\pi/2$
   (c) By Corollary 1(b) to Theorem 17-26, the sum of the measures of the angles of PAOB is 2π. So, by part (b), $m(\angle P) + m(\angle O) = \pi$. Hence, by Theorem 17-22, $\angle P$ and $\angle O$ are supplementary.
   (d) $m(\angle P) = \pi - k/r$.

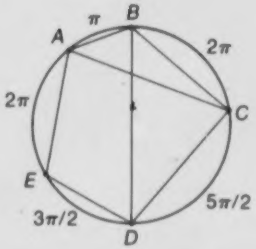
2. Given the information from Exercise 1, find OP and AP in terms of 'r' given that
   (a) $\angle AOB$ is a right angle. (b) $\triangle AOB$ is equilateral.
3. Suppose that l and m are secants of a circle with center O and radius r, that $l \cap m = \{P\}$, and that l and m intersect the circle in points A and B, and C and D, respectively, as shown in the picture at the right. Given that $m(\widehat{BD}) = k_1$ and $m(\widehat{AC}) = k_2$, compute $m(\angle APC)$ in terms of 'r', 'k_1', and 'k_2'. [*Hint*: Use Theorem 17–25.]



4. Given the information from Exercise 3, suppose that Q is the point of intersection of chords $\overline{AD}$ and $\overline{BC}$. Find the measures of $\angle BQD$ and $\angle BQA$ in terms of 'r', 'k_1', and 'k_2'. [*Hint*: Use Theorem 17–25.]
5. Suppose that $ABCDE$ is a regular pentagon inscribed in a circle of radius 10.
   (a) What is $m(\widehat{AB})$?
   (b) Given that $\overline{AC} \cap \overline{BE} = \{P\}$, find $m(\angle BPC)$.
   (c) Given that $\overline{AC} \cap \overline{BD} = \{Q\}$, find $m(\angle BQC)$.
   (d) Determine whether $\triangle BPQ$ is isosceles.
   (e) Let R be the point of intersection of $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$. Find $m(\angle ARB)$.



6. Consider the inscribed pentagon $ABCDE$ pictured at the right. Given the measures of the arcs as indicated in the picture, answer the following.
   (a) What is the radius of the given circle?
   (b) Give the measures of each of the angles of $ABCDE$.
   (c) Let R be the point of intersection of $\overline{AC}$ and $\overline{BD}$. What is $m(\angle BRC)$?
   (d) Let Q be the point of intersection of $\overleftrightarrow{AE}$ and $\overleftrightarrow{CD}$. What is $m(\angle AQC)$?



Answers for Part F [cont.]

2. (a) $OP = r\sqrt{2}$, $AP = r$ (b) $OP = 2r/\sqrt{3}$, $AP = r/\sqrt{3}$
3. In $\triangle ADP$, $m(\angle A) + m(\angle P)$ is the measure of an exterior angle at D [Theorem 17-25]. By Theorem 17-27, $m(\angle A) = k_1/(2r)$ and $m(\angle ADC) = k_2/(2r)$. So, $m(\angle P) = (k_2 - k_1)/(2r)$.
4. $\angle BQD$ is an exterior angle of $\triangle AQB$ and, so, its measure is the sum of the measures of $\angle A$ and $\angle B$. Hence, by Theorem 17-27, $m(\angle BQD) = (k_1 + k_2)/(2r)$. $\angle BQA$ is a supplement of $\angle BQD$. So, $m(\angle BQA) = \pi - (k_1 + k_2)/(2r)$.
5. (a) 4π (b) $2\pi/5$ (c) $3\pi/5$
   (d) In $\triangle BPQ$, $\angle P \cong \angle Q$. So, $\triangle BPQ$ is isosceles with base $\overline{PQ}$.
   (e) $\pi/5$
6. (a) $9/2$
   (b) $m(\angle A) = 2\pi/3$, $m(\angle B) = 2\pi/3$, $m(\angle C) = \pi/2$, $m(\angle D) = 5\pi/9$, $m(\angle E) = 11\pi/18$
   (c) $11\pi/18$
   (d) $\pi/6$

TC 358 (1)

The sum of the degree-measures of the angles of a convex quadrilateral is 360; that of a convex pentagon is 540.

Suggestions for the exercises of section 17.08:
(i) Use Part A to demonstrate degree-measure to your class.
(ii) Use Parts B and C as homework.

From Exercises 3 and 4 we have two useful theorems on measuring angles:

> **Theorem 17–28** The measure of an angle whose vertex is exterior to a circle of radius r and whose sides intersect the circle is the quotient by r of half the difference of the measures of the intercepted arcs.

> **Theorem 17–29** The measure of an angle whose vertex is interior to a circle of radius r is the quotient by r of half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

17.08 Degree–Measures of Angles

There is another measure for angles which is commonly used. It is known as *degree-measure*. We define this concept in terms of the now-familiar radian-measure as follows:

> **Definition 17–6** Given an angle, $\angle A$, whose radian-measure is k, the degree-measure of $\angle A$ is $180k/\pi$.

For convenience, we shall often use '$^{\circ}m(\angle A)$' in place of 'the degree-measure of $\angle A$'.

By definition, an angle of $\pi/2$ radians has as its degree-measure $180 \cdot \frac{\pi}{2}\Big/\pi$, or 90. That is, an angle of $\pi/2$ radians is an angle of 90°. Since angles whose radian-measure is $\pi/2$ are right angles, we have that the degree-measure of a right angle is 90. Alternately, a right angle is an angle of 90°.

In the last section, we learned that the sum of the radian-measures of the angles of a triangle is π. Making use of the relation between radian-measure and degree-measure given in Definition 17–6, it is easy to see that the sum of the degree-measures of the angles of a triangle is 180. [What is the sum of the degree-measures of the angles of a convex quadrilateral? Of a convex pentagon?]

Exercises

Part A

1. In each of the following, you are given the radian-measure of an angle. Give its degree-measure.

(a) $\pi/3$	**(b)** $\pi/4$	**(c)** $2\pi/3$	**(d)** $2\pi/5$
(e) $5\pi/6$	**(f)** $\pi/6$	**(g)** $3\pi/4$	**(h)** $5/2$

TC 358 (2)

Answers for Part A

1. (a) 60 (b) 45 (c) 120 (d) 72
   (e) 150 (f) 30 (g) 135 (h) $450/\pi$

TC 359

2. (a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $3\pi/4$
   (e) $5\pi/6$ (f) $2\pi/3$ (g) $\pi/2$ (h) $\pi/8$
3. (a) 55 (b) 60, 30 (c) $^{\circ}m(\angle B) = 65$, $^{\circ}m(\angle C) = 25$
4. 40, 60, 80
5. (a) 60 (b) 90 (c) 108 (d) 120 (e) 135 (f) 144
6. (a) 180, 360, 540, 720, 1080, 1440 [respectively] (b) 360
7. (a) 180 (b) 90
8. (a) Yes. Such an angle has degree-measure 45.
   (b) Yes. Such an angle has degree-measure 90.

Answers for Part B

1. (a) $\pi/4$; 45 [Making use of $\triangle ABC$ pictured in Part B, we have that $m(\angle A) = m(\angle B)$, $m(\angle C) = \pi/2$, and $m(\angle A) + m(\angle B) + m(\angle C) = \pi$. So, $2m(\angle A) = \pi/2$. Hence, $m(\angle A) = \pi/4 = m(\angle B)$.]
   (b) $\pi/3$; 60
   (c) $\pi/6$ and $\pi/3$; 30 and 60 [Reflecting $\triangle PQR$ in $\overleftrightarrow{QR}$ maps P onto P′, and QPP′ is an equilateral triangle. Using the result in part (b), $m(\angle Q) = \pi/6$ and $m(\angle P) = \pi/3$.]
2. (a) $1/\sqrt{2}$; $1/\sqrt{2}$ [This makes use of the results in Exercise 1 and the fact that congruent angles have the same radian-measures, degree-measures, cosine values, and sine values. So, for example, any angle whose radian-measure is $\pi/4$ is congruent to $\angle ABC$ of the isosceles $\triangle ABC$ described in Exercise 1.]
   (b) $1/2$; $\sqrt{3}/2$ (c) $\sqrt{3}/2$; $1/2$

TC 360

3. (a) $\sqrt{3}/2$; $1/2$ (b) $1/\sqrt{2}$; $1/\sqrt{2}$
   (c) $1/2$; $\sqrt{3}/2$ (d) 0; 1
4. (a) $-1/\sqrt{2}$; $1/\sqrt{2}$ [An angle of $3\pi/4$ radians is a supplement of an angle of $\pi/4$ radians. So, its cosine value is opposite, and its sine value is the same as, that of an angle of $\pi/4$ radians.]
   (b) $-1/2$; $\sqrt{3}/2$ (c) $-\sqrt{3}/2$; $1/2$

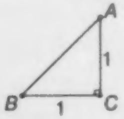
Answers for Part C

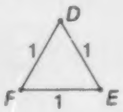
1. $\angle BGF$ is a plane angle of $\angle A\text{-}HG\text{-}F$ and has radian-measure $\pi/6$. So, $\angle A\text{-}HG\text{-}F$ has radian-measure $\pi/6$ and degree-measure 30. A plane angle of $\angle A\text{-}FG\text{-}H$ has radian-measure $\pi/4$. So, $\angle A\text{-}FG\text{-}H$ is a dihedral angle of $\pi/4$ radians and of 45°.
2. $\pi/3$ [As indicated by the dashed right triangle in the picture, a plane angle of $\angle P\text{-}AB\text{-}C$ is an angle whose cosine is $1/2$ and, so, is an angle of $\pi/3$ radians.]

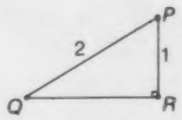
2. The following are degree-measures for angles. Give the corresponding radian-measures.
   (a) 30 (b) 45 (c) 60 (d) 135
   (e) 150 (f) 120 (g) 90 (h) $22\frac{1}{2}$
3. Given that $\triangle ABC$ is a right triangle with hypotenuse $\overline{BC}$.
   (a) Find ${}^\circ m(\angle B)$ given that ${}^\circ m(\angle C) = 35$.
   (b) Find ${}^\circ m(\angle B)$ and ${}^\circ m(\angle C)$, given that ${}^\circ m(\angle B) = 2 \cdot {}^\circ m(\angle C)$.
   (c) If $\overline{AM}$ is the median from A and ${}^\circ m(\angle MAC) = 25$, find ${}^\circ m(\angle B)$ and ${}^\circ m(\angle C)$.
4. Suppose that α, β, and γ are the degree-measures of $\angle A$, $\angle B$, and $\angle C$, respectively, of $\triangle ABC$ and that $\alpha : \beta : \gamma = 2 : 3 : 4$. Find α, β, and γ.
5. Find the degree-measure of an angle of each of the following *regular* polygons.
   (a) equilateral triangle (b) square (c) pentagon
   (d) hexagon (e) octagon (f) decagon
6. (a) What is the sum of the degree-measures of the angles of each of the polygons described in Exercise 5?
   (b) What is the sum of the degree-measures of the exterior angles, one at each vertex, of the polygons described in Exercise 5?
7. What is the sum of the degree-measures of
   (a) two supplementary angles? (b) two complementary angles?
8. (a) Is there an angle which is its own complement? If so, what is its degree-measure? If not, explain.
   (b) Is there an angle which is its own supplement? If so, what is its degree-measure? If not, explain.

Part B

Here are pictures of an isosceles right triangle, $\triangle ABC$, an equilateral triangle, $\triangle DEF$, and a right triangle, $\triangle PQR$, one of whose legs is half as long as its hypotenuse.







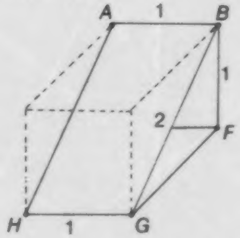
1. What are the radian-measures and degree-measures of the acute angles of
   (a) an isosceles right triangle?
   (b) an equilateral triangle?
   (c) a right triangle like $\triangle PQR$? [*Hint*: Consider the reflection in $\overleftrightarrow{QR}$.]
2. What are the cosine and sine values of an angle whose radian-measure is
   (a) $\pi/4$? (b) $\pi/3$? (c) $\pi/6$?

3. What are the cosine and sine values of an angle whose degree-measure is
   (a) 30? (b) 45? (c) 60? (d) 90?
4. Give the cosine and sine values of angles whose radian-measures are the following.
   (a) $3\pi/4$ (b) $2\pi/3$ (c) $5\pi/6$
   [*Hint*: Consider supplements of the angles described in Exercise 2.]

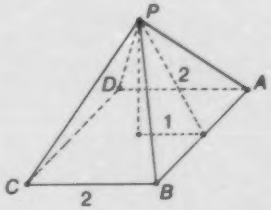
Part C

In Chapter 15, we discussed the notions of a dihedral angle and the plane angle of a dihedral angle. It is customary to assign as the measure of a dihedral angle the measure of its plane angles. So, for example, if a plane angle of a dihedral angle is an angle of $\pi/3$ radians then the dihedral angle is one of $\pi/3$ radians. Its corresponding degree-measure is 60 for its plane angles are angles of 60°.

1. Given the rectangular box some of whose dimensions are shown in the picture at the right, find the radian-measures and degree-measures of $\angle A\text{-}HG\text{-}F$ and $\angle A\text{-}FG\text{-}H$. [$\angle A\text{-}HG\text{-}F$ as indicated in the picture by the heavy lines.]



2. Consider the pyramid $P\text{-}ABCD$ pictured at the right. Its base $ABCD$ is a square whose sides have measure 2 and its triangular faces are isosceles triangles whose altitudes have measure 2. Find the radian-measure of the dihedral angle $\angle P\text{-}AB\text{-}C$.



17.09 Areas of Circular Regions

Earlier, we learned how to compute area-measures of certain plane regions. We shall now consider the problem of computing the area-measure of a circular region. As in the case of computing the measure of an arc, we shall make use of inscribed and circumscribed polygons.

Given that $ABCDEF$ is a polygon inscribed in a circle $\mathscr{K}$ and that $A'B'C'D'E'F'$ is the related circumscribed polygon, as shown in

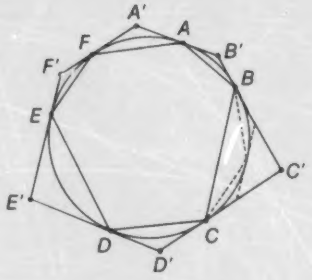


Fig. 17–21

Figure 17–21. It is reasonable to expect that the area-measures of the given inscribed polygon and circumscribed polygon are approximations to the area-measure of $\mathscr{K}$. [The areas of such inscribed and circumscribed polygons are sometimes called *lower approximations* and *upper approximations*, respectively. Why 'lower' and 'upper'?] And, by increasing the number of sides of the inscribed polygon, as indicated by the dashed lines in Figure 17–21, we obtain a better lower approximation. [Can we also obtain a better upper approximation?] If we observe, now, that each circumscribed polygon of $\mathscr{K}$ has an area-measure which is greater than the area-measure of an inscribed polygon of $\mathscr{K}$, we see that the set of all area-measures of polygons inscribed in $\mathscr{K}$ has an upper bound and, so, has a least upper bound. Hence, it is natural to define the area-measure of $\mathscr{K}$ to be that least upper bound. We formalize this in:

> Definition 17–7 The area-measure of a circle $\mathscr{K}$ is the least upper bound of the set of all area-measures of polygons inscribed in $\mathscr{K}$.

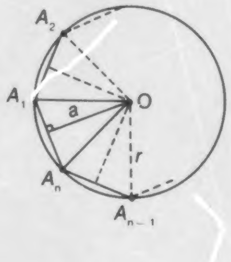


Fig. 17–22

It can be shown that the set of area-measures of regular polygons inscribed in a given circle has the same least upper bound as does the set of area-measures of polygons inscribed in that circle. So, to get some idea as to how to compute the area-measure of a circle, it is enough to consider the regular polygons inscribed in the circle. In Fig. 17–22, we show regular polygon $A_1A_2 \ldots A_{n-1}A_n$ inscribed in a circle with center O and radius r. Each side of the polygon is the base of an isosceles triangle whose vertex, O, is the center of the circle. There are n such isosceles triangles

Answers to questions in the text.

The area-measure of a polygon inscribed in a circle is a lower approximation to the area-measure of the circle because the region it bounds is a proper subset of the circular region and, so, has an area-measure which is less than that of the circle.

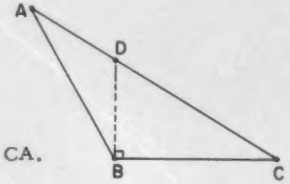
The area-measure of a polygon circumscribed about a circle is an upper approximation to the area-measure of the circle because the region it bounds contains as a proper subset the circular region and, so, has an area-measure which is greater than that of the circle.

A better upper approximation is obtained by increasing the number of sides of the circumscribed polygon, as indicated by the dashed tangent line in Figure 17-21.

*

Sample Quiz

1. In $\triangle ABC$, the radian-measure of $\angle A$ is $3\pi/4$ and that of $\angle B$ is twice that of $\angle C$.
   (a) What is the radian-measure of $\angle B$ and of $\angle C$?
   (b) What are the degree-measures of $\angle A$, $\angle B$, and $\angle C$?
   (c) Given that $AC = 6$, show that $BC = 6\sqrt{2}$ and $AB = 3(\sqrt{6} - \sqrt{2})$.
2. In $\triangle ABC$, suppose that the degree-measure of $\angle B$ is 105, that D is the point of $\overline{AC}$ such that $\angle DBC$ is a right angle and that $\triangle ADB$ is isosceles with base $\overline{AB}$.
   (a) What are the degree-measures of $\angle A$ and $\angle C$?
   (b) If $BC = 6$, compute BD, CD, and CA.



Key to Sample Quiz

1. (a) $\pi/6$; $\pi/12$ (b) 135; 30; 15
   (c) $\frac{BC}{\sin\angle A} = \frac{AC}{\sin\angle B}$, by the sine law, so that $BC = \frac{AC \sin\angle A}{\sin\angle B}$ $\frac{6 \cdot \sqrt{2}/2}{1/2} = 6\sqrt{2}$. Let D be the foot of the perpendicular from C to $\overleftrightarrow{AB}$. Then, $CD = 3\sqrt{2} = AD$ and $BD = 3\sqrt{6}$. So, $AB = 3\sqrt{6} - 3\sqrt{2} = 3(\sqrt{6} - \sqrt{2})$. [Alternately, since $\angle C$ is half as large as $\angle B$, $\sin\angle C = \sqrt{(1 - \cos\angle B)/2} = \sqrt{(1 - \sqrt{3}/2)/2}$ $= \sqrt{2 - \sqrt{3}}/2$. So, $AB = \frac{AC \sin\angle C}{\sin\angle B} = \frac{6\sqrt{2 - \sqrt{3}}/2}{1/2} = 6\sqrt{2 - \sqrt{3}}$. And, since $\sqrt{2 - \sqrt{3}} = (\sqrt{6} - \sqrt{2})/2$, $AB = 3(\sqrt{6} - \sqrt{2})$.]
2. (a) 15; 60 (b) $6\sqrt{3}$; 12; $12 + 6\sqrt{3}$.

and they are all congruent. [Explain.] Given that the measure of the base of each of these isosceles triangles is s and that the altitude to the base is a, it follows that the area-measure of any one of the isosceles triangles is $\frac{1}{2}as$. So, the area-measure of the given n-sided regular polygon is $n \cdot \frac{1}{2}as$ or, more conveniently,

$$(*) \qquad \tfrac{1}{2}a(ns).$$

Notice that for larger values of 'n', (*) yields better approximations to the area-measure of the given circle. Now, the least upper bound of the set of all the measures of altitudes a is r and the least upper bound of the set of all the perimeters, ns, of the inscribed regular polygons is $2\pi r$. This being so, it can be shown that the least upper bound of the set of all the products $\frac{1}{2}a(ns)$ is $\frac{1}{2}r(2\pi r)$, or simply, πr^2. Thus, we have:

The area-measure of a circle of radius r is πr^2.

Exercises

Part A

1. Compute the area-measures of circles whose radii are:
   (a) 3 **(b)** 4/9 **(c)** $\sqrt{3/\pi}$ **(d)** $/\pi$
2. **(a)** Give a formula for computing the area-measure of a circle in terms of the diameter, d, of the circle.
   (b) Make use of your formula to compute the area-measures of circles whose diameters are 6, 8/9, and $2/\pi$.
3. Find the radii of circles whose area-measures are the following.
   (a) 18π **(b)** $18/\pi$ **(c)** 18 **(d)** $9\pi^2/25$
4. Find the area-measures of circles whose circumferences are the following.
   (a) 18π **(b)** $18/\pi$ **(c)** 18 **(d)** $18\pi^2$
5. In each of the following, you are given the radii of circles $\mathscr{K}_1$ and $\mathscr{K}_2$, respectively. Find the ratio of the area-measure of $\mathscr{K}_1$ to that of $\mathscr{K}_2$.
   (a) 5, 7 **(b)** 5, 10 **(c)** 5, 3 **(d)** r_1, r_2

*

Given an arc—say, $\widehat{AB}$—of a circle with center O and radius r, the region bounded by the arc and the radii $\overline{OA}$ and $\overline{OB}$ is called *a circular*

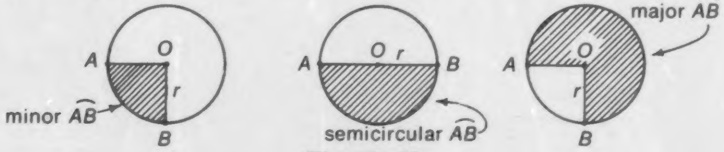


Fig. 17–23

Explanation called for in the text. There are n such isosceles triangles as a polygon with n vertices has n sides. They are all congruent by s.s.s.

Suggestions for the exercises of section 17.09:
(i) Part A, and the discussion preceding it, should be teacher directed.
(ii) Parts B and C may be assigned for homework.

Answers for Part A

1. (a) 9π (b) $16\pi/81$ (c) $3/\pi$ (d) $/\pi$
2. (a) $K = \pi d^2/4$
   (b) 9π; $16\pi/81$; $/\pi$
3. (a) $3\sqrt{2}$ (b) $3/(\pi\sqrt{2})$ (c) $3\sqrt{2/\pi}$ (d) $3\sqrt{\pi}/5$
4. (a) $8/\pi$ (b) $81/\pi^3$ (c) $81/\pi$ (d) $81\pi^3$
5. (a) 35/49 (b) 1/4 (c) 25/9 (d) r_1^2/r_2^2

sector [or: a sector] of the circle. Making use of inscribed regular polygons, we can obtain the area-measure of a circular sector in terms of r and the measure of the bounding arc.

To obtain a formula for the area-measure of a sector of a circle determined by an arc $\overset{\frown}{AB}$ of that circle, consider the regular polygon $AA_1A_2 \ldots A_n$ inscribed in a circle of radius r. Those vertices of the polygon which are points of $\overset{\frown}{AB}$ together with A and perhaps [but not necessarily] B determine the bases of congruent isosceles triangles whose vertex is the center O of the circle. As in the case of the full circle, we see that the area-measure of the sector is approximated by the sum of the area-measures of those isosceles triangles which are contained in the given sector. So, given that the measure of the bases of these isosceles triangles is s, that the altitude to the base is a, and that there are k of these isosceles triangles contained in the sector, we see that the area-measure of the sector is approximated by:

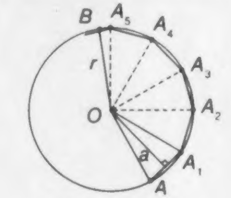


Fig. 17–24

$$\tfrac{1}{2}a(ks)$$

Since the least upper bound of the set of all the measures of altitudes a is r and the least upper bound of the set of all sums of measures of bases included in the sector is $m(\overset{\frown}{AB})$, it can be shown that the least upper bound of the set of all products $\frac{1}{2}a(ks)$ is $\frac{1}{2}r \cdot m(\overset{\frown}{AB})$. In short,

(**) the area-measure of a sector of a circle of radius r and determined by an arc with measure l is $\frac{1}{2}rl$.

For example, assume that in the circle of radius 6 shown at the right, the measure of $\overset{\frown}{ACB}$ is 8. The area-measure of sector $OACB$ is $\frac{1}{2}\cdot 6 \cdot 8$, or 24. Note that, in this example, the measure of $\overset{\frown}{ADB}$ is $12\pi - 8$. So, the area-measure of sector $OADB$ is $\frac{1}{2}\cdot 6 \cdot (12\pi - 8)$, or $36\pi - 24$. This last result "checks" in that the sum of the area-measures of the given sectors is the area-measure of the circle.

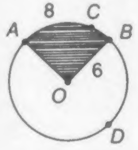


Fig. 17–25

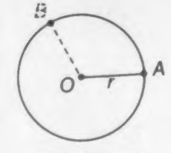


Fig. 17–26

Given a circle $\mathscr{K}$ with center O and radius r, let A be any point of $\mathscr{K}$. Then, any point of $\mathscr{K}$ different from A determines two arcs of $\mathscr{K}$ and, so, determines two sectors of $\mathscr{K}$. The sum of the area-measures of the two sectors determined by two points A and B of $\mathscr{K}$ is, of course, πr^2. Thus, it is enough to know the area-measure of one of those sectors [and the radius] in order to compute the area-measure of the other. If $\overline{AB}$ is a diameter, then the sectors have the same area-measure. If $\overline{AB}$ is not a diameter, then $\angle AOB$ intercepts minor $\overset{\frown}{AB}$ and the area-measure of the sector in the interior of $\angle AOB$ is an increasing function of the radian-measure of $\angle AOB$ as well as of the degree-measure of $\angle AOB$.

We have already discussed the assignment of length-measures to circular arcs. There is another kind of measure—called *degree-measure*—which is at times convenient to assign to circular arcs. It is done as follows:

Given two points A and B of a circle with center O,
(a) if $\overset{\frown}{AB}$ is a minor arc, $^\circ m(\overset{\frown}{AB})$ is $^\circ m(\angle AOB)$,
(b) if $\overset{\frown}{AB}$ is a semicircular arc, $^\circ m(\overset{\frown}{AB})$ is 180, and
(c) if $\overset{\frown}{AB}$ is a major arc, $^\circ m(\overset{\frown}{AB})$ is $360 - {}^\circ m(\angle AOB)$.

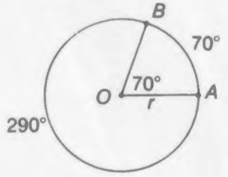


Fig. 17–27

Thus, if $\angle AOB$ is a central angle of 70°, it follows that the degree-measure of minor $\overset{\frown}{AB}$ is 70 and the degree-measure of major $\overset{\frown}{AB}$ is $360 - 70$, or 290. That is, minor $\overset{\frown}{AB}$ is an arc of 70° and major $\overset{\frown}{AB}$ is an arc of 290°. [What are the degree-measures of the major and minor arcs of a circle determined by a central angle which is a right angle? Is this the case regardless of the radius of the circle? Explain.]

In the example described in Fig. 17–27, we see that A and B determine two sectors of the given circle. Furthermore, the length-measure of minor $\overset{\frown}{AB}$ is $\frac{70}{360}\cdot 2\pi r$. [Why?] So, the area-measure of the minor sector OAB is $\frac{1}{2}r \cdot \frac{70}{360}\cdot 2\pi r$, or $\frac{70}{360}\pi r^2$. [What is the area-measure of the major sector OAB?] Reasoning in a similar fashion with regard to an arc of a°, we see that the area-measure of the sector determined by that arc is $\frac{a}{360}\cdot \pi r^2$, where r is the radius of the circle.

The degree-measures of the circular arcs determined by a central right angle are 90 [minor arc] and 270 [major arc]. Since the definition of degree-measure for an arc $\widehat{AB}$ refers only to whether the arc is semicircular, minor, or major and, in the last two cases to $°m(\angle ADB)$, the degree-measure of an arc is independent of its radius. [It may be worth noting that arcs are similar if and only if they have the same degree-measure.]

Since $°m(\angle AOB) = 70$, $m(\angle AOB) = 70(2\pi/360)$ and, so, $m(\widehat{AB}) = \frac{70}{360} \cdot 2\pi r$.

The area-measure of the major sector OAB is $\frac{290}{360} \cdot 2\pi r$.

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Answers for Part B

1. (a) $15\pi/2$ (b) 18π (c) 25 (d) 50π (e) $25\pi/2$ (f) $25/2$
2. (a) $63\pi/4$ (b) $100\pi/3$ (c) 50π
   (d) $405\pi/8$ (e) $243\pi/4$ (f) $175\pi/2$
3. (a) 90 (b) 120 (c) 200 (d) 216 (e) 330 (f) 324
4. (a) 90 (b) 120 (c) 160 (d) 144 (e) 30 (f) 38

Answers for Part C

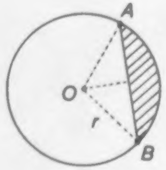
1. (a) $100(\pi/3 - \sqrt{3}/4)$ (b) $9\pi - 18$ (c) $32\pi/3 - 16\sqrt{3}$
2. (a) $100\pi/3$
   (b) 4
   (c) $100\pi/3 - 25\sqrt{3}$
   (d) $25\sqrt{3} - 25\pi/3$ [or: $(225/\sqrt{3} - 25\pi)/3$]

Part B

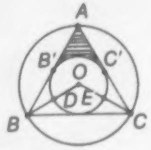
1. In each of the following, compute the area-measure of the sector from the information given about its arc, $\widehat{AB}$, and radius, r.
   (a) $m(\widehat{AB}) = 3\pi$, $r = 5$ (b) $m(\widehat{AB}) = 6\pi$, $r = 6$
   (c) $m(\widehat{AB}) = 10$, $r = 5$ (d) $\widehat{AB}$ is semicircular, $r = 10$
   (e) $\widehat{AB}$ is semicircular, $r = 5$ (f) $m(\widehat{AB}) = 5$, $r = 5$
2. In each of the following, compute the area-measure of the sector from the information given about its arc, $\widehat{AB}$, and radius, r.
   (a) $°m(\widehat{AB}) = 70$, $r = 9$ (b) $°m(\widehat{AB}) = 120$, $r = 10$
   (c) $°m(\widehat{AB}) = 180$, $r = 10$ (d) $°m(\widehat{AB}) = 225$, $r = 9$
   (e) $°m(\widehat{AB}) = 270$, $r = 9$ (f) $°m(\widehat{AB}) = 315$, $r = 10$
3. What is the degree-measure of an arc of a circle whose sector has an area-measure which is
   (a) $\frac{1}{4}$ that of the circle? (b) $\frac{1}{3}$ that of the circle?
   (c) $\frac{5}{9}$ that of the circle? (d) 60% that of the circle?
   (e) $\frac{11}{12}$ that of the circle? (f) 90% that of the circle?
4. What are the degree-measures of the central angles determined by the end points of the arcs described in Exercise 3?

Part C

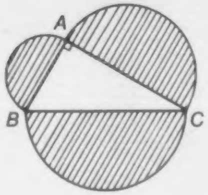
1. A *circular segment* is a region bounded by an arc of a circle and its chord. Given the circular segment bounded by minor $\widehat{AB}$ and $\overline{AB}$ shown at the right, compute its area-measure when
   (a) $r = 10$ and $°m(\widehat{AB}) = 120$;
   (b) $\triangle AOB$ is a right triangle and $r = 6$;
   (c) $\overline{AB}$ is the side of a regular *hexagon* inscribed in the circle and $r = 8$.



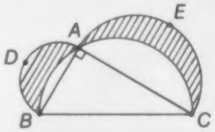
2. Consider the circles inscribed and circumscribed about equilateral $\triangle ABC$. Given that O is the common center of these circles and that $OB = 10$ compute the following.
   (a) Area-measure of minor sector OBC.
   (b) Ratio of area-measures of sectors OBC and ODE.
   (c) Area-measure of segment bounded by minor $\widehat{AB}$ and $\overline{AB}$.
   (d) Area-measure of shaded region $AB'C'$.



3. Given the semicircular regions whose diameters are the sides of right triangle, $\triangle ABC$, whose hypotenuse is $\overline{CB}$, show that the area-measure of the region on the hypotenuse is the sum of the area-measures of the regions on the legs of $\triangle ABC$.



4. Suppose that $\widehat{BDA}$, $\widehat{BAC}$, and $\widehat{AEC}$ are semicircular arcs. Show that the sum of the area-measures of the shaded regions is the area-measure of $\triangle ABC$.



17.10 Chapter Summary

Vocabulary Summary

sphere
radius
semicircle
circumcircle
secant line
central angle
upper bound
least upper bound
radian-measure
regular polygon
circle
diameter
major arc
incircle
internally tangent circles
inscribed angle
lower bound
greatest lower bound
degree-measure
pentagon
chord
circumference
minor arc
tangent line
externally tangent circles
intercepted arc
inscribed polygonal line
circumscribed polygonal line
convex polygon
hexagon

Additional Postulates

5_{13}. Each nonempty subset of $\mathscr{R}$ which has an upper bound has a least upper bound.

5. $\mathscr{R}$ is a complete ordered field.

Definitions

17-1. The sphere with center C and radius $r > 0$ is $\{X : \|X - C\| = r\}$.

17-2. The circle of π with center $C \in \pi$ and radius $r > 0$ is

$$\{X : X \in \pi \text{ and } \|X - C\| = r\}.$$

Answers for Part C [cont.]

3. $\pi(BC)^2/2 = \pi(CA)^2/2 + \pi(AB)^2/2$ because $(BC)^2 = (CA)^2 + (AB)^2$.

4. Let K_1, K_2, and K_3 be the area-measures of the semicircular regions and let K be that of the triangular region. By Exercise 3, $K_1 + K_2 = K_3$. From the figure, the sum of the area-measures of the shaded regions is $K_1 + K_2 - (K_3 - K)$. So, this sum is K.

17-3. Given a coplanar line l and circle $\mathscr{K}$, (a) l is a tangent of $\mathscr{K}$ if and only if $l \cap \mathscr{K}$ consists of exactly one point, and (b) l is a secant of $\mathscr{K}$ if and only if $l \cap \mathscr{K}$ consists of two points.

17-4. The measure of a circular arc is the least upper bound of the set of measures of polygonal lines inscribed in the arc.

17-5. The radian-measure of an angle is the ratio of the measure of the arc intercepted by the angle, on any circle for which it is a central angle, to the radius of the arc.

17-6. Given an angle, $\angle A$, whose radian-measure is k, the degree-measure of $\angle A$ is $180k/\pi$.

Other Theorems

17-1. Four noncoplanar points belong to one and only one sphere.

17-2. Three noncollinear points belong to one and only one circle.

Corollary. Any triangle has a unique circumscribed circle.

17-3. The intersection of a plane π and the sphere with center C and radius r is either the empty set, or a set consisting of a single point, or a circle. Specifically, if F is the foot of the perpendicular from C to π and d is the distance between C and π then the intersection is $\varnothing$ if $d > r$, is $\{F\}$ if $d = r$, and is the circle of π with center F and radius $\sqrt{r^2 - d^2}$ if $d < r$.

17-4. The intersection of coplanar line l and circle with center C and radius r is either the empty set, or a set consisting of a single point, or a set consisting of two points. Specifically, if F is the foot of the perpendicular from C to l and $d = CF$ then the intersection is $\varnothing$ if $d > r$, is $\{F\}$ if $d = r$, and consists of the two points of l whose distance from F is $\sqrt{r^2 - d^2}$ if $d < r$.

17-5. Two spheres [or: coplanar circles] with centers C_1 and C_2 and radii r_1 and r_2 intersect if and only if $|r_1 - r_2| \leq \|C_1 - C_2\| \leq r_1 + r_2$.

17-6. Two spheres [or: circles] are congruent if and only if they have the same radius.

Corollary. Any two spheres [or: circles] are similar in the ratio of their radii.

17-7. Any isometry maps a circle onto a congruent circle, mapping minor arcs onto minor arcs, major arcs onto major arcs, and semicircles onto semicircles, mapping centers on centers and endpoints on endpoints.

17-8. Given two noncongruent chords of the same circle, the shorter of the chords is farther from the center of the circle.

17-9. Minor [or: Major] arcs of congruent circles are congruent if and only if the chords which subtend them are congruent.

17-10. Any arc $\widehat{ABC}$ is a union $\widehat{AB} \cup \{B\} \cup \widehat{BC}$ where $\widehat{AB}$ and $\widehat{BC}$ have no common point and each may be either a minor arc, a semicircle, or a major arc.

17-11. A coplanar line is tangent to a circle at a given point of the circle if and only if the line contains the point and is perpendicular to the radius at that point.

17-12. An angle inscribed in a major arc is half as large as its corresponding central angle; an angle inscribed in a minor arc is a supplement of an angle half as large as its corresponding central angle.

Corollary 1. Any two inscribed angles which intercept the same arc are congruent.

Corollary 2. If $ABCD$ is a convex quadrilateral inscribed in a circle, each two opposite angles of $ABCD$ are supplementary.

17-13. If two chords of a circle intersect, the point of intersection divides each chord into segments such that the product of the measures of the segments of one chord is the product of the measures of the segments of the other.

17-14. If two secants of a circle intersect at a point in the exterior of a circle, the product of the distances between the exterior point and the points of intersection on one of the secants is the product of the distances between the exterior point and the points of intersection on the other secant.

Corollary. If a secant and a tangent of a circle intersect at a point exterior to the circle then the product of the distances between the exterior point and the points of intersection on the secant is the square of the distance between the exterior point and the point of tangency.

17-15. If a secant and tangent of a circle intersect in a point of the circle, the angle between the secant and the tangent is either congruent to, or a supplement of, an angle half as large as its corresponding central angle according as the center of the circle is exterior or interior to the former angle.

17-16. If $\widehat{AP}$ and $\widehat{PB}$ are arcs [minor, semicircular, or major] which have no point in common then $m(\widehat{APB}) = m(\widehat{AP}) + m(\widehat{PB})$.

Corollary. The circumference of a circle is the sum of the measures of any two arcs which have the same endpoints.

17-17. The ratio of the circumference of one circle is that of a second is the ratio of the diameter of the first to that of the second.

Corollary. If a circle has diameter d and circumference c then $c = \pi d$.

17-18. If two arcs are similar then the ratio of their measures is the ratio of their radii.

Corollary. Congruent arcs have the same measure.

17-19. If the ratio of the measures of two arcs in the ratio of their radii then the arcs are similar.

Corollary. Arcs which have the same radius and the same measure are congruent.

17-20. Angles are congruent if and only if they have the same radian-measure.

17-21. If D is interior to $\angle ABC$ then $m(\angle ABC) = m(\angle ABD) + m(\angle DBC)$.

17-22. Angles are supplementary if and only if the sum of their measures is π.

17-23. Given a number k such that $0 < k < \pi$, and given a half-line r, there is one and only one angle whose radian-measure is k, which has r as one of its sides, and whose other side is in a given side of the line which contains r.

17-24. Given that $m(\angle ABC) = k$. Then, (a) $\angle ABC$ is right if and only if $k = \pi/2$, (b) $\angle ABC$ is acute if and only if $0 < k < \pi/2$, and (c) $\angle ABC$ is obtuse if and only if $\pi/2 < k < \pi$.

17-25. The measure of an exterior angle of a triangle is the sum of the measures of the opposite interior angles.

17-26. The sum of the measures of the angles of a convex polygon of n sides is $(n - 2)\pi$.

Corollary 1. The sum of the measures of the angles (a) of a triangle is π; (b) of a convex quadrilateral is 2π.

Corollary 2. The sum of the measures of the exterior angles, one at each vertex, of a convex polygon is 2π.

Corollary 3. The measures of an angle of a regular polygon of n sides is $(n - 2)\pi/n$.

17-27. The measure of an angle inscribed in a circle of radius r is half the measure of the arc intercepted by that angle divided by r.

17-28. The measure of an angle whose vertex is exterior to a circle of radius r and whose sides intersect the circle is the quotient by r of half the difference of the measures of the intercepted arcs.

17-29. The measure of an angle whose vertex is interior to a circle of radius r is the quotient by r of half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

Chapter Test

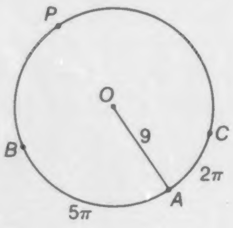
1. Given that $\angle A$ is an angle of $7\pi/12$ radians.
   - **(a)** What is the degree-measure of a supplement of $\angle A$?
   - **(b)** Does $\angle A$ have a complement? If so, give its degree-measure. If not, explain.
   - **(c)** If $\angle A$ is the vertex angle of an isosceles triangle, $\triangle ABC$, what is $°m(\angle B)$?
   - **(d)** Find the radian-measures of $\angle B$ and $\angle C$ of $\triangle ABC$ given that $m(\angle B)/m(\angle C) = \frac{7}{3}$.

Answers for Chapter Test

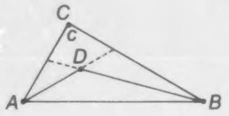
1. (a) 75 [The radian-measure of a supplement of $\angle A$ is $5\pi/12$. So, the degree-measure of a supplement of $\angle A$ is $\frac{5\pi}{12} \cdot \frac{180}{\pi}$, or 75.]
   - (b) No, for $\angle A$ is obtuse and only acute angles have complements.
   - (c) 75/2, or $37\frac{1}{2}$. [The radian-measure of $\angle B$ is $\frac{1}{2} \cdot \frac{5\pi}{12}$.]
   - (d) $m(\angle B) = 7\pi/24$ and $m(\angle C) = \pi/8$ [From the given information, $m(\angle B) = 7a$ and $m(\angle C) = 3a$, for some a, and $7a + 3a = 5\pi/12$. From the latter, $a = \pi/24$ so that $m(\angle B) = 7 \cdot \pi/24$ and $m(\angle C) = 3 \cdot \pi/24$.]

2. (a) 105 [$°m(\angle B) = 50 = °m(\angle ACB)$ so that $°m(\angle FCB) = 25$. So, $°m(\angle BFC) = 180 - (50 + 25) = 105$.]
   - (b) FC, AC, AF. [The degree-measures of $\angle A$, $\angle F$, and $\angle C$, in $\triangle AFC$, are 80, 75, and 25, respectively, and the larger of two sides of a triangle is opposite the larger of their opposite angles.
3. (a) $\pi/3$ [The radian-measure of the central angle intercepting any arc whose end points are two consecutive hour marks is $\pi/6$.]
   - (b) $5\pi/6$ (c) $\pi/6$
4. (a) 100 [The radian-measure of $\angle AOB$ is $5\pi/9$, and $\frac{5\pi}{9} \cdot \frac{180}{\pi} = 100$.]
   - (b) 20 [$°m(\angle APC) = \frac{1}{2} \cdot °m(\angle AOC) = \frac{1}{2} \cdot \frac{2\pi}{9} \cdot \frac{180}{\pi}$.]
   - (c) 70 [$m(\overset{\frown}{CP}) = 7\pi$. So, $°m(\angle PBC) = \frac{1}{2} \cdot °m(\angle POC) = \frac{1}{2} \cdot \frac{7\pi}{9} \cdot \frac{180}{\pi} = 70$.]
5. $90 + \frac{1}{2}c$ [$°m(\angle ADB) = 180 - [°m(\angle DAB) + °m(\angle DBA)] = 180 - [\frac{1}{2}°m(\angle CAB) + \frac{1}{2}°m(\angle CBA)] = 180 - \frac{1}{2}[°m(\angle CAB) + °m(\angle CBA)] = 180 - \frac{1}{2}[180 - c] = 90 + \frac{1}{2}c$.]
6. (a) 8, 8 [$AC = AS + SC = \sqrt{5^2 - 3^2} + \sqrt{5^2 - 3^2} = 8$. Similarly for AB.]
   - (b) In $\triangle ABC$, T and S are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. So, $\overline{ST} \parallel \overline{CB}$.
   - (c) A is equidistant from C and B, for $AC = AB = 8$. Also O is equidistant from C and B. So, $\overleftrightarrow{AO}$ is contained in the perpendicular bisector of $\overline{CB}$.
   - (d) $ST = 24/5$ and $CB = 48/5$. [Let M be the point of intersection of $\overline{ST}$ and $\overline{AO}$. Then, $\overline{SM}$ is the altitude from S in right triangle, $\triangle ASO$, and $SM = \frac{1}{2}ST$. So, $SM = 3 \cdot 4/5$ so that $ST = 24/5$. Since $CB = 2ST$, $CB = 48/5$.]
7. 18 [Each exterior angle of the given polygon has degree-measure 20. Since the sum of the degree-measures of the exterior angles, one at each vertex, is 360, and the number of sides is the number of vertices, there are 360/20, or 18, sides.]
8. (a) $4\sqrt{3}$ [$\triangle GAB$ is equilateral and $\overline{GH}$ is an altitude. So, $GH = 8\sqrt{3}/2 = 4\sqrt{3}$.]
   - (b) $96\sqrt{3}$ [As suggested in the figure, the given region is composed of six equilateral triangular regions. So, the area-measure is $6 \cdot \frac{1}{2} \cdot 8 \cdot 4\sqrt{3}$, or $96\sqrt{3}$.]

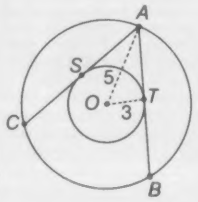
2. Given isosceles triangle, $\triangle ABC$, with base $\overline{BC}$, assume that $\overline{CF}$ is the angle bisector from C and that $^\circ m(\angle B) = 50$.
   (a) Find $^\circ m(\angle CFB)$.
   (b) Arrange the sides of $\triangle AFC$ in order from longest to shortest.
3. What is the radian-measure of the angle formed by the minute hand and hour hand of a clock at
   (a) 2 o'clock? (b) 7 o'clock? (c) 11 o'clock?
4. Given a circle with center O and radius 9, assume that $m(\widehat{AB}) = 5\pi$ and $m(\widehat{AC}) = 2\pi$, as shown in the picture.
   (a) Find $^\circ m(\angle AOB)$.
   (b) Find $^\circ m(\angle APC)$, where $\angle APC$ is an inscribed angle which intercepts $\widehat{AC}$.
   (c) Given that $\overline{AP}$ is a diameter, find $^\circ m(\angle PBC)$.



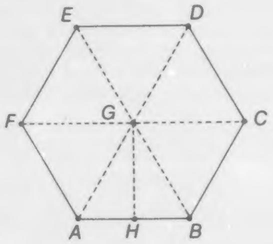
5. Suppose that the angle bisectors of $\angle A$ and $\angle B$ of $\triangle ABC$ intersect in the point D, as shown at the right, and that $^\circ m(\angle C) = c$. Find $^\circ m(\angle ADB)$ in terms of 'c'.



6. Given that coplanar circles $\mathscr{K}_1$ and $\mathscr{K}_2$ have the same center, O, that the radii of $\mathscr{K}_1$ and $\mathscr{K}_2$ are 3 and 5, respectively, and that $\overline{AC}$ and $\overline{AB}$ are chords of $\mathscr{K}_2$ which are tangent to $\mathscr{K}_1$ at S and T, as shown at the right.
   (a) Find AC and AB.
   (b) Show that $\overline{ST} \parallel \overline{CB}$.
   (c) Show that $\overleftrightarrow{AO}$ is contained in the perpendicular bisector of $\overline{CB}$.
   (d) Find ST and CB.



7. Given that each of the angles of a regular polygon is an angle of 160°, how many sides has the regular polygon?
8. Assume that $ABCDEF$ is a regular hexagon whose sides have measures 8. Given that G is the center of the hexagon find the following.
   (a) GH, where H is the foot of the perpendicular from G to $\overline{AB}$.
   (b) The area-measure of hexagon $ABCDEF$.



The principal purpose of this Background Topic is to introduce students to the notions of even functions and odd functions, and to call to their attention certain properties of these functions. This is done in Part B. Part A deals with the power functions for nonnegative integral exponents. It is relevant to Part B in that, in parts (a) and (b) of Exercise 2, it is shown that the power functions with even exponents are among those functions which, in Part B, are called even functions, and that the power functions with odd exponents are odd functions. Part A also serves to give students a little practice in mathematical induction and a foundation for the main laws of exponents.

There are three explanations asked for in the preamble for Part A. First, since $1 = 0 + 1$, $a^1 = a^{0+1} = a^0 \cdot a = 1 \cdot a = a$ [the middle two '='s are justified by the two parts of the recursive definition (1)] and since $2 = 1 + 1$, $a^2 = a^{1+1} = a^1 \cdot a = a \cdot a$ [the last '=' is justified by the previously proved theorem '$a^1 = a$'].

For the second explanation, the first '=' is justified by the fact that $m + 0 = m$, the second by the fact that, for any c, $c = c \cdot 1$, the third by the first part of (1).

For the third explanation, the first '=' is justified by the fact that $m + (p + 1) = (m + p) + 1$, the second by the second part of (1), the third by the assumption '$a^{m+p} = a^m \cdot a^p$' [the so-called 'inductive hypothesis'], the fourth by the associative principle for multiplication, and the fifth by the second part of (1). [Note that in the first application of (1) we need to know that, for $m, p \in Nn$, $m + p \in Nn$. This we have proved in earlier exercises.]

The 'Why?' is to be answered by reference to the induction postulate (Nn_3). We have shown that '$a^{m+n} = a^m \cdot a^n$' holds for the value 0 of 'n' and that if it holds for any given value $p \in Nn$ of 'n' then it also holds for the value $p + 1$. So, by (Nn_3) we are assured that '$a^{m+n} = a^m \cdot a^n$' holds for any value of 'n' in Nn. We have seen, in fact, that this is the case whatever nonnegative integral value 'm' may be given. Hence, (2).

Background Topic

Part A

We have, earlier, adopted the recursive definition:

$$(1) \qquad \begin{cases} a^0 = 1 \\ a^{n+1} = a^n \cdot a \qquad [n \in Nn] \end{cases}$$

[Note that it follows from this that, for any real number $a, a^1 = a$ and $a^2 = a \cdot a$. Explain.] On the basis of this definition we can use mathematical induction to prove:

$$(2) \qquad m, n \in Nn \Longrightarrow a^{m+n} = a^m \cdot a^n, \text{ and}$$
$$(3) \qquad m, n \in Nn \Longrightarrow a^{mn} = (a^m)^n$$

To prove (2), we begin by noting that, for $m \in Nn$, $a^{m+0} = a^m = a^m \cdot 1 = a^m \cdot a^0$. [Explain.] The next step is to show that, assuming, for $m, p \in Nn$, that $a^{m+p} = a^m \cdot a^p$, it follows that $a^{m+(p+1)} = a^m \cdot a^{p+1}$. This is easy, since $a^{m+(p+1)} = a^{(m+p)+1} = a^{m+p} \cdot a = a^m \cdot a^p \cdot a = a^m(a^p \cdot a) = a^m \cdot a^{p+1}$. [Explain.] We can now conclude by mathematical induction, that (2) is a theorem. [Why?]

1. Prove (3). [*Hint*: Use mathematical induction. At a point in the second step you will find it convenient to use (1).]
2. (a) Prove that $(-a)^2 = a^2$. [*Hint*: $(-a)^2 = -a \cdot -a = \ldots.$]
   (b) Prove that, for $n \in Nn$, $(-a)^{2n} = a^{2n}$. [*Hint*: Use part (a) and theorem (2).]
   (c) Prove that, for $n \in Nn$, $(-a)^{2n+1} = -a^{2n+1}$.
3. Show that, for $n \in Nn$,
   (a) $1^n = 1$,
   (b) $(-1)^{2n} = 1$,
   (c) $(-1)^{2n+1} = -1$.
4. Use mathematical induction to prove:

$$(4) \qquad m \in Nn \Longrightarrow (ab)^m = a^m \cdot b^m$$

Part B

Suppose that, for a given $n \in Nn$, $f(x) = x^{2n}$ and $g(x) = x^{2n+1}$, for each real number x. In Exercise 2 of Part A you have shown that, for each x,

$$\text{(i) } f(-x) = f(x) \qquad \text{and} \qquad \text{(ii) } g(-x) = -g(x).$$

Functions which, like f and g, satisfy either (i) or (ii) are of special interest in some parts of mathematics. Functions like f are called *even functions* and those like g are called *odd functions*. [Guess the reason for using the words 'even' and 'odd' to describe such functions.]

1. Suppose you were given an even function and asked to draw its graph for arguments between −10 and 10. Knowing that the func-

Answers for Part A

1. To begin with, for $m \in Nn$, $a^{m0} = a^0 = 1 = (a^m)^0$ [because $m0 = 0$, and by the first part of (1), used twice]. Suppose then that, for $m, p \in Nn$, $a^{mp} = (a^m)^p$. It follows that

$$a^{m(p+1)} = a^{mp+m} = a^{mp} \cdot a^m = (a^m)^p \cdot a^m = (a^m)^{p+1}$$

[because $m(p + 1) = mp + m$, and by (2), by the inductive hypothesis, and by the second part of (1)]. Hence, by mathematical induction, for $m, n \in Nn$, $a^{mn} = (a^m)^n$.

2. (a) $(-a)^2 = -a \cdot -a = --(a \cdot a) = a \cdot a = a^2$ [by the second "little theorem" proved in the preamble and three real number theorems dealing with oppositing]. [Note that in applying (2) we have used the fact that, for $m, p \in Nn$, $mp \in Nn$.]
   (b) For $n \in Nn$ [and since $2 \in Nn$], $(-a)^{2n} = [(-a)^2]^n = [a^2]^n = a^{2n}$ [because of (2), part (a), and (2)].
   (c) For $n \in Nn$ [and since $2 \in Nn$], $(-a)^{2n+1} = (-a)^{2n} \cdot -a = -(a^{2n} \cdot a) = -a^{2n+1}$ [because of (1), part (b), a theorem about oppositing, and (1)].
3. (a) To begin with $1^0 = 1$. Suppose, for some $p \in Nn$, that $1^p = 1$. Then $1^{p+1} = 1^p \cdot 1 = 1 \cdot 1 = 1$. Hence, by mathematical induction, for $n \in Nn$, $1^n = 1$.
   (b) By part (b) of Exercise 2, $(-1)^{2n} = 1^{2n} = 1$, by part (a) of the present exercise.
   (c) Similarly, $(-1)^{2n+1} = -1^{2n+1} = -1$.
4. To begin with, $(ab)^0 = 1 = 1 \cdot 1 = a^0 \cdot b^0$. Suppose, for some $p \in Nn$, that $(ab)^p = a^p \cdot b^p$. It follows that $(ab)^{p+1} = (ab)^p(ab) = (a^p \cdot b^p)(ab) = (a^p \cdot a)(b^p \cdot b) = a^{p+1} \cdot b^{p+1}$. Hence, by mathematical induction, for $m \in Nn$, $(ab)^m = a^m \cdot b^m$.

tion is even would save you about half the work that you would otherwise have to do. Explain how this comes about.

2. Repeat Exercise 1 with 'odd' in place of 'even'.

3. Illustrate your answers for Exercises 1 and 2 by drawing some graphs which you are sure are graphs of even functions and some which you are sure are graphs of odd functions. Also, draw some graphs which represent functions which are neither even nor odd.

4. **(a)** What can you say about the value of an odd function at the argument 0?

(b) Is there a function [defined for all real numbers] which is both even and odd?

5. **(a)** Is the absolute value function even or odd? If so, tell which.

(b) Is the signum function even or odd? If so, tell which.

(c) Is the integral part function even or odd? If so, tell which.

6. Suppose that f_1 and f_2 are even functions and that g_1 and g_2 are odd functions. For each of the functions described below, tell whether the function is even or odd, or may be neither.

(a) $h_1(x) = f_1(x) + f_2(x)$ **(b)** $h_2(x) = g_1(x) + g_2(x)$
(c) $h_3(x) = f_1(x) + g_1(x)$ **(d)** $h_4(x) = f_1(x)f_2(x)$
(e) $h_5(x) = f_1(x)g_1(x)$ **(f)** $h_6(x) = g_1(x)g_2(x)$

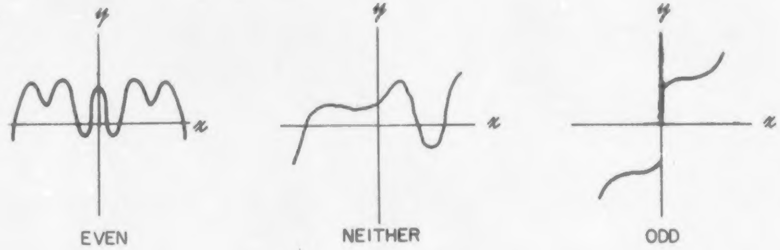
***7.** Suppose that h is any function whose domain is $\mathscr{R}$ and whose range is a subset of $\mathscr{R}$. Show that there is an even function f and an odd function g such that, for each $x \in \mathscr{R}$, $h(x) = f(x) + g(x)$. [*Hint*: What kind of function is the function k such that, for each x, $k(x) = h(x) + h(-x)$?]

Answers for Part B

1. Since an even function has the same value for an argument and its opposite, it would be enough to draw the graph for arguments between 0 and 10, 0 included, and then copy this by reflection in the vertical axis. [Graphing in the x, y-plane, the y-axis is a line of symmetry for any even function.]

2. In the case of an odd function, draw the graph of arguments between 0 and 10, 0 included, and copy this in the third quadrant according to the rule $(x, y) \rightarrow (-x, -y)$. [The origin is a center of symmetry for any odd functions.]

3.



EVEN NEITHER ODD

4. (a) If f is an odd function then $f(0) = 0$. [For, if f is odd, $f(-0) = -f(0)$, while in any case, $f(-0) = f(0)$.]

(b) There is such a function, and only one such. Its value is 0 for each real number.

5. (a) The absolute value function is even. $[|-a| = |a|]$

(b) The signum function is odd. $[\mathrm{sgn}(-x) = -\mathrm{sgn}(x)]$

(c) The integral part function is neither even nor odd. $[\llbracket 1/2 \rrbracket = 0, \llbracket -1/2 \rrbracket = -1]$

6. (a) h_1 is even. $[h_1(-a) = f_1(-a) + f_2(-a) = f_1(a) + f_2(a) = h_1(a)]$

(b) h_2 is odd. $[h_2(-a) = g_1(-a) + g_2(-a) = -g_1(a) + -g_2(a) = -[g_1(a) + g_2(a)] = -h_2(a)]$

(c) If f_1 is the function of Exercise 4(a) then $h_3 = g_1$ and h_3 is odd. If g_1 is the function of Exercise 4(a) then $h_3 = f_1$, and h_3 is even. In fact, by Exercise 7, for proper choice of f_1 and g_1, h_3 may be any real-valued function whose domain is $\mathscr{R}$.

(d) h_4 is even. $[h_4(-a) = f_1(-a)f_2(-a) = f_1(a)f_2(a) = h_4(a)]$

(e) h_5 is odd. $[h_5(-a) = f_1(-a)g_1(-a) = f_1(a) \cdot -g_1(a) = -[f_1(a)g_1(a)] = -h_5(a)]$

(f) h_6 is even. $[h_6(-a) = g_1(-a)g_2(-a) = -g_1(a) \cdot -g_2(a) = g_1(a)g_2(a) = h_6(a)]$

7. Since $h(a) = [h(a) + h(-a)]/2 + [h(a) - h(-a)]/2$, let $f(x) = h(x) + h(-x)]/2$ and $g(x) = [h(x) - h(-x)]/2$, for each x. It is easy to show that f is even and that g is odd.

Chapter Eighteen

Oriented Planes and Sensed Angles

18.01 Specifying an Orientation for a Line

If we "look" at a line we can distinguish two senses of orientation,

Fig. 18–1

say, left and right. But, what is "left" and what is "right" really de-

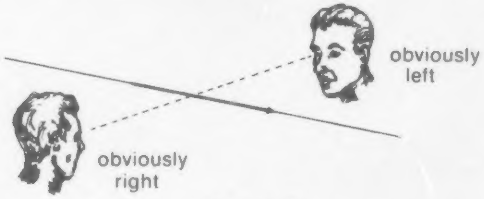


Fig. 18–2

pends on one's point of view. We can, however, *specify* the orientation of a line l in a precise manner. To do this mathematically, we merely select from $[l]$ all those vectors which have a given sense, and agree to say that those selected vectors are *positively sensed* and that the remaining non-$\vec{0}$ vectors in $[l]$ are *negatively sensed.* Once we have specified which of the vectors in $[l]$ are positively sensed, we shall say that we have oriented the line l.

For example, if A and B are two points of l, we may orient l by agreeing that the vectors in $[B - A]^+$ are positively sensed. [Fig. 18–3(a).] Alternately, we may orient l by agreeing that the vectors in $[A - B]^+$

Given a line ℓ and one of the two unit vectors — say, $\vec{u}$ — in $[\ell]$ we can define the sensed distance from a point P of ℓ to a point Q of ℓ to be $(Q - P):\vec{u}$. [See page 363 of volume 1 and recall that we have, in the present volume, defined $\vec{0}:\vec{u}$ to be 0.] Since there are two unit vectors in $[\ell]$, there are two possible definitions of sensed distance for an ordered pair, (P, Q), of points of ℓ. The sensed distance from P to Q according to either of these definitions is precisely the opposite of the sensed distance from P to Q according to the other definition. We can remove this ambiguity by agreeing to speak of sensed distances, from one point to a second, on an oriented line. One orients a line ℓ by choosing one of the two proper senses in $[\ell]$ and agreeing to speak of members of this sense as being "positively sensed" and of members of the other proper sense in $[\ell]$ as being "negatively sensed". The line ℓ, together with the chosen proper sense, is an oriented line. Given an oriented line we can define the sensed distance from P to Q on this oriented line to be $(Q - P):\vec{u}$, where $\vec{u}$ is the positively sensed unit vector in the direction of the line.

We can specify an orientation for a line ℓ by stipulating that some given proper translation $\vec{a}$ in $[\ell]$ is to have the positive sense. Note that a translation $\vec{b}$ in $[\ell]$ has the same sense as $\vec{a}$ — and, so, has the positive sense — if and only if $\vec{a} \cdot \vec{b} > 0$. For, since $\vec{b} \in [\ell] = [\vec{a}]$, $\vec{b} = \vec{a}b$ for some real number b, and $\vec{a} \cdot \vec{b} = \|\vec{a}\|^2 b$ and is greater than 0 if and only if $b > 0$ — that is, if and only if $\vec{b} \in [\vec{a}]^+$.

To obtain a similar notion of orientation for planes, it is convenient to note that orienting a line ℓ amounts to choosing one of two classes of bases for $[\ell]$, where the bases $(\vec{a})$ and $(\vec{b})$ for $[\ell]$ belong to the same class if and only if $\vec{a}$ and $\vec{b}$ have the same sense, or, equivalently, if and only if $\vec{a} \cdot \vec{b} > 0$. One of the principal jobs of the present chapter is to show that the bases for a given bidirection $[\pi]$ can be separated into two classes, each of which is associated with a sense of rotation — "clockwise" or "counterclockwise" — in π, and to develop an algebraic criterion for determining whether or not two bases for $[\pi]$ belong to the same class.

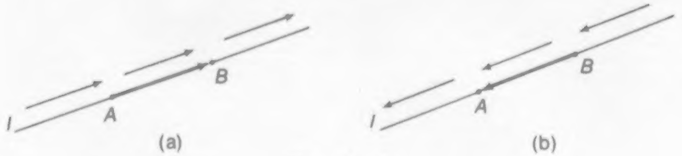


Fig. 18–3

are positively sensed. [Fig. 18–3(b)] The choices just described for orienting l yield opposite orientations of l. And, these are the only two possible choices.

Notice that orienting a line l amounts to partitioning the non-$\vec{0}$ members of $[l]$ into two *sense classes*—the class of positively sensed members of $[l]$ and the class of negatively sensed members of $[l]$. If l is an oriented line and $\vec{a}$ and $\vec{b}$ are non-$\vec{0}$ members of $[l]$ then either $\vec{a}$ and $\vec{b}$ belong to the same sense class of l or $\vec{a}$ and $\vec{b}$ belong to opposite sense classes of l.

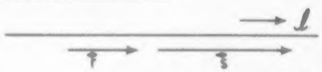
Exercises

1. On your paper, draw a picture of an oriented line l. Draw an arrow to indicate the positive sense class of l.
   (a) Draw pictures of two vectors $\vec{r}$ and $\vec{s}$, which belong to the positive sense class of l. [In this case, $\vec{r}$ and $\vec{s}$ are said to be positively sensed.] What can you say about $\vec{r} \cdot \vec{s}$?
   (b) Draw pictures of two negatively sensed vectors, $\vec{p}$ and $\vec{q}$. What can you say about $\vec{p} \cdot \vec{q}$?
   (c) Draw pictures of two vectors, $\vec{a}$ and $\vec{b}$, such that $\vec{a}$ is positively sensed and $\vec{b}$ is negatively sensed. What can you say about $\vec{a} \cdot \vec{b}$?
2. Is there a vector in $[l]$ which is neither positively sensed nor negatively sensed? Explain.
3. Complete the following with statements about $\vec{a} \cdot \vec{b}$.
   (a) Given an oriented line l and bases $(\vec{a})$ and $(\vec{b})$ for $[l]$, $\vec{a}$ and $\vec{b}$ belong to the same sense class of l if and only if ________.
   (b) Given an oriented line l and bases $(\vec{a})$ and $(\vec{b})$ for $[l]$, $\vec{a}$ and $\vec{b}$ belong to opposite sense classes of l if and only if ________.

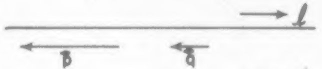
*

From the last exercise, we see that the non-$\vec{0}$ members of $[l]$ can be sorted into two sense classes by putting any two of them in the same sense class if [and only if] their dot product is positive. This is illustrated in Fig. 18–4.

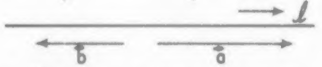
Answers for Exercises

1. (a)


$\vec{r} \cdot \vec{s} > 0$

(b)


$\vec{p} \cdot \vec{q} > 0$

(c)


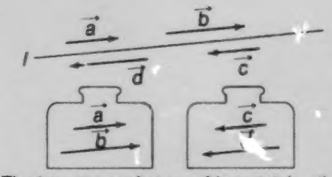
$\vec{a} \cdot \vec{b} < 0$

2. Since $\vec{0}$ does not belong to either of the proper sense classes contained in $[l]$, $\vec{0}$ is neither positively sensed or negatively sensed.

3. (a) $\vec{a} \cdot \vec{b} > 0$ (b) $\vec{a} \cdot \vec{b} < 0$

*

To explain the results in Exercise 3 note that, for $\vec{a} \neq \vec{0}$, $\vec{b}$ belongs to the same sense class as $\vec{a}$ if and only if $\vec{b} = \vec{a}b$ for some $b > 0$ and, so, $\vec{a} \cdot \vec{b} = \vec{a} \cdot (\vec{a}b) = (\vec{a} \cdot \vec{a})b > 0$. Also, $\vec{a}$ and $\vec{b}$ are in opposite sense classes if and only if $\vec{b} = \vec{a}b$ for some $b < 0$ and, so, $\vec{a} \cdot \vec{b} = (\vec{a} \cdot \vec{a})b < 0$.



The two sense classes of l, containing the non-0 members of $[l]$.

Fig. 18–4

$\vec a \cdot \vec b > 0$. So, $\vec a$ and $\vec b$ are in the same sense class.
$\vec a \cdot \vec c < 0$. So, $\vec a$ and $\vec c$ are in opposite sense classes.
$\vec c \cdot \vec d > 0$. So, $\vec c$ and $\vec d$ are in the same sense class.

To orient a line amounts to choosing one of its two sense classes to be the positive sense class.

18.02 Specifying an Orientation for a Plane

If we "look" at a plane, we can distinguish two senses of rotation. Either of these senses of rotation can be indicated on a picture of the plane by drawing a curved arrow. To orient a plane amounts to specifying that one of these senses of rotation is the positive sense of rotation.

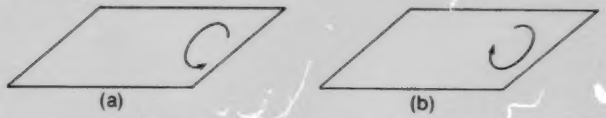

(a) (b)

Fig. 18–5

One sometimes reads something like: "We shall orient π by choosing the counterclockwise sense of rotation in π as positive." This is, of course, nonsense. In the first place, planes are not material objects which one can see and on which one can place clocks. In the second place, if they were, what appeared clockwise and what appeared counterclockwise would depend, as illustrated in Fig. 18–6, on one's point of view. What does make sense is to say that in drawing pictures of an oriented plane, it may be convenient to draw things so that the chosen positive sense of rotation appears in the picture as counterclockwise. [Fig. 18–5(a)] This convention is similar to the one according to which one often pictures an oriented line by choosing the positive sense of translation to be that from left to right.

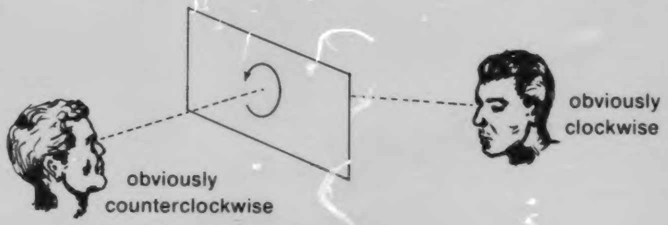


Fig. 18–6

There are many ways of distinguishing the two senses of rotation in the bidirection of a plane π. One of these bears a close analogy to the means we used in distinguishing the two senses of translation in the direction of a line l. There we found that the non-$\vec 0$ members of $[l]$ could be sorted into two sense classes by putting $\vec a$ and $\vec b$ in the same sense class if and only if $\vec a \cdot \vec b > 0$. Since each non-0 member of $[l]$ constitutes a basis for $[l]$, this suggests that what we need to do is to sort the bases for $[\pi]$ into two sense classes in such a way that the members of each class correspond in some intuitive manner to one of the two senses of rotation.

The intuitive portion of this task comes easily enough to mind. Given two linearly independent members, $\vec a$ and $\vec b$, of $[\pi]$ it is natural to associate the basis $(\vec a, \vec b)$ with the direction of rotation which, in Fig. 18–7, appears as counterclockwise, and to associate the basis $(\vec b, \vec a)$ with the direction of rotation which appears as clockwise. This suggests that whatever method we decide upon to sort the bases for $[\pi]$ into two sense classes, we should want the bases $(\vec a, \vec b)$ and $(\vec b, \vec a)$ to be in opposite sense classes.

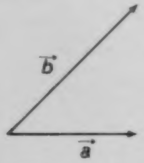
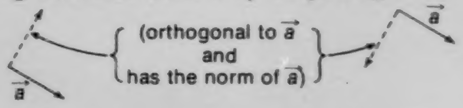


Fig. 18–7

The problem which remains is that of formulating an algebraic criterion for determining when two bases for a given bidirection belong to the same sense class. In order to obtain such a criterion, it is convenient first to introduce another way of describing the two orientations of a plane.

We already know that, given a non-$\vec 0$ vector $\vec a$ of $[\pi]$, there are exactly two vectors in $[\pi]$ each of which is orthogonal to $\vec a$ and has the same norm as does $\vec a$. In pictorial terms, we can think of either of these vectors as being obtained from $\vec a$ by subjecting $\vec a$ to a quarter-turn



(a) **Fig. 18–8** (b)

Just as we can choose an orientation for a line ℓ by choosing one non-$\vec{0}$ translation from $[\ell]$ and decreeing that this translation has the positive sense, so we can choose an orientation of a plane π by choosing one basis for $[\pi]$ and decreeing that this basis has the positive sense. In the case of a line we know how to determine whether two non-$\vec{0}$ translations in $[\ell]$ have the same sense — they do if and only if their dot product is positive. In the case of a plane we have yet to develop such a criterion. As an intermediate step in developing such a criterion we introduce the notion of a perping operation. Such an operation on $[\pi]$ is a way of assigning uniformly to each non-$\vec{0}$ translation $\vec{a} \in [\pi]$ an orthogonal translation $\vec{a}^{\perp} \in [\pi]$ which has the same norm as $\vec{a}$. "Uniformly" means, intuitively, that for any non-$\vec{0}$ translations $\vec{a}$ and $\vec{c}$ in $[\pi]$, the bases $(\vec{a}, \vec{a}^{\perp})$ and $(\vec{c}, \vec{c}^{\perp})$ will have the same sense. In (3) on page 380 we have an algebraic condition which forces this kind of uniformity on a perping operation. Using it we are able to prove that, for any bidirection there are just two perping operations. And, we can then formally orient a plane π by choosing one of these to be "the" perping operation in $[\pi]$. We can then agree that a basis $(\vec{a}, \vec{b})$ of $[\pi]$ is positively sensed if and only if $\vec{a}^{\perp} \cdot \vec{b} > 0$. Using this we arrive, in Theorem 18-4 on page 387, at a dot product condition for determining when two bases for $[\pi]$ have the same sense.

* * *

Suggestions for the exercises of section 18.02:

(i) Part A should be developed in class. (You might also develop part of Part B.)

(ii) The remainder of Part B, and Part C, may be assigned for homework. Be sure to subsequently discuss these exercises in class.

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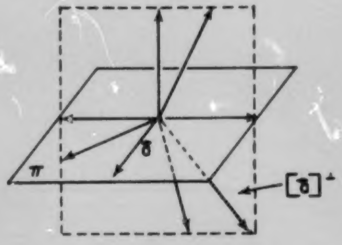
Answers for Part A

1. There are infinitely many vectors orthogonal to $\vec{a}$. They are just the members of $[\vec{a}]^{\perp}$.



2. There are infinitely many vectors which are orthogonal to $\vec{a}$ and have the same norm as $\vec{a}$. The set of them is $\{\vec{x}\colon \vec{x} \cdot \vec{a} = 0$ and $\vec{x} \cdot \vec{x} = \vec{a} \cdot \vec{a}\}$.

3. Just two vectors of $[\pi]$ are orthogonal to $\vec{a}$ and have the same norm as $\vec{a}$.

4. [Some students may choose to draw $\vec{a}^{\perp}$ upward and some downward. In the first case the sense of $(\vec{a}, \vec{b})$ is positive in (a) and (c) and $\vec{a}^{\perp} \cdot \vec{b} > 0$, and is negative in (b) and (d) and $\vec{a}^{\perp} \cdot \vec{b} < 0$. In the second case the sense of $(\vec{a}, \vec{b})$ is negative in (a) and (c) and $\vec{a}^{\perp} \cdot \vec{b} < 0$, and is positive in (b) and (d) and $\vec{a}^{\perp} \cdot \vec{b} > 0$. In both cases the drawing of $\vec{a}^{\perp}$ should be the same length as the drawing of $\vec{a}$.]

—counterclockwise, as in Fig. 18–8(a), or clockwise, as in Fig. 18–8(b). Intuitively, there are two singulary operations defined on $[\pi]$, one of which maps any member of $[\pi]$ on the vector obtained by subjecting it to a "counterclockwise quarter-turn", while the other maps any member of $[\pi]$ on the vector obtained by subjecting it to a "clockwise quarter-turn". One way of orienting π is to specify one of these two operations as the "preferred way" of selecting a vector in $[\pi]$ which is orthogonal to a given vector in $[\pi]$. In fact, if we indicate the preferred one of these singulary operations by '$^{\perp}$' [read '$\vec{a}^{\perp}$' as '$\vec{a}$ perp'] then it is intuitively clear that a basis $(\vec{a}, \vec{b})$ for $[\pi]$ is to be regarded as positively sensed if and only if $\vec{a}^{\perp} \cdot \vec{b} > 0$. [How can you tell from the picture in Fig. 18–9 that $\vec{a}^{\perp} \cdot \vec{b} > 0$?] Our problem, now, is to characterize the two perping operations on $[\pi]$ in such a way that we can prove that there are two such operations and to discover their less obvious properties.

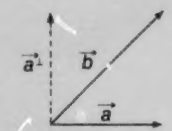


Fig. 18–9

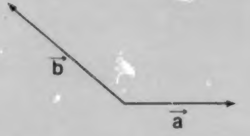
Exercises

Part A

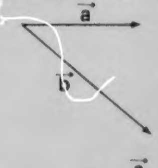
Given π, suppose that $\vec{a}$ is a non-$\vec{0}$ member of $[\pi]$.

1. How many vectors are orthogonal to $\vec{a}$? Give a description of the set of all such vectors, and draw a picture to illustrate your answer.
2. How many vectors are orthogonal to $\vec{a}$ and have the same norm as does $\vec{a}$? Give a description of the set of all such vectors.
3. How many vectors in $[\pi]$ are orthogonal to $\vec{a}$ and have the same norm as does $\vec{a}$?
4. Choose as $\vec{a}^{\perp}$ one of the vectors described in Exercise 3. On your paper, make a copy of each of the following bases, draw an arrow describing $\vec{a}^{\perp}$, and tell (i) whether $(\vec{a}, \vec{b})$ is positively sensed or negatively sensed and (ii) whether $\vec{a}^{\perp} \cdot \vec{b}$ is positive or negative.

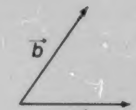
(a)



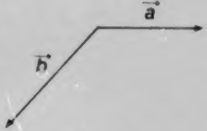
(b)



(c)

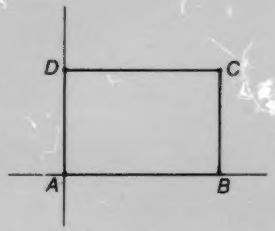


(d)



5. Consider the basis for $[\pi]$ pictured in Exercise 4(a).
   (a) Suppose that this basis is positively sensed. Make a copy of this basis and draw an arrow which describes $\vec{a}^{\perp}$.
   (b) Suppose that this basis is negatively sensed. Draw an arrow which describes $\vec{a}^{\perp}$.
   (c) What is the relation between the arrows you drew in parts (a) and (b)?

Part B



Consider the rectangle, *ABCD*, pictured at the right. We know that $\overline{AD} \perp \overline{AB}$, so that $B - A = \vec{i}b$ and $D - A = \vec{j}d$, for some b and d and orthonormal basis $(\vec{i}, \vec{j})$ for the bidirection of the plane of *ABCD*.

1. Express $C - A$ as a linear combination of $\vec{i}$ and $\vec{j}$.
2. Let $E = A + \vec{j}b$ and $G = A + \vec{i} \cdot -d$.
   (a) Make a copy of the given picture of *ABCD*, locate points *E* and *G* in your picture, and locate *F* such that *AEFG* is a rectangle.
   (b) Express $F - A$ as a linear combination of $\vec{i}$ and $\vec{j}$, where *F* is the point described in part (a).
   (c) Show that $F - A$ and $C - A$ have the same norm and are orthogonal.
3. Let $\vec{i_1}$ and $\vec{j_1}$ be unit vectors in $[C - A]^+$ and $[F - A]^+$, respectively. Show that $(\vec{i_1}, \vec{j_1})$ is an orthonormal basis for the plane of *ABCD*.
4. (a) Does it appear that $(\vec{i_1}, \vec{j_1})$ and $(\vec{i}, \vec{j})$ give the same, or opposite, orientations to the plane of *ABCD*? Is this the case no matter what your point of view? Explain your answer.
   (b) Assume that $(\vec{i}, \vec{j})$ is positively sensed. So, $\vec{i}^{\perp} = \vec{j}$. Based on your answer for part (a), would you say that $\vec{i_1}^{\perp} = \vec{j_1}$ or that $\vec{i_1}^{\perp} = -\vec{j_1}$? Why?
5. Let $P = A + \vec{i} \cdot -b$ and $Q = A + \vec{j} \cdot -d$.
   (a) Locate *P* and *Q* in your picture, and locate *R* such that *APRQ* is a rectangle.
   (b) Show that $\|R - A\| = \|F - A\|$ and $(R - A) \perp (F - A)$.
   (c) What can you say about $R - A$ and $C - A$?
6. Let $J = A + \vec{i}d$ and $L = A + \vec{j} \cdot -b$.
   (a) Locate *J* and *L* in your picture, and locate *K* such that *AJKL* is a rectangle.
   (b) Show that $\|K - A\| = \|C - A\|$ and $(K - A) \perp (C - A)$.

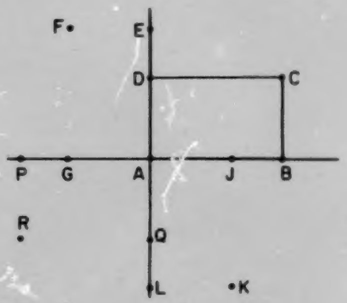
Answers for Part A [cont.]

5. [In (a) $\vec{a}^{\perp}$ should be drawn upward; in (b) $\vec{a}^{\perp}$ should be drawn downward. The two choices of $\vec{a}^{\perp}$ should be opposites of each other.]

Answers for Part B

1. $C - A = \vec{i}b + \vec{j}d$

2.



[Some of the points in this figure are referred to in Exercises 5 and 6.]

(b) $F - A = \vec{i} \cdot -d + \vec{j}b$

(c) $\|F - A\|^2 = b^2 + d^2$
$= \|C - A\|^2;$
$(F - A) \cdot (C - A)$
$= -db + bd = 0$

3. Since $C - A$ and $F - A$ are linear combinations of $\vec{i}$ and $\vec{j}$, where $(\vec{i}, \vec{j})$ is a basis for the plane of ABCD, both $C - A$ and $F - A$ belong to the bidirection of this plane. So, $\vec{i_1}$ and $\vec{j_1}$ are unit vectors in this bidirection and, since $C - A$ and $F - A$ are orthogonal, so are $\vec{i_1}$ and $\vec{j_1}$.

4. (a) $(\vec{i_1}, \vec{j_1})$ and $(\vec{i}, \vec{j})$ are associated with the same sense of rotation in the plane of ABCD — counterclockwise as viewed from in front of the paper and clockwise as viewed from behind the paper. So, choosing either $(\vec{i_1}, \vec{j_1})$ or $(\vec{i}, \vec{j})$ as positively sensed would establish the same orientation of the plane.
   (b) $\vec{i_1}^{\perp} = \vec{j_1}$ because the perping operation for which $(\vec{i}, \vec{j})$ is positively sensed is, by part (a), the same as the perping operation for which $(\vec{i_1}, \vec{j_1})$ is positively sensed.

5. (a) [See figure in answer for Exercise 2(a).]
   (b) Since $R - A = (P - A) + (Q - A) = \vec{i} \cdot -b + \vec{j} \cdot -d$ and $F - A = \vec{i} \cdot -d + \vec{j}b$ it follows that $\|R - A\|^2 = b^2 + d^2 = \|F - A\|^2$ and $(R - A) \cdot (F - A) = -b \cdot -d + -db = 0$.
   (c) $R - A$ and $C - A$ are opposites.

6. (a) [See figure in answer for Exercise 2(a).]
   (b) Since $K - A = (J - A) + (L - A) = \vec{i}d + \vec{j} \cdot -b$ and $C - A = \vec{i}b + \vec{j}d$ it follows that $\|K - A\|^2 = d^2 + b^2 = \|C - A\|^2$ and $(K - A) \cdot (C - A) = db + -bd = 0$.

Answers for Part B [cont.]

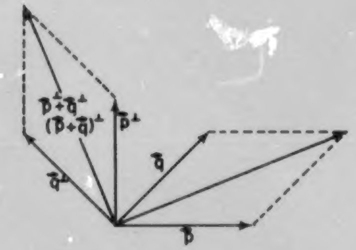
6. (c) [Similar to answer for Exercise 3.]

   (d) $(\vec{i}_2, \vec{j}_2)$ and $(\vec{i}, \vec{j})$ are associated with opposite senses of rotation in the plane of ABCD. As viewed from the front $(\vec{i}_2, \vec{j}_2)$ is associated with the clockwise sense of rotation and $(\vec{i}, \vec{j})$ is associated with the counterclockwise sense; as viewed from behind the plane, $(\vec{i}_2, \vec{j}_2)$ is associated with the counterclockwise sense of rotation and $(\vec{i}, \vec{j})$ is associated with the clockwise sense.

   (e) Since $(\vec{i}_2, \vec{j}_2)$ and $(\vec{i}, \vec{j})$ are oppositely sensed and $\vec{i}^{\perp} = \vec{j}$, $\vec{i}_2^{\perp} = -\vec{j}_2$.

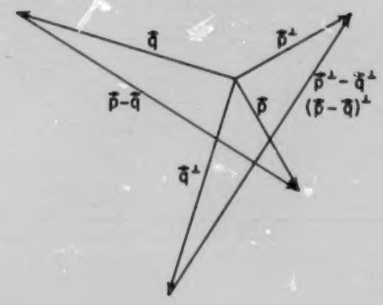
Answers for Part C

1.


$\vec{p}^{\perp}+\vec{q}^{\perp}$
$(\vec{p}+\vec{q})^{\perp}$
$\vec{p}^{\perp}$
$\vec{q}$
$\vec{q}^{\perp}$
$\vec{p}$

[Students should discover that $(\vec{p} + \vec{q})^{\perp} = \vec{p}^{\perp} + \vec{q}^{\perp}$.]

2.



[Students should discover that $(\vec{p} - \vec{q})^{\perp} = \vec{p}^{\perp} - \vec{q}^{\perp}$.]

(c) Given that $\vec{i}_2$ and $\vec{j}_2$ are unit vectors in $[C - A]^+$ and $[K - A]^+$, respectively, show that $(\vec{i}_2, \vec{j}_2)$ is an orthonormal basis for the plane of *ABCD*.

(d) Does it appear that $(\vec{i}_2, \vec{j}_2)$ gives the same, or opposite, orientation to the plane of *ABCD* as does $(\vec{i}, \vec{j})$? Is this the case no matter what your point of view? Explain.

(e) Would you say that $\vec{i}_2^{\perp} = \vec{j}_2$ or that $\vec{i}_2^{\perp} = -j_2$? Explain.

*

We are searching for algebraic criteria with which to characterize the two perping operations on $[\pi]$. As we have seen, perping is to be a singulary operation on $[\pi]$ such that, for any $\vec{a} \in [\pi]$,

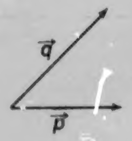
$$\vec{a}^{\perp} \in [\pi], \; \vec{a}^{\perp} \cdot \vec{a} = 0, \text{ and } \|\vec{a}^{\perp}\| = \|\vec{a}\|.$$

—that is, such that $\vec{a}^{\perp} \in [\pi]$, is orthogonal to $\vec{a}$, and has the same norm as does $\vec{a}$. These conditions are by no means sufficient to give us what we wish. For, in theory at least, these conditions can be satisfied by choosing, quite independently for each separate $\vec{a}$ in $[\pi]$, one of the two vectors in $[\pi]$ which is orthogonal to $\vec{a}$ and which has the same norm as does $\vec{a}$, and then taking this vector to be $\vec{a}^{\perp}$. In view of what we are after, it is reasonable to subject perping to some other requirements. We investigate these in the next exercises.

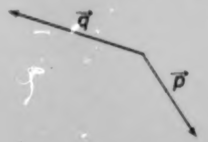
Part C

Suppose that, intuitively speaking, we have chosen a perping operation on $[\pi]$ such that, for any $\vec{a} \in [\pi]$, $\vec{a}^{\perp}$ is obtained by subjecting $\vec{a}$ to a counterclockwise quarter-turn.

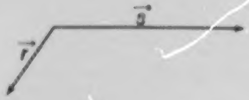
1. Given vectors $\vec{p}$ and $\vec{q}$ in $[\pi]$, which are pictured at the right, draw pictures of each of the following.
   (a) $\vec{p} + \vec{q}$ (b) $\vec{p}^{\perp}$ (c) $\vec{q}^{\perp}$
   (d) $\vec{p}^{\perp} + \vec{q}^{\perp}$ (e) $(\vec{p} + \vec{q})^{\perp}$



2. Given the vectors $\vec{p}$ and $\vec{q}$, shown at the right, draw pictures of each of the following.
   (a) $\vec{p} - \vec{q}$ (b) $\vec{p}^{\perp}$ (c) $-\vec{q}^{\perp}$
   (d) $\vec{p}^{\perp} - \vec{q}^{\perp}$ (e) $(\vec{p} - \vec{q})^{\perp}$



3. Given the vectors $\vec{r}$ and $\vec{s}$, shown at the right, draw pictures of each of the following.
   (a) $\vec{r}2 - \vec{s}\frac{1}{2}$ (b) $\vec{r}^{\perp}2$
   (c) $-\vec{s}^{\perp}\frac{1}{2}$ (d) $\vec{r}^{\perp}2 - \vec{s}^{\perp}\frac{1}{2}$
   (e) $(\vec{r}2 - \vec{s}\frac{1}{2})^{\perp}$



4. Compare your results in parts (d) and (e) of each of Exercises 1, 2, and 3.
5. Complete the following sentences and be prepared to illustrate your answers.
   (a) $(\vec{a} + \vec{b})^{\perp} =$ __ (b) $(\vec{a}a)^{\perp} =$ __
   (c) $\vec{a}^{\perp} - \vec{b}^{\perp} =$ __ (d) $(\vec{a}a + \vec{b}b)^{\perp} =$ __
6. Show that each of the results in parts (a)–(c) of Exercise 5 follows from part (d) of that exercise.

18.03 The Perping Operations

It is clear on intuitive grounds that there are two senses of rotation associated with a given bidirection $[\pi]$.

In seeking to characterize the perping operations on $[\pi]$ we found two properties that we wished those singulary operations to have. They are:

$$(1) \qquad \vec{a}^{\perp} \cdot \vec{a} = 0$$

$$(2) \qquad \|\vec{a}^{\perp}\| = \|\vec{a}\|$$

A third property of those operations is suggested in Part C, above. It is:

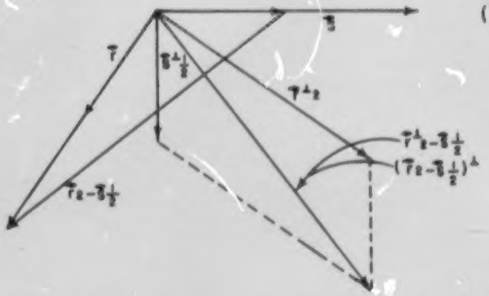
$$(3) \qquad \text{For } \vec{a}, \vec{b} \in [\pi],\ (\vec{a}a + \vec{b}b)^{\perp} = \vec{a}^{\perp}a + \vec{b}^{\perp}b.$$

It is interesting to note that there are exactly two singulary operations on $[\pi]$ which have these three properties. This is not difficult to establish, and we shall do so shortly. But, first, we adopt the following definition and investigate some of its consequences.

> **Definition 18–1** A singulary operation $^{\perp}$ on $[\pi]$ is a perping operation on $[\pi]$ if and only if, for $\vec{a}$ and $\vec{b}$ in $[\pi]$,
> (a) $\vec{a}^{\perp} \cdot \vec{a} = 0$,
> (b) $\|\vec{a}^{\perp}\| = \|\vec{a}\|$, and
> (c) $(\vec{a}a + \vec{b}b)^{\perp} = \vec{a}^{\perp}a + \vec{b}^{\perp}b$.

Answers for Part C [cont.]

3.


$\vec{r}$
$\vec{s}$
$\vec{s}^{\perp}\frac{1}{2}$
$\vec{r}^{\perp}2$
$\vec{r}^{\perp}2-\vec{s}\frac{1}{2}$
$(\vec{r}2-\vec{s}\frac{1}{2})^{\perp}$
$\vec{r}2-\vec{s}\frac{1}{2}$

[Students should discover that $(\vec{r}2 - \vec{s}\frac{1}{2})^{\perp} = \vec{r}^{\perp}2 - \vec{s}^{\perp}\frac{1}{2}$.]

4. In each exercise, parts (d) and (e) refer to the same translation.
5. (a) $\vec{a}^{\perp} + \vec{b}^{\perp}$ (b) $\vec{a}^{\perp}a$
   (c) $\vec{a}^{\perp} - \vec{b}^{\perp}$ (d) $\vec{a}^{\perp}a + \vec{b}^{\perp}b$
6. For (a), take $a = 1 = b$; for (b), take $b = 0$; for (c) take $a = 1$, $b = -1$.

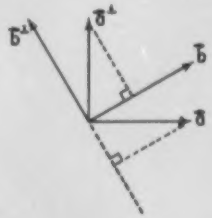
Suggestions for the exercises of section 18.03:
(i) Part A may be used for class discussion to insure an understanding of Definition 18-1.
(ii) Part B should also be developed in class.
(iii) Part C may be assigned as homework.

Answers for Part A

1. (a) 0 (b) $\vec{0}$
   (c) 0 (d) $\|\vec{a}\|\sqrt{a^2 + b^2}$
   (e) $\vec{a}\cdot\vec{b}^{\perp} + \vec{b}\cdot\vec{a}^{\perp}$ [or: 0] (f) 0

2. (a) By (c) of Definition 18-1, the products in (e) and (f) are the same. So, $\vec{a}\cdot\vec{b}^{\perp} + \vec{b}\cdot\vec{a}^{\perp} = 0$ and, hence, $\vec{a}\cdot\vec{b}^{\perp} = -\vec{a}^{\perp}\cdot\vec{b}$.

   (b)



Considering two right triangles in the figure, $|\vec{a}^{\perp}\cdot\vec{b}|/\|\vec{b}\|$ $= |\vec{a}\cdot\vec{b}^{\perp}|/\|\vec{b}^{\perp}\|$. Note that $\|\vec{b}^{\perp}\| = \|\vec{b}\|$ and, in the figure, $\vec{a}^{\perp}\cdot\vec{b} > 0$ and $\vec{a}\cdot\vec{b}^{\perp} < 0$. [There are other figures, but any one can be treated similarly to this.]

   (c) Since $\vec{a}\cdot\vec{b}^{\perp}$ is the product of $\|\vec{a}\|$, $\|\vec{b}^{\perp}\|$, and the cosine of an angle whose sides have the senses of $\vec{a}$ and $\vec{b}^{\perp}$, while $\vec{a}^{\perp}\cdot\vec{b}$ is the product of $\|\vec{a}^{\perp}\|$, $\|\vec{b}\|$, and the cosine of an angle whose sides have the senses of $\vec{a}^{\perp}$ and $\vec{b}$, it follows from part (a) [and the fact that $\|\vec{a}\| = \|\vec{a}^{\perp}\|$ and $\|\vec{b}\| = \|\vec{b}^{\perp}\|$] that the cosines of these angles are opposites and, so, that the angles are supplementary.

3. If $\vec{a}a + \vec{a}^{\perp}b = \vec{0}$ then $0 = \|\vec{a}a + \vec{a}^{\perp}b\|^2 = a^2 + b^2$ and, so, $a = 0 = b$.

4. (a) For any $\vec{b} \in [\pi]$, $\vec{b}^{\perp} \in [\pi]$. So, given $\vec{a} \in [\pi]$, $\vec{a}^{\perp} \in [\pi]$ and, since $\vec{a}^{\perp} \in [\pi]$, $(\vec{a}^{\perp})^{\perp} \in [\pi]$.

   (b) For $\vec{a} \neq \vec{0}$, $(\vec{a}, \vec{a}^{\perp})$ is an orthogonal basis for $[\pi]$. So, $\vec{a}^{\perp\perp} = \vec{a}a + \vec{a}^{\perp}b$ for some a and b and it follows that $\vec{a}^{\perp}\cdot\vec{a}^{\perp\perp} = \vec{a}^{\perp}\cdot(\vec{a}^{\perp}b) = (\vec{a}^{\perp}\cdot\vec{a}^{\perp})b = \|\vec{a}\|^2 b$. Since $\vec{a}^{\perp}\cdot\vec{a}^{\perp\perp} = 0$ it follows that $b = 0$ and, so, that $\vec{a}^{\perp\perp} = \vec{a}a$. So, for $\vec{a} \neq \vec{0}$, $\vec{a}^{\perp\perp} \in [\vec{a}]$. On the other hand, $\vec{0}^{\perp\perp} = \vec{0}$.

   (c) By Exercise 2(a), $\vec{a}^{\perp\perp}\cdot\vec{a} = -\vec{a}^{\perp}\cdot\vec{a}^{\perp}$. So, by Definition 18-1(b), $\vec{a}^{\perp\perp}\cdot\vec{a} = -\vec{a}\cdot\vec{a}$.

   (d) In the notation of part (b), for $\vec{a} \neq \vec{0}$, $\vec{a}^{\perp\perp} = \vec{a}a$ and, so, $\vec{a}^{\perp\perp} + \vec{a} = \vec{a}(a + 1)$. Hence, $(\vec{a}^{\perp\perp} + \vec{a})\cdot\vec{a} = (\vec{a}\cdot\vec{a})(a + 1)$ and, since, by (c), $(\vec{a}^{\perp\perp} + \vec{a})\cdot\vec{a} = 0$ it follows that $a = -1$ and, so, that $\vec{a}^{\perp\perp} = -\vec{a}$. In case $\vec{a} = \vec{0}$, $\vec{a}^{\perp\perp} = \vec{0}$ and, certainly, $\vec{a}^{\perp\perp} = -\vec{a}$.

5. (a) By Exercise 2(a), $\vec{a}^{\perp}\cdot\vec{b}^{\perp} = -\vec{a}^{\perp\perp}\cdot\vec{b} = \vec{a}\cdot\vec{b}$ by Exercise 4(d).

   (b) $\angle A \cong \angle B$

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[We adopt the convention according to which saying that $^{\perp}$ is a singulary operation on $[\pi]$ implies that, for $\vec{a} \in [\pi]$, $\vec{a}^{\perp} \in [\pi]$.]

Exercises

Part A

Suppose that $^{\perp}$ is a perping operation on $[\pi]$.

1. Complete each of the following.
   (a) $\vec{a}\cdot\vec{a}^{\perp} =$ __ (b) $\vec{0}^{\perp} =$ __
   (c) $\|\vec{0}^{\perp}\| =$ __ (d) $\|\vec{a}a + \vec{a}^{\perp}b\| =$ __
   (e) $(\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b}^{\perp}) =$ __ (f) $(\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b})^{\perp} =$ __
2. (a) Show that $\vec{a}\cdot\vec{b}^{\perp} = -\vec{a}^{\perp}\cdot\vec{b}$. [*Hint*: Make use of Exercises 1(e) and 1(f).]
   (b) Draw a picture of a pair of linearly independent vectors, $\vec{a}$ and $\vec{b}$, and illustrate part (a).
   (c) Consider two angles of π, the first of which has sides in the senses of $\vec{a}$ and $\vec{b}^{\perp}$ and the second of which has sides in the senses of $\vec{a}^{\perp}$ and $\vec{b}$. What does part (a) tell us about these angles?
3. Show that, if $\vec{a} \neq \vec{0}$, $(\vec{a}, \vec{a}^{\perp})$ is linearly independent. [*Hint*: Suppose that $\vec{a}a + \vec{a}^{\perp}b = \vec{0}$, for some a and b, and compute $\|\vec{a}a + \vec{a}^{\perp}b\|^2$.]
4. (a) Give an argument that, for $\vec{a} \in [\pi]$, both $\vec{a}^{\perp}$ and $\vec{a}^{\perp\perp}$ are in $[\pi]$. [*Note*: $\vec{a}^{\perp\perp} = (\vec{a}^{\perp})^{\perp}$.]
   (b) Give an argument that $\vec{a}$ and $\vec{a}^{\perp\perp}$ are linearly dependent vectors.
   (c) Show that $\vec{a}^{\perp\perp}\cdot\vec{a} = -\vec{a}\cdot\vec{a}$.
   (d) Show that $\vec{a}^{\perp\perp} = -\vec{a}$. [*Hint*: By (c), $(\vec{a}^{\perp\perp} + \vec{a})\cdot\vec{a} = 0$. Now, use the result in (b).]
5. (a) Show that $\vec{a}^{\perp}\cdot\vec{b}^{\perp} = \vec{a}\cdot\vec{b}$. [*Hint*: Use Exercises 2(a) and 4(d).]
   (b) Given that $(\vec{a}, \vec{b})$ is linearly independent, let $\angle A$ and $\angle B$ be angles of π whose sides are in the senses of $\vec{a}$ and $\vec{b}$, and of $\vec{a}^{\perp}$ and $\vec{b}^{\perp}$, respectively. What does part (a) tell us about $\angle A$ and $\angle B$?

Some of the results in Part A about the singulary operation perping are worth stating. We do so in:

> **Theorem 18-1** Given that $^{\perp}$ is a perping operation on $[\pi]$, if $\vec{a}$ and $\vec{b}$ belong to $[\pi]$, then
> (a) $\vec{a}\cdot\vec{b}^{\perp} = -\vec{a}^{\perp}\cdot\vec{b}$,
> (b) $\vec{a}^{\perp\perp} = -\vec{a}$, and
> (c) $\vec{a}^{\perp}\cdot\vec{b}^{\perp} = \vec{a}\cdot\vec{b}$.

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Part B

Suppose that $^\perp$ is a perping operation on $[\pi]$.

1. Given that $(\vec{a}, \vec{b})$ is a basis for $[\pi]$ and that $\vec{a} \perp \vec{b}$, show that $\vec{a}^\perp \in [\vec{b}]$ and $\vec{b}^\perp \in [\vec{a}]$.
2. Given that $(\vec{i}, \vec{j})$ is an orthonormal basis for $[\pi]$, show the following.
   (a) Either $\vec{i}^\perp = \vec{j}$ or $\vec{i}^\perp = -\vec{j}$ (b) Either $\vec{j}^\perp = \vec{i}$ or $\vec{j}^\perp = -\vec{i}$
   [*Hint*: Use Exercise 1.]
3. Given that $(\vec{i}, \vec{j})$ is an orthonormal basis for $[\pi]$, make use of Theorem 18–1 and Exercise 2 to show the following.
   (a) If $\vec{i}^\perp = \vec{j}$ then $\vec{j}^\perp = -\vec{i}$.
   (b) If $\vec{i}^\perp = -\vec{j}$ then $\vec{j}^\perp = \vec{i}$.
   (c) Either $(\vec{i} + \vec{j})^\perp = \vec{i} - \vec{j}$ or $(\vec{i} + \vec{j})^\perp = \vec{j} - \vec{i}$.

We are now in a position to prove that there are exactly two perping operations on $[\pi]$. We do this as follows:

Suppose that $(\vec{i}, \vec{j})$ is an orthonormal basis for $[\pi]$. By Exercise 3, above, we have that

$$(\vec{i}^\perp = \vec{j} \text{ and } \vec{j}^\perp = -\vec{i}) \text{ or } (\vec{i}^\perp = -\vec{j} \text{ and } \vec{j}^\perp = \vec{i}).$$

Also, for any $\vec{a} \in [\pi]$, we have that

$$\vec{a} = \vec{i}(\vec{a} \cdot \vec{i}) + \vec{j}(\vec{a} \cdot \vec{j}),$$

so that

$$\vec{a}^\perp = \vec{i}^\perp(\vec{a} \cdot \vec{i}) + \vec{j}^\perp(\vec{a} \cdot \vec{j}).$$

Now, if $\vec{i}^\perp = \vec{j}$ and $\vec{j}^\perp = -\vec{i}$ then

$$\begin{aligned}\vec{a}^\perp &= \vec{j}(\vec{a} \cdot \vec{i}) - \vec{i}(\vec{a} \cdot \vec{j})\\ &= -\vec{i}(\vec{a} \cdot \vec{j}) + \vec{j}(\vec{a} \cdot \vec{i}).\end{aligned}$$

And, if $\vec{i}^\perp = -\vec{j}$ and $\vec{j}^\perp = \vec{i}$ then

$$\vec{a}^\perp = \vec{i}(\vec{a} \cdot \vec{j}) - \vec{j}(\vec{a} \cdot \vec{i}).$$

Hence, either

$$(*)\quad \begin{cases}\text{for any } \vec{a} \in [\pi],\ \vec{a}^\perp = -\vec{i}(\vec{a} \cdot \vec{j}) + \vec{j}(\vec{a} \cdot \vec{i})\\ \qquad\qquad\text{or}\\ \text{for any } \vec{a} \in [\pi],\ \vec{a}^\perp = \vec{i}(\vec{a} \cdot \vec{j}) - \vec{j}(\vec{a} \cdot \vec{i}).\end{cases}$$

It is easy to check that the operations on $[\pi]$ defined by the alternatives in (*) are different, and each satisfies the three conditions in Definition 18–1. [Do so.] Hence, each of the operations defined in (*) are perping operations on $[\pi]$, and they are the only two such operations.

Answers for Part B

1. Since $\vec{a}^\perp \in [\pi]$, $\vec{a}^\perp = \vec{a}a + \vec{b}b$ for some a and b. Since $\vec{a} \cdot \vec{a}^\perp = 0$ it follows that $\vec{a} \cdot (\vec{a}a) + \vec{a} \cdot (\vec{b}b) = 0$ and so, since $\vec{a} \cdot \vec{b} = 0$ that $(\vec{a} \cdot \vec{a})a = 0$. Since $\vec{a} \neq \vec{0}$, $a = 0$. Hence, $\vec{a}^\perp = \vec{b}b \in [\vec{b}]$. Similarly, $\vec{b}^\perp \in [\vec{a}]$.
2. (a) By Exercise 1, $\vec{i}^\perp \in [\vec{j}]$. Since, like $\vec{i}$, $\vec{i}^\perp$ is a unit vector it follows that $\vec{i}^\perp$ is one of the two unit vectors, $\vec{j}$ and $-\vec{j}$, in $[\vec{j}]$.
   (b) If $(\vec{i}, \vec{j})$ is an orthonormal basis then so is $(\vec{j}, \vec{i})$ and the argument for part (a) shows that $\vec{j}^\perp = \vec{i}$ or $\vec{j}^\perp = -\vec{i}$.
3. (a) If $\vec{i}^\perp = \vec{j}$ then $\vec{i}^{\perp\perp} = \vec{j}^\perp$. But, by Theorem 18-1(b), $\vec{i}^{\perp\perp} = -\vec{i}$. So, if $\vec{i}^\perp = \vec{j}$ then $\vec{j}^\perp = -\vec{i}$.
   (b) If $(\vec{i}, \vec{j})$ is an orthonormal basis for $[\pi]$ then so is $(\vec{i}, -\vec{j})$. So, by part (a), if $\vec{i}^\perp = -\vec{j}$ then $(-\vec{j})^\perp = -\vec{i}$. But, by definition, $(-\vec{j})^\perp = -\vec{j}^\perp$. So, if $\vec{i}^\perp = -\vec{j}$ then $\vec{j}^\perp = \vec{i}$.
   (c) By definition, $(\vec{i} + \vec{j})^\perp = \vec{i}^\perp + \vec{j}^\perp$. By Exercise 2 and parts (a) and (b) either $\vec{i}^\perp = \vec{j}$ and $\vec{j}^\perp = -\vec{i}$ or $\vec{i}^\perp = -\vec{j}$ and $\vec{j}^\perp = \vec{i}$. In the former case $\vec{i}^\perp + \vec{j}^\perp = \vec{j} - \vec{i}$ and in the latter case $\vec{i}^\perp + \vec{j}^\perp = -\vec{j} + \vec{i} = \vec{i} - \vec{j}$. So, in any case, $(\vec{i} + \vec{j})^\perp$ is either $\vec{i} - \vec{j}$ or $\vec{j} - \vec{i}$.

*

The operations described in (*) are different since, for the first, $\vec{i}^\perp = \vec{j}$ and, for the second, $\vec{i}^\perp = -\vec{j}$ [and $-\vec{j} \neq \vec{j}$]. We shall show that the first of them — that defined by:

$$\vec{a}^\perp = -\vec{i}(\vec{a} \cdot \vec{j}) + \vec{j}(\vec{a} \cdot \vec{i})$$

— satisfies Definition 18-1. Since $\vec{a} = \vec{i}(\vec{a} \cdot \vec{i}) + \vec{j}(\vec{a} \cdot \vec{j})$ and $(\vec{i}, \vec{j})$ is orthonormal,

$$\vec{a}^\perp \cdot \vec{a} = -(\vec{a} \cdot \vec{j})(\vec{a} \cdot \vec{i}) + (\vec{a} \cdot \vec{i})(\vec{a} \cdot \vec{j}) = 0.$$

Moreover,

$$||\vec{a}^\perp||^2 = (\vec{a} \cdot \vec{j})^2 + (\vec{a} \cdot \vec{i})^2 = (\vec{a} \cdot \vec{i})^2 + (\vec{a} \cdot \vec{j})^2 = ||\vec{a}||^2.$$

Finally,

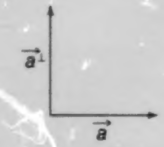
$$\begin{aligned}(\vec{a}a + \vec{b}b)^\perp &= -\vec{i}[(\vec{a}a + \vec{b}b) \cdot \vec{j}] + \vec{j}[(\vec{a}a + \vec{b}b) \cdot \vec{i}]\\ &= [-\vec{i}(\vec{a} \cdot \vec{j}) + \vec{j}(\vec{a} \cdot \vec{i})]a + [-\vec{i}(\vec{b} \cdot \vec{j}) + \vec{j}(\vec{b} \cdot \vec{i})]b\\ &= \vec{a}^\perp a + \vec{b}^\perp b.\end{aligned}$$

We have just proved:

> **Theorem 18–2** There are exactly two perping operations on $[\pi]$. Furthermore, if $(\vec{i}, \vec{j})$ is an orthonormal basis for $[\pi]$ and ${}^{\perp}$ is a perping operation on $[\pi]$ then, for any $\vec{a} \in [\pi]$,
>
> $$\vec{a}^{\perp} = -\vec{i}(\vec{a} \cdot \vec{j}) + \vec{j}(\vec{a} \cdot \vec{i}),$$
>
> or for any $\vec{a} \in [\pi]$,
>
> $$\vec{a}^{\perp} = \vec{i}(\vec{a} \cdot \vec{j}) - \vec{j}(\vec{a} \cdot \vec{i}).$$

Part C

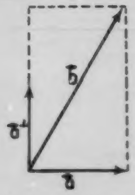
Suppose that we have chosen one of the two perping operations on $[\pi]$, and that the picture at the right describes vectors $\vec{a}$ and $\vec{a}^{\perp}$. We shall say that $(\vec{a}, \vec{a}^{\perp})$ is positively sensed. Make a copy of the basis $(\vec{a}, \vec{a}^{\perp})$ on your paper.



1. (a) Given that $\vec{b} = \vec{a} + \vec{a}^{\perp}2$, draw a picture of $\vec{b}$ and tell whether or not $(\vec{a}, \vec{b})$ is positively sensed.
   (b) Compute $\vec{a}^{\perp} \cdot \vec{b}$. Tell whether or not $\vec{a}^{\perp} \cdot \vec{b}$ is greater than 0.
2. (a) Given that $\vec{c} = -\vec{a} + \vec{a}^{\perp}2$, draw a picture of $\vec{c}$ and tell whether or not $(\vec{a}, \vec{c})$ is positively sensed.
   (b) Compute $\vec{a}^{\perp} \cdot \vec{c}$, and tell whether or not it is greater than 0.
3. (a) Consider $(\vec{b}, \vec{c})$, where $\vec{b}$ and $\vec{c}$ are as described in Exercises 1 and 2. Tell whether or not $(\vec{b}, \vec{c})$ is positively sensed.
   (b) Compute $\vec{b}^{\perp} \cdot \vec{c}$, and tell whether or not it is greater than 0.
4. (a) Given that $\vec{d} = \vec{a}a + \vec{a}^{\perp}b$, for some $b > 0$. Tell whether or not $(\vec{a}, \vec{d})$ is positively sensed. Would you change your answer if $a = 0$? If $a < 0$?
   (b) Compute $\vec{a}^{\perp} \cdot \vec{d}$, and tell whether or not it is greater than 0. Does your answer depend on the choice of values for 'a'?
5. (a) Given that $\vec{e} = \vec{a}a + \vec{a}^{\perp}b$, for some $b < 0$. Tell whether or not $(\vec{a}, \vec{e})$ is positively sensed. Would you change your answer if $a = 0$? If $a < 0$?
   (b) Compute $\vec{a}^{\perp} \cdot \vec{e}$, and tell whether or not it is greater than 0. Does your answer depend on the choice of values for 'a'?
6. (a) Given that $\vec{f} = \vec{a}a$, for some $\vec{a}$, tell whether or not $(\vec{a}, \vec{f})$ is a basis for $[\pi]$. Does your answer depend on the choice of values for 'a'?
   (b) Compute $\vec{a}^{\perp} \cdot \vec{f}$. Does your answer depend on the choice of values for 'a'?

Answers for Part C

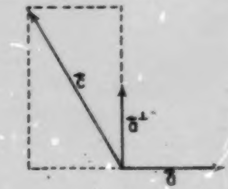
1. (a)



$(\vec{a}, \vec{b})$ is positively sensed.

(b) $\vec{a}^{\perp} \cdot \vec{b} = \vec{a}^{\perp} \cdot (\vec{a} + \vec{a}^{\perp}2) = ||\vec{a}||^2 2 > 0$

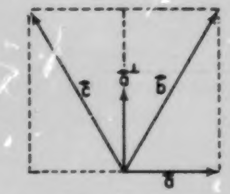
2. (a)



$(\vec{a}, \vec{c})$ is positively sensed.

(b) $\vec{a}^{\perp} \cdot \vec{c} = \vec{a}^{\perp} \cdot (-\vec{a} + \vec{a}^{\perp}2) = ||\vec{a}||^2 2 > 0$

3. (a)



$(\vec{b}, \vec{c})$ is positively sensed.

(b)
$$\begin{aligned}\vec{b}^{\perp} \cdot \vec{c} &= (\vec{a} + \vec{a}^{\perp}2)^{\perp} \cdot (-\vec{a} + \vec{a}^{\perp}2)\\ &= (\vec{a}^{\perp} - \vec{a}2) \cdot (-\vec{a} + \vec{a}^{\perp}2)\\ &= ||\vec{a}||^2 4 > 0\end{aligned}$$

4. (a) $(\vec{a}, \vec{d})$ is positively sensed.; No.; No.
   (b) $\vec{a}^{\perp} \cdot \vec{d} = ||\vec{a}||^2 b > 0$ [since $\vec{a} \neq \vec{0}$ and $b > 0$]; No.

5. (a) $(\vec{a}, \vec{e})$ is not positively sensed.; No.; No.
   (b) $\vec{a}^{\perp} \cdot \vec{e} = ||\vec{a}||^2 b < 0$; No.
6. (a) $(\vec{a}, \vec{f})$ is not a basis for $[\pi]$.; No.
   (b) $\vec{a}^{\perp} \cdot \vec{f} = \vec{a}^{\perp} \cdot (\vec{a}a) = 0$; No.

18.04 Using a Perping Operation to Orient a Plane

Recall that we are after a mathematical procedure for orienting a a plane that is analogous to that for orienting a line. In other words, we are after a procedure for determining when two bases for a bidirection belong to the same sense class.

So far, we have made good progress towards this goal. We know that there are exactly two perping operations which give us orthogonal bases for a bidirection. We know, too, that by choosing either of these perping operations as that one which determines the positive sense of rotation, each basis for the bidirection is either positively sensed or negatively sensed. And, the exercises just completed suggest a way of sorting bases for a bidirection into two sense classes. We make use of the latter in:

> **Definition 18–2** Choosing a perping operation, $^{\perp}$, on $[\pi]$,
> (a) a basis $(\overrightarrow{a}, \overrightarrow{b})$ for $[\pi]$ belongs to the sense class determined by $^{\perp}$ and, so, is *positively sensed* if and only if $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b} > 0$, and
> (b) a basis $(\overrightarrow{a}, \overrightarrow{b})$ for $[\pi]$ is *negatively sensed* if and only if $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b} < 0$.

For example, given the bases $(\overrightarrow{a}, \overrightarrow{b})$ and $(\overrightarrow{a}, \overrightarrow{c})$ for $[\pi]$ and $\overrightarrow{a}^{\perp}$ as shown in Fig. 18–10, we see from the picture that $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b} > 0$ and that $\overrightarrow{c}^{\perp} \cdot \overrightarrow{c} < 0$. So $(\overrightarrow{a}, \overrightarrow{b})$ is positively sensed and $(\overrightarrow{a}, \overrightarrow{c})$ is negatively sensed.

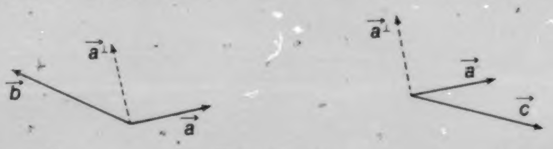


Fig. 18–10

[How can you tell from the picture that $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b} > 0$ and that $\overrightarrow{a}^{\perp} \cdot \overrightarrow{c} < 0$? What can you say about $\overrightarrow{a} \cdot \overrightarrow{b}$? About $\overrightarrow{a} \cdot \overrightarrow{c}$?]

For convenience, we also introduce:

> **Definition 18–3** Choosing a perping operation on $[\pi]$, bases $(\overrightarrow{a}, \overrightarrow{b})$ and $(\overrightarrow{c}, \overrightarrow{d})$ for $[\pi]$ belong to the same sense class if and only if both are positively sensed or both are negatively sensed.

Choosing a perping operation, $^{\perp}$, on $[\pi]$ we know that if $\overrightarrow{a}$ is a non-0 vector in $[\pi]$ then $(\overrightarrow{a}, \overrightarrow{a}^{\perp})$ is a basis for $[\pi]$. In fact, since $\overrightarrow{a}^{\perp} \cdot \overrightarrow{a}^{\perp} > 0$, we know that $(\overrightarrow{a}, \overrightarrow{a}^{\perp})$ is a positively sensed basis for $[\pi]$. Now, for any $\overrightarrow{b} \in [\pi]$, there are real numbers a and b such that

$$\overrightarrow{b} = \overrightarrow{a}a + \overrightarrow{a}^{\perp}b.$$

Moreover, $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b} = \|\overrightarrow{a}\|^2 b$. [Explain.] So, $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b} > 0$ if and only if $b > 0$. Thus, we have the following:

> **Theorem 18–3** Given a basis $(\overrightarrow{a}, \overrightarrow{b})$ for $[\pi]$, and choosing a perping operation, $^{\perp}$, on $[\pi]$,
> (a) $(\overrightarrow{a}, \overrightarrow{b})$ is positively sensed if and only if
> $$\exists_x \exists_{y>0}\ \overrightarrow{b} = \overrightarrow{a}x + \overrightarrow{a}^{\perp}y;\text{ and}$$
> (b) $(\overrightarrow{a}, \overrightarrow{b})$ is negatively sensed if and only if
> $$\exists_x \exists_{y<0}\ \overrightarrow{b} = \overrightarrow{a}x + \overrightarrow{a}^{\perp}y.$$

Theorem 18–3 tells us that the bases which are determined by a chosen perping operation, $^{\perp}$, on $[\pi]$ are just the pairs $(\overrightarrow{a}, \overrightarrow{a}a + \overrightarrow{a}^{\perp}b)$, for $\overrightarrow{a}$ a non-$\overrightarrow{0}$ member of $[\pi]$ and $b > 0$. In particular, an orthonormal basis $(\overrightarrow{i}, \overrightarrow{j})$ for $[\pi]$ is positively sensed if and only if $\overrightarrow{j} = \overrightarrow{i}a + \overrightarrow{i}^{\perp}b$, for some $b > 0$. And, the latter is the case if and only if $\overrightarrow{j} = \overrightarrow{i}^{\perp}$. Hence, an orthonormal basis $(\overrightarrow{i}, \overrightarrow{j})$ for $[\pi]$ is positively sensed if and only if $\overrightarrow{j} = \overrightarrow{i}^{\perp}$. And, for such a basis, it follows that, if $\overrightarrow{a} \in [\pi]$ then

$$(*)\qquad \begin{cases} \overrightarrow{a} = \overrightarrow{i}(\overrightarrow{a} \cdot \overrightarrow{i}) + \overrightarrow{j}(\overrightarrow{a} \cdot \overrightarrow{j}) \\ \overrightarrow{a}^{\perp} = -\overrightarrow{i}(\overrightarrow{a} \cdot \overrightarrow{j}) + \overrightarrow{j}(\overrightarrow{a} \cdot \overrightarrow{i}). \end{cases}$$

Let us consider, for a moment, an oriented line l. We know that bases $(\overrightarrow{a})$ and $(\overrightarrow{b})$ for $[l]$ belong to the same sense class if and only if both $\overrightarrow{a}$ and $\overrightarrow{b}$ are positively sensed or both $\overrightarrow{a}$ and $\overrightarrow{b}$ are negatively sensed. And, the latter is the case if and only if $\overrightarrow{a} \cdot \overrightarrow{b} > 0$. Note that the latter is a criterion we can use to determine whether or not $\overrightarrow{a}$ and $\overrightarrow{b}$ belong to the same sense class without referring to a picture of l and, perhaps more significantly, without referring to the particular orientation chosen on l.

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Note that by Definition 18-2, for $\vec{a} \neq \vec{0}$, $(\vec{a}, \vec{a}^{\perp})$ is positively sensed. For, $(\vec{a}, \vec{a}^{\perp})$ is a linearly independent sequence of members of $[\pi]$ and $\vec{a}^{\perp} \cdot \vec{a}^{\perp} = ||\vec{a}||^2 > 0$.

* * *

Suggestions for the exercises of section 18.04:

(i) Part A and the preceding discussion should be teacher directed.
(ii) After suitable examples, Parts B and C may be assigned for homework.

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Answers for Part A

1. (a) $\vec{c}^{\perp} = -\vec{i}(\vec{c} \cdot \vec{j}) + \vec{j}(\vec{c} \cdot \vec{i})$, $\vec{d} = \vec{i}(\vec{d} \cdot \vec{i}) + \vec{j}(\vec{d} \cdot \vec{j})$

   (b) $\vec{c}^{\perp} \cdot \vec{d} = -(\vec{c} \cdot \vec{j})(\vec{d} \cdot \vec{i}) + (\vec{c} \cdot \vec{i})(\vec{d} \cdot \vec{j})$

   (c) Since $\vec{i} = \vec{a}/||\vec{a}||$ and $\vec{j} = \vec{i}^{\perp}$ it follows that $\vec{j} = \vec{a}^{\perp}/||\vec{a}||$. Making these replacements for '$\vec{i}$' and '$\vec{j}$' in (b) and simplifying ["factoring out" $||\vec{a}||$] yields the desired result.

2. [Both (a) and (b) follow directly from the result in Exercise 1(c) on noting that $||\vec{a}||^2 > 0$. For it follows that $\vec{c}^{\perp} \cdot \vec{d}$ is positive or negative according as $-(\vec{c} \cdot \vec{a}^{\perp})(\vec{d} \cdot \vec{a}) + (\vec{c} \cdot \vec{a})(\vec{d} \cdot \vec{a}^{\perp})$ is positive or negative.]

3. (a) Since $(\vec{a}, \vec{b})$ is a basis, $\vec{b} \notin [\vec{a}]$ and, so, $b \neq 0$.

   (b), (c) With $\vec{b} = \vec{a}a + \vec{a}^{\perp}b$, $\vec{a}^{\perp} \cdot \vec{b} = \vec{a}^{\perp} \cdot (\vec{a}^{\perp})b = ||\vec{a}||^2 b$. So, since $||\vec{a}||^2 > 0$, $\vec{a}^{\perp} \cdot \vec{b}$ is positive or negative according as b is positive or negative.

   (d) With $\vec{b} = \vec{a}a + \vec{a}^{\perp}b$, $\vec{a} \cdot \vec{b} = (\vec{a} \cdot \vec{a})a$ and $\vec{a}^{\perp} \cdot \vec{b} = (\vec{a}^{\perp} \cdot \vec{a}^{\perp})b = (\vec{a} \cdot \vec{a})b$.

   So, $\vec{a} = (\vec{a} \cdot \vec{b})/(\vec{a} \cdot \vec{a})$, $\vec{b} = (\vec{a}^{\perp} \cdot \vec{b})/(\vec{a} \cdot \vec{a})$ and, hence,
   $$\vec{b} = \vec{a}(\vec{a} \cdot \vec{b})/(\vec{a} \cdot \vec{a}) + \vec{a}^{\perp}(\vec{a}^{\perp} \cdot \vec{b})/(\vec{a} \cdot \vec{a}).$$
   Solving this last for '$\vec{a}^{\perp}$' gives the desired result.

4. From Exercise 3(d) we see that
   $$\vec{a}^{\perp} \cdot \vec{d} = ((\vec{b} \cdot \vec{d})(\vec{a} \cdot \vec{a}) - (\vec{a} \cdot \vec{d})(\vec{a} \cdot \vec{b}))/(\vec{a}^{\perp} \cdot \vec{b})$$
   and
   $$\vec{a}^{\perp} \cdot \vec{c} = ((\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{a}) - (\vec{a} \cdot \vec{c})(\vec{a} \cdot \vec{b}))/(\vec{a}^{\perp} \cdot \vec{b}).$$
   So,
   $$\begin{aligned} &(\vec{a} \cdot \vec{c})(\vec{a}^{\perp} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{a}^{\perp} \cdot \vec{c}) \\ &= ((\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d})(\vec{a} \cdot \vec{a}) - (\vec{a} \cdot \vec{c})(\vec{a} \cdot \vec{d})(\vec{a} \cdot \vec{b}))/(\vec{a}^{\perp} \cdot \vec{b}) \\ &\quad - ((\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{a}) - (\vec{a} \cdot \vec{d})(\vec{a} \cdot \vec{c})(\vec{a} \cdot \vec{b}))/(\vec{a}^{\perp} \cdot \vec{b}) \\ &= ((\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{c}))(a \cdot \vec{a})/(\vec{a}^{\perp} \cdot \vec{b}). \end{aligned}$$
   Since $\vec{a} \cdot \vec{a} > 0$, each of parts (a) and (b) follows from the corresponding part of Exercise 2.

5. This follows at once from Exercise 1(c) and the result of the computation in Exercise 4 once we know that $\vec{a} \cdot \vec{a} \neq 0$ and $\vec{a}^{\perp} \cdot \vec{b} \neq 0$. These follow from the assumption that $(\vec{a}, \vec{b})$ is a basis. [As to the last, if $\vec{a}^{\perp} \cdot \vec{b} = 0$ then $\vec{b} \perp \vec{a}^{\perp}$ and, so, $\vec{b} \in [\vec{a}]$.]

Now, in the case of an oriented plane, we wish to find an algebraic criterion for determining whether or not bases $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ belong to the same sense class. And, as with lines, we wish to find such a criterion which doesn't depend on the particular orientation chosen on π. That is, we wish to find a way to complete the sentence:

> Bases $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ for $[\pi]$ belong
> to the same sense class if and only if. . .

which is independent of the perping operation chosen to orient π. The results we have obtained so far will be of great use in obtaining such a criterion. In the exercises below, one such criterion is obtained.

Exercises

Part A

Given that $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ are bases for $[\pi]$, let $\vec{i} = \vec{a}/||\vec{a}||$ and $\vec{j} = \vec{i}^{\perp}$.

1. (a) Express both $\vec{c}^{\perp}$ and $\vec{d}$ as linear combinations of $\vec{i}$ and $\vec{j}$. [*Hint*: See (*), above.]
   (b) Compute $\vec{c}^{\perp} \cdot \vec{d}$.
   (c) Show that
   $$\vec{c}^{\perp} \cdot \vec{d} = \frac{-(\vec{c} \cdot \vec{a}^{\perp})(\vec{d} \cdot \vec{a}) + (\vec{c} \cdot \vec{a})(\vec{d} \cdot \vec{a}^{\perp})}{||\vec{a}||^2}.$$

2. Make use of the results in Exercise 1 to show the following.
   (a) $\vec{c}^{\perp} \cdot \vec{d} > 0 \longleftrightarrow (\vec{a} \cdot \vec{c})(\vec{a}^{\perp} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{a}^{\perp} \cdot \vec{c}) > 0$
   (b) $\vec{c}^{\perp} \cdot \vec{d} < 0 \longleftrightarrow (\vec{a} \cdot \vec{c})(\vec{a}^{\perp} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{a}^{\perp} \cdot \vec{c}) < 0$

3. We know that $(\vec{a}, \vec{a}^{\perp})$ is a basis for $[\pi]$, so that $\vec{b} = \vec{a}a + \vec{a}^{\perp}b$, for some a and b. Show the following.
   (a) $b \neq 0$ (b) $\vec{a}^{\perp} \cdot \vec{b} > 0 \longleftrightarrow b > 0$
   (c) $\vec{a}^{\perp} \cdot \vec{b} < 0 \longleftrightarrow b < 0$ (d) $\vec{a}^{\perp} = (\vec{b}(\vec{a} \cdot \vec{a}) - \vec{a}(\vec{a} \cdot \vec{b}))/(\vec{a}^{\perp} \cdot \vec{b})$

4. Use the results from Exercises 2 and 3 to prove the following.
   (a) If $\vec{a}^{\perp} \cdot \vec{b} > 0$ and $\vec{c}^{\perp} \cdot \vec{d} > 0$ then
   $(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) > 0$.
   (b) If $\vec{a}^{\perp} \cdot \vec{b} < 0$ and $\vec{c}^{\perp} \cdot \vec{d} < 0$ then
   $(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) > 0$.

5. Show that
   $$(\vec{a}^{\perp} \cdot \vec{b})(\vec{c}^{\perp} \cdot \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{c})$$

6. Show that $(\overrightarrow{a, b})$ and $(\overrightarrow{c, d})$ belong to the same sense class if and only if

$$(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) > 0.$$

[*Hint*: This should be an easy job by now.]

Taking advantage of determinants, the results of the exercises just completed can be summarized in:

> **Theorem 18–4** Given bases $(\overrightarrow{a, b})$ and $(\overrightarrow{c, d})$ for $[\pi]$, $(\overrightarrow{a, b})$ and $(\overrightarrow{c, d})$ belong to the same sense class if and only if
>
> $$\begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix} > 0.$$

Part B

Given that $(\overrightarrow{i, j})$ is an orthonormal basis for $[\pi]$, let $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ be such that

$$\vec{a} = \vec{i}2 + \vec{j}4, \vec{b} = \vec{i}6 - \vec{j}2, \vec{c} = -\vec{i}2 + \vec{j}6, \text{ and } \vec{d} = \vec{i}5 + \vec{j}5.$$

Do the following problems. [*Note*: Computing the required dot products is an easy task.]

1. Determine which of the following belong to the same sense class, and which belong to opposite sense classes of $[\pi]$.
   (a) $(\overrightarrow{a, b}), (\overrightarrow{c, d})$ (b) $(\overrightarrow{a, c}), (\overrightarrow{d, b})$ (c) $(\overrightarrow{a, b}), (\overrightarrow{b, d})$
   (d) $(\overrightarrow{b, c}), (\overrightarrow{d, a})$ (e) $(\overrightarrow{a, c}), (\overrightarrow{b, d})$ (f) $(\overrightarrow{d, a}), (\overrightarrow{b, c})$
2. Let $A - O = \vec{a}, B - O = \vec{b}, C - O = \vec{c}$, and $D - O = \vec{d}$. Compute the cosine of each of the following.
   (a) $\angle AOB$ (b) $\angle COD$ (c) $\angle DOB$
   (d) $\angle BOC$ (e) $\angle AOC$ (f) $\angle AOD$
3. List the angles mentioned in Exercise 2 according to size from smallest to largest.
4. (a) Give the two possible linear combinations of $\vec{i}$ and $\vec{j}$ for $\vec{d}^{\perp}$.
   (b) Show that, in any case, $\vec{d}^{\perp} \cdot \vec{c} = -\vec{d}^{\perp} \cdot \vec{b}$.
   (c) What does the result in (b) tell you about $(\overrightarrow{d, c})$ and $(\overrightarrow{d, b})$? Does this agree with your results in Exercises 1(a) and 1(c)?
   (d) What does the result in (b) tell you about B and C in relation to $\overleftrightarrow{OD}$? What else must you know in order to tell whether or not D is interior to $\angle BOC$?

Answers for Part A [cont.]

6. By Definition 18-2, $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ belong to the same sense class if and only if $(\vec{a}^{\perp} \cdot \vec{b})(\vec{c}^{\perp} \cdot \vec{d}) > 0$. By Exercise 5, this is the case if and only if $(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) > 0$. [Note that, by a similar argument, $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ belong to opposite sense classes if and only if $(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) < 0$.]

Answers for Part B

1. (a) same (b) same (c) opposite (d) same
   (e) opposite [Compare with (b).] (f) same [Compare with (d).]
2. (a) $\sqrt{2}/10$ (b) $\sqrt{5}/5$ (c) $\sqrt{10}/5$
   (d) $-3/5$ (e) $\sqrt{2}/2$ (f) $3\sqrt{10}/10$
3. $\angle AOD$, $\angle AOC$, $\angle DOB$, $\angle COD$, $\angle AOB$, $\angle BOC$ [I.e., (f), (e), (c), (b), (a), (d). The order of the cosines found in Exercise 2 should be easy enough to determine by inspection. For example, $3\sqrt{10}/10$ is nearly 1 and is certainly larger than $\sqrt{2}/2$ which is about 0.7. Also, $\sqrt{10}/5 = \sqrt{2}/\sqrt{5}$ is certainly less than $\sqrt{2}/2$.]
4. (a) $\vec{d}^{\perp}$ is either $-\vec{i}5 + \vec{j}5$ or $\vec{i}5 - \vec{j}5$.
   (b) In the first case $\vec{d}^{\perp} \cdot \vec{c} = 40$ and $\vec{d}^{\perp} \cdot \vec{b} = -40$ and in the second $\vec{d}^{\perp} \cdot \vec{c} = -40$ and $\vec{d}^{\perp} \cdot \vec{b} = 40$.
   (c) $(\vec{d}, \vec{c})$ and $(\vec{d}, \vec{b})$ have opposite senses. Yes.; since $(\vec{c}, \vec{d})$ and $(\vec{b}, \vec{d})$ have opposite senses, one has the same sense as $(\vec{a}, \vec{b})$ and the other has the sense opposite to $(\vec{a}, \vec{b})$.
   (d) B and C are on opposite sides of $\overleftrightarrow{OD}$.; It is sufficient to know that $\vec{b} \cdot \vec{d} > 0$ and $\vec{c} \cdot \vec{d} > 0$. This is the case.

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Answers for Part C

1. If either a_2 or b_2 were 0 then $(\vec{a}, \vec{c})$ or $(\vec{b}, \vec{c})$ would be linearly dependent contrary to the assumption that A, B, and C are vertices of a triangle.
2. $\vec{c}^{\perp} \cdot \vec{a} = (\vec{c}^{\perp} \cdot \vec{c}^{\perp})a_2 = (\vec{c} \cdot \vec{c})a_2$, $\vec{c}^{\perp} \cdot \vec{b} = (\vec{c} \cdot \vec{c})b_2$
3. (a), (b) [Solve the equations (*) for '$\vec{c}^{\perp}$'.]
   (c) [Follow the hint.]
4. (a) From parts (a) and (c) of Exercise 3 it follows that, since $(\vec{a}, \vec{c})$ is linearly independent, $/a_2 = -/b_2$. So, $a_2 = -b_2$.
   (b) By part (a) and Exercises 1 and 2 it follows that one of $\vec{c}^{\perp} \cdot \vec{a}$ and $\vec{c}^{\perp} \cdot \vec{b}$ is positive and the other is negative. So, $(\vec{c}, \vec{a})$ and $(\vec{c}, \vec{b})$ are oppositely sensed.
5. (a) $\vec{d} = \vec{c}(b_1 + \frac{1}{2}) + \vec{c}^{\perp}b_2$
   (b) $\vec{c}^{\perp} \cdot \vec{d} = (\vec{c} \cdot \vec{c})b_2$ and $\vec{c}^{\perp} \cdot \vec{a} = (\vec{c} \cdot \vec{c})a_2$. Since $a_2 = -b_2$ it follows that $(\vec{c}, \vec{d})$ and $(\vec{c}, \vec{a})$ are oppositely sensed. [Alternatively, the determinant of Theorem 18-4 turns out, in this case, to be $(\vec{c} \cdot \vec{c})^2 a_2 b_2$ and, so, to be negative.]
6. By Theorem 15-17, $\vec{e} = \vec{b} + \vec{c}b/(a + b)$. So, $\vec{c}^{\perp} \cdot \vec{e} = \vec{c}^{\perp} \cdot \vec{b}$. Since, by Exercise 4(b), $(\vec{c}, \vec{d})$ and $(\vec{c}, \vec{a})$ are oppositely sensed, so are $(\vec{c}, \vec{e})$ and $(\vec{c}, \vec{a})$.

Sample Quiz

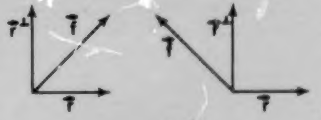
1. Let $(\vec{r}, \vec{s})$ and $(\vec{a}, \vec{b})$ be bases for $[\sigma]$, where σ is an oriented plane.
   (a) Give a determinant criterion for judging when $(\vec{r}, \vec{s})$ and $(\vec{a}, \vec{b})$ belong to the same sense class.
   (b) Consider the basis $(\vec{r}, \vec{r}^{\perp})$ for $[\sigma]$. Draw a picture which shows a vector $\vec{f}$ such that $\vec{r}^{\perp} \cdot \vec{f} > 0$.
   (c) Make use of your criterion from (a) to judge whether or not $(\vec{r}, \vec{r}^{\perp})$ and $(\vec{r}, \vec{f})$ belong to the same sense class.
2. Let $(\vec{i}, \vec{j})$ be an orthonormal basis for an oriented plane and let $a = \vec{i}5 + \vec{j}2$ and $b = \vec{i} \cdot -2 + \vec{j}3$.
   (a) Determine whether or not $(\vec{i}, \vec{a})$ and $(\vec{j}, \vec{b})$ belong to the same sense class.
   (b) Given that $(\vec{i}, \vec{j})$ is positively sensed, express $\vec{a}^{\perp}$ as a linear combination of $\vec{i}$ and $\vec{j}$.
   (c) Given that $(\vec{i}, \vec{j})$ is negatively sensed, express $\vec{b}^{\perp}$ as a linear combination of $\vec{i}$ and $\vec{j}$.

Key to Sample Quiz

1. (a) $(\vec{r}, \vec{s})$ and $(\vec{a}, \vec{b})$ belong to the same sense class if and only if

$$\begin{vmatrix} \vec{r} \cdot \vec{a} & \vec{r} \cdot \vec{b} \\ \vec{s} \cdot \vec{a} & \vec{s} \cdot \vec{b} \end{vmatrix} > 0. \quad \text{[Any comparable determinant is just as good.]}$$

(b) Here are two sample answers, both of which are acceptable:



(c)
$$\begin{vmatrix} \vec{r} \cdot \vec{r} & \vec{r} \cdot \vec{f} \\ \vec{r}^{\perp} \cdot \vec{r} & \vec{r}^{\perp} \cdot \vec{f} \end{vmatrix} = (\vec{r} \cdot \vec{r})(\vec{r}^{\perp} \cdot \vec{f}) - (\vec{r} \cdot \vec{f})(\vec{r}^{\perp} \cdot \vec{r}) = (\vec{r} \cdot \vec{r})(\vec{r}^{\perp} \cdot \vec{f}) > 0$$

since both $\vec{r} \cdot \vec{r}$ and $\vec{r}^{\perp} \cdot \vec{f}$ are positive. Hence, $(\vec{r}, \vec{r}^{\perp})$ and $(\vec{r}, \vec{f})$ belong to the same sense class.

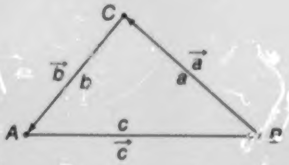
2. (a) They belong to the same sense class, for
$$\begin{vmatrix} \vec{i} \cdot \vec{j} & \vec{i} \cdot \vec{b} \\ \vec{a} \cdot \vec{j} & \vec{a} \cdot \vec{b} \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 2 & -4 \end{vmatrix} = 4 > 0.$$

(b) $\vec{a}^{\perp} = \vec{i}^{\perp}5 + \vec{j}^{\perp}2$ and, since $\vec{j} = \vec{i}^{\perp}$ and $\vec{j}^{\perp} = -\vec{i}$, it follows that $\vec{a}^{\perp} = -\vec{i}2 + \vec{j}5$.

(c) $\vec{b}^{\perp} = \vec{i}^{\perp} \cdot -2 + \vec{j}^{\perp}3$ and, since $\vec{i} = \vec{j}^{\perp}$ and $\vec{i}^{\perp} = -\vec{j}$, $\vec{b}^{\perp} = \vec{i}3 + \vec{j}2$.

Part C

Consider the triangle, $\triangle ABC$, pictured at the right, where, as usual, $B - A = \vec{c}$, $C - B = \vec{a}$, and $A - C = \vec{b}$. Since $\vec{c} \neq \vec{0}$, $(\vec{c}, \vec{c}^{\perp})$ is a basis for $[\overleftrightarrow{ABC}]$ and, for some a_1, a_2, b_1, and b_2,



$$(*) \quad \vec{a} = \vec{c}a_1 + \vec{c}^{\perp}a_2 \text{ and } \vec{b} = \vec{c}b_1 + \vec{c}^{\perp}b_2.$$

Do the following problems.

1. Show that $a_2 \neq 0$ and $b_2 \neq 0$.
2. Evaluate $\vec{c}^{\perp} \cdot \vec{a}$ and $\vec{c}^{\perp} \cdot \vec{b}$.
3. Verify each of the following.
   (a) $\vec{c}^{\perp} = \vec{a}/a_2 - \vec{c}(a_1/a_2)$
   (b) $\vec{c}^{\perp} = \vec{b}/b_2 - \vec{c}(b_1/b_2)$
   (c) $\vec{c}^{\perp} = -\vec{a}/b_2 - \vec{c}(1/b_2 + b_1/b_2)$ [*Hint*: Use (b) and the fact that $\vec{b} = -\vec{a} - \vec{c}$.]
4. (a) Use the results of Exercises 3(a) and 3(c) to show that $a_2 = -b_2$.
   (b) Show that $(\vec{c}, \vec{a})$ and $(\vec{c}, \vec{b})$ are oppositely sensed.
5. Suppose that $\overline{CD}$ is the median of $\triangle ABC$ from C. Let $\vec{d} = D - C$.
   (a) Express $\vec{d}$ as a linear combination of $\vec{c}$ and $\vec{c}^{\perp}$.
   (b) Show that $(\vec{c}, \vec{d})$ and $(\vec{c}, \vec{a})$ belong to opposite sense classes.
6. Suppose that $\overline{CE}$ is the angle bisector of $\triangle ABC$ from C. Let $\vec{e} = E - C$.
   (a) Express $\vec{e}$ as a linear combination of $\vec{c}$ and $\vec{c}^{\perp}$. [*Hint*: In what ratio does E divide $\overline{AB}$?]
   (b) Show that $(\vec{c}, \vec{e})$ and $(\vec{c}, \vec{a})$ belong to opposite sense classes.

*

In this section, we have learned how to specify the two senses of rotation in a plane and how to determine whether two bases for the bidirection of the plane belong to the same sense class. We now have a mathematical criterion at our disposal which enables us to specify which of the two senses of rotation is positive, or counterclockwise. With this criterion we can, for example, describe in mathematical terms precisely how to "travel around" a circle [or any other simple plane closed figure] in a clockwise or counterclockwise fashion. We shall have occasion, in Chapter 19, to do just that and, in so doing, we shall study some new functions on the real numbers and extend our knowledge of both Euclidean geometry and algebra. Before doing this, however, it will be convenient to take a second look at the concept of angle, and to extend our knowledge of this concept. The remainder of this chapter is devoted to that end.

18.05 Sensed Angles

Consider a basis, $(\overrightarrow{b}, \overrightarrow{c})$, for $[\pi]$. Given that $\overrightarrow{r} \in [$[illegible]$]^+$ and $\overrightarrow{s} \in [\overrightarrow{c}]^+$, $(\overrightarrow{r}, \overrightarrow{s})$ is also a basis for $[\pi]$. And, all such bases $(\overrightarrow{r}, \overrightarrow{s})$ belong to the same sense class. [Verify this.] Now, given that $B = A + \overrightarrow{b}$ and $C = A + \overrightarrow{c}$, $\overleftrightarrow{AC}$ are rays with the same vertex, and the ordered pair of rays, $(\overleftrightarrow{AB}, \overleftrightarrow{AC})$, is called *a sensed angle*. Furthermore, the sense of $(\overleftrightarrow{AB}, \overleftrightarrow{AC})$ is the sense class. [Verify this.] Now, given that $B = A + \overrightarrow{b}$ and $C = A + \overrightarrow{c}$, $\overleftrightarrow{AC}$ are its sense is the sense class which contains $(\overrightarrow{c}, \overrightarrow{b})$.

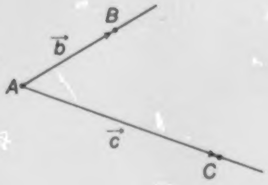


Fig. 18–11

We formalize the notion of sensed angle in:

> **Definition 18–4** A sensed angle is an ordered pair of rays with the same vertex.

Given that (r, s) is a sensed angle, the ray r is sometimes called *the initial side*, and s is called *the terminal side*, of (r, s). Notice that when r and s are noncollinear rays [with the same vertex], $r \cup s$ is an angle. Notice, also, that Definition 18–4 does not restrict sensed angles to those ordered pairs whose initial and terminal sides are noncollinear rays. Thus, for any ray r, (r, r) and $(r, -r)$ *are sensed angles* [even though $r \cup r$ and $r \cup -r$ are not angles].

For convenience, we introduce the following:

> **Definition 18–5** Given rays r and s with a common vertex.
> (a) (r, s) is null if and only if $s = r$.
> (b) (r, s) is right if and only if $s \perp r$.
> (c) (r, s) is straight if and only if $s = -r$.

For a sensed angle—say, (r, s)—which is neither null nor straight, there is exactly one plane which contains the angle $r \cup s$. Furthermore, all bases $(\overrightarrow{r}, \overrightarrow{s})$, where $\overrightarrow{r}$ and $\overrightarrow{s}$ are vectors in the senses of rays r and s, respectively, belong to the same sense class of the bidirection of that plane. This sense class is called *the sense of* (r, s), and is formalized in:

> **Definition 18–6** Given that sensed angle (r, s) is neither null nor straight, the sense of (r, s) is the sense class which contains the basis $(\overrightarrow{r}, \overrightarrow{s})$, where $\overrightarrow{r} \in [r]^+$ and $\overrightarrow{s} \in [s]^+$.

To show that if $\vec{r} \in [\vec{b}]^+$ and $\vec{s} \in [\vec{c}]^+$ then $(\vec{b}, \vec{c})$ and $(\vec{r}, \vec{s})$ belong to the same sense class we can use either the determinant condition or the definition. In either case we assume that $\vec{r} = \vec{b}b$ and $\vec{s} = \vec{c}c$ where $b > 0$ and $c > 0$. Then

$$\begin{vmatrix} \vec{b}\cdot\vec{r} & \vec{c}\cdot\vec{r} \\ \vec{b}\cdot\vec{s} & \vec{c}\cdot\vec{s} \end{vmatrix} = \begin{vmatrix} (\vec{b}\cdot\vec{b})b & (\vec{c}\cdot\vec{b})b \\ (\vec{b}\cdot\vec{c})c & (\vec{c}\cdot\vec{c})c \end{vmatrix} = \begin{vmatrix} \vec{b}\cdot\vec{b} & \vec{c}\cdot\vec{b} \\ \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c} \end{vmatrix}(bc).$$

Since the third of these determinants is positive by the Schwarz inequality it follows that $(\vec{b}, \vec{c})$ and $(\vec{r}, \vec{s})$ have the same sense if and only if $bc > 0$. So, since $b > 0$ and $c > 0$, they have the same sense.

Alternatively,

$$\vec{r}^{\perp}\cdot\vec{s} = (\vec{b}b)^{\perp}\cdot(\vec{c}c) = (\vec{b}^{\perp}b)\cdot(\vec{c}c) = (\vec{b}^{\perp}\cdot\vec{c})bc$$

and, as before, it follows that $(\vec{b}, \vec{c})$ and $(\vec{r}, \vec{s})$ have the same sense if and only if $bc > 0$.

The notion of angle introduced in Chapter 15 is, as we have seen, a useful one in many geometrical contexts. In some cases, however, [particularly in trigonometry], we need another notion of angle which can be used in distinguishing between senses of rotation and which includes notions of null angles and straight angles. [See Definition 18-5.] The notion of sensed angles introduced in Definition 18-4 serves these purposes. Note that a sensed angle is not [like an angle or, more generally, a geometric figure] a set of points. It is an ordered pair whose components are sets of points. Note, also, that we have found it convenient to take the sides of a sensed angle to be rays, although the sides of an [ordinary] angle are defined to be half-lines.

Each angle "determines" two sensed angles which are neither null nor straight; each sensed angle which is neither null nor straight "determines" one angle.

Note that the vertex of any sensed angle (r, s) is uniquely determined while no unique vertex can be singled out for $r \cup s$ in case $s = -r$. This, in addition to their general lack of utility, is why straight angles were not introduced in Chapter 15.

* * *

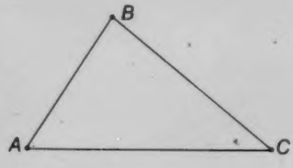
Suggestions for the exercises of section 18.05:

(i) Part A and the accompanying discussion should be directed by the teacher.
(ii) Parts B and C may be assigned for homework.
(iii) Part D may be developed in class, along with some of Part E.
(iv) The remainder of Part E, and Part F, may be assigned as homework.

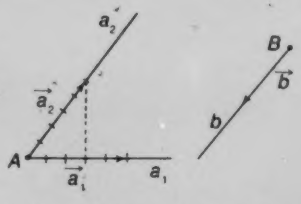
Exercises

Part A

1. Given an angle, $\angle PQR$, what are the two sensed angles whose sides contain those of $\angle PQR$? In each case, give the initial and terminal sides of the sensed angle. What can you say about the senses of the two sensed angles?
2. Consider $\triangle ABC$, pictured at the right. $(\overrightarrow{AB}, \overrightarrow{AC})$ is a sensed angle of $\triangle ABC$.
   - **(a)** Give the other sensed angles of $\triangle ABC$ which have the same sense as $(\overrightarrow{AB}, \overrightarrow{AC})$.
   - **(b)** Give the sensed angles of $\triangle ABC$ which have senses opposite to that of $(\overrightarrow{AB}, \overrightarrow{AC})$.



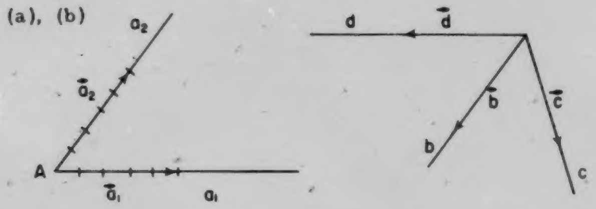
3. Given sensed angle (a_1, a_2) with vertex A and coplanar ray b with vertex B, as pictured at the right. Assume that $\cos \angle A = \frac{3}{5}$ and that $\vec{a}_1$, $\vec{a}_2$, and $\vec{b}$ are unit vectors in the senses of the rays a_1, a_2, and b, respectively. Copy this picture on your paper.
   - **(a)** Draw a picture of sensed angle (b, c) which has the sense of (a_1, a_2) and is such that $b \cup c$ is congruent to $\angle A$. What is $\sin (b \cup c)$?
   - **(b)** Draw a picture of sensed angle (b, d) which has the sense opposite to that of (a_1, a_2) and is such that $b \cup d$ is congruent to $\angle A$. What is $\sin (b \cup d)$?
   - **(c)** Suppose that the orientation of the plane is chosen so that (a_1, a_2) is positively sensed. What is $\vec{a}_1^{\perp} \cdot \vec{a}_2$? What is $\vec{b}^{\perp} \cdot \vec{c}$, where $\vec{c}$ is the unit vector in $[c]^+$? What is $\vec{b}^{\perp} \cdot \vec{d}$, where $\vec{d}$ is the unit vector in $[d]^+$? What can you say about $\vec{b}^{\perp} \cdot \vec{c}$ and $\vec{b}^{\perp} \cdot \vec{d}$?
   - **(d)** Suppose that the orientation of the plane is chosen so that (a_1, a_2) is negatively sensed. Answer the questions in part (c). Which of your answers are the same as those of part (c)?



4. Which of the following are true and which are false? [r and s are rays with the same vertex.]
   - **(a)** If (r, s) is a sensed angle and $\vec{r} \in [r]^+$ and $\vec{s} \in [s]^+$, then $(\vec{r}, \vec{s})$ is linearly independent.
   - **(b)** If (r, s) is neither null nor straight and $\vec{r} \in [r]^+$ and $\vec{s} \in [s]^+$, then $(\vec{r}, \vec{s})$ is linearly independent.
   - **(c)** The sensed angle (r, s) is null if and only if $[r]^+ = [s]^+$.
   - **(d)** If $r = s$ then $[r]^+ = [s]^+$.
   - **(e)** If $[r]^+ = [s]^+$ then $r = s$.
   - **(f)** The sensed angle (r, s) is straight if and only if $[r]^+ = \{\vec{x}: \exists_{\vec{s}} (\vec{s} \in [s]^+ \text{ and } \vec{x} = -\vec{s})\}$.

Answers for Part A

1. The two sensed angles are $(\overrightarrow{QP}, \overrightarrow{QR})$ and $(\overrightarrow{QR}, \overrightarrow{QP})$. The senses of these angles are opposites of each other.
2. (a) $(\overrightarrow{CA}, \overrightarrow{CB})$, $(\overrightarrow{BC}, \overrightarrow{BA})$
   (b) $(\overrightarrow{AC}, \overrightarrow{AB})$, $(\overrightarrow{CB}, \overrightarrow{CA})$, $(\overrightarrow{BA}, \overrightarrow{BC})$
3. (a), (b)



   (c) $\vec{a}_1^{\perp} \cdot \vec{a}_2 = 4/5$; $\vec{b}^{\perp} \cdot \vec{c} = 4/5$; $\vec{b}^{\perp} \cdot \vec{d} = -4/5$; $\vec{b}^{\perp} \cdot \vec{c}$ and $\vec{b}^{\perp} \cdot \vec{d}$ are opposites.
   (d) $\vec{a}_1^{\perp} \cdot \vec{a}_2 = -4/5$; $\vec{b}^{\perp} \cdot \vec{c} = -4/5$; $\vec{b}^{\perp} \cdot \vec{d} = 4/5$; $\vec{b}^{\perp} \cdot \vec{c}$ and $\vec{b}^{\perp} \cdot \vec{d}$ are opposites. The answers to the last question are the same.
4. (a) False. (b) True. (c) True. (d) True.
   (e) True. (f) True. (g) False. (h) True.

(g) The sensed angles (r, s) and (s, r) have opposite senses.

(h) If sensed angle (r, s) is neither null nor straight, then (r, s) and (s, r) have opposite senses.

5. Suppose that (r, s) and (p, q) are sensed angles of a plane π, and that $\vec{r} \in [r]^+$, $\vec{s} \in [s]^+$, $\vec{p} \in [p]^+$, and $\vec{q} \in [q]^+$. Complete each of the following in terms of a condition on a particular determinant, and explain.

(a) (r, s) and (p, q) have the same sense $\longleftrightarrow$ __

(b) (r, s) and (p, q) have opposite senses $\longleftrightarrow$ __

(c) (r, s) and (p, q) are neither null nor straight $\longleftrightarrow$ __

(d) At least one of (r, s) and (p, q) is either null or straight $\longleftrightarrow$ __

6. Suppose that (r, s) and (p, q) are sensed angles in parallel planes. Which of your answers in Exercise 5 remain unchanged?

The results of Exercises 5 and 6, above, are summarized in:

> **Theorem 18–5** Given that (r, s) and (p, q) are sensed angles, that $\vec{r} \in [r]^+$, $\vec{s} \in [s]^+$, $\vec{p} \in [p]^+$, and $\vec{q} \in [q]^+$, and that the sides of (r, s) and of (p, q) are contained in parallel planes, (r, s) and (p, q) have the same sense if and only if
>
> $$\begin{vmatrix} \vec{r} \cdot \vec{p} & \vec{r} \cdot \vec{q} \\ \vec{s} \cdot \vec{p} & \vec{s} \cdot \vec{q} \end{vmatrix} > 0.$$
>
> They have opposite senses if and only if this determinant is less than 0. And, one of the sensed angles is either null or straight if and only if the determinant is 0.

As was the case with angles, it is convenient to study relationships among sensed angles in terms of relationships among dot products of unit vectors in the senses of their sides. In the case of angles, the cosine function was used to order them according to size. In the case of sensed angles with noncollinear sides, it is intuitively clear that the cosines of the related angles can be used in the very same way to order these sensed angles according to [absolute] size. Remembering to include the "extreme" cases—the null and straight sensed angles—we adopt:

> **Definition 18–7** Given sensed angle (r, s),
>
> $$\cos(r, s) = (\vec{r} \cdot \vec{s})/(\|\vec{r}\| \, \|\vec{s}\|),$$
>
> where $\vec{r} \in [r]^+$ and $\vec{s} \in [s]^+$.

[Read 'cos(r, s)' as 'the cosine of [sensed angle] r, s'.]

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Answers for Part A [cont.]

5. The answers all refer to the determinant

$$\begin{vmatrix} \vec{r} \cdot \vec{p} & \vec{r} \cdot \vec{q} \\ \vec{s} \cdot \vec{p} & \vec{s} \cdot \vec{q} \end{vmatrix}.$$

(a) the determinant is positive

(b) the determinant is negative

(c) the determinant is not zero

(d) the determinant is zero

6. All answers remain unchanged.

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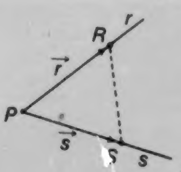
Part B

1. Show that, for any sensed angle (r, s), $\cos(r, s) = \cos(s, r)$.
2. Under what conditions is $\cos(r, s) = \cos(r \cup s)$?
3. Complete each of the following sentences.
   (a) Given that (r, s) is null, $\cos(r, s) =$ ___.
   (b) Given that (r, s) is ___, $\cos(r, s) = 0$.
   (c) Given that $\cos(r, s) = -1$, (r, s) is ___.
   (d) If $\cos(r, s) = 0$ then (r, s) is ___.
   (e) (r, s) is ___ only if $\cos(r, s) = -1$.
4. Suppose that R and S are points on the sides of (r, s) such that $PR = PS = 1$, and then $R - P = \vec{r}$ and $S - P = \vec{s}$, as shown in the picture at the right. Complete the following.



   (a) $RS = 1$ if and only if $\vec{r} \cdot \vec{s} =$ ___.
   (b) $RS = \frac{1}{2}$ if and only if $\vec{r} \cdot \vec{s} =$ ___.
   (c) $RS =$ ___ if and only if $\vec{r} \cdot \vec{s} = 0$.
   (d) Assume that $T = P + \vec{s}2$ and $RS = 1$. Then, $RT =$ ___.
   (e) Assume that $A = P + \vec{r}2$, $B = P + \vec{s}3$ and $\vec{r} \cdot \vec{s} = \frac{7}{8}$. Then, $AS =$ ___, $RB =$ ___, and $AB =$ ___.

*

We know that two figures are congruent if and only if there is an isometry which maps one of them onto the other. This is of no direct help to us if we wish to talk about "congruent" sensed angles, for sensed angles are not "figures". Sensed angles are *ordered pairs* of rays, and rays are figures. Knowing that isometries map rays onto rays, lines onto lines, and angles onto angles with the same cosine, it is reasonable to talk about congruence of sensed angles in terms of their cosines. We introduce this notion in:

> **Definition 18-8** Given sensed angles (p, q) and (r, s), whose sides are contained in parallel planes, (p, q) and (r, s) are congruent if and only if both are null angles, both are straight angles, or both have have the same sense and $\cos(p, q) = \cos(r, s)$.

Part C

1. Give arguments to support each of the following statements concerning sensed angles.
   (a) (r, s) is congruent to (r, s). [For short: $(r, s) \cong (r, s)$]
   (b) $(r, s) \cong (t, u) \longrightarrow (t, u) \cong (r, s)$
   (c) $[(r, s) \cong (a, b)$ and $(a, b) \cong (p, q)] \longrightarrow (r, s) \cong (p, q)$
   (d) $(r, s) \cong (-r, -s)$
   (e) $(r, -s) \cong (-r, s)$

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Answers for Part B

1. This is the case because both dot multiplication and multiplication of real numbers are commutative.
2. If r and s are noncollinear rays with a common vertex.
3. (a) 1 (b) right (c) straight (d) right (e) straight
4. (a) 1/2 (b) 7/8 (c) $\sqrt{2}/2$ (d) $\sqrt{3}$
   (e) $\sqrt{3/2}$, $\sqrt{19}/2$, $\sqrt{5/2}$

Answers for Part C

1. (a) If (r, s) is null or straight then so is (r, s). Otherwise (r, s) has the same sense and the same cosine as (r, s).

   (b) If (r, s) and (t, u) are both null or both are straight then so are (t, u) and (r, s). If (r, s) and (t, u) have the same sense and the same cosine then so do (t, u) and (r, s). Hence, in any case, if $(r, s) \cong (t, u)$ then $(t, u) \cong (r, s)$.

   (c) If (r, s) and (a, b) are both null or both straight and $(a, b) \cong (p, q)$ then (a, b) and (p, q) are both null or both straight, respectively and, so, (r, s) and (p, q) are both null or straight. If (r, s) and (a, b) have the same sense and the same cosine and $(a, b) \cong (p, q)$ then (a, b) is neither null nor straight and, so, (a, b) and (p, q) have the same sense and cosine. So, in this case, (r, s) and (p, q) have the same sense and cosine. Hence, in any case, if $(r, s) \cong (a, b)$ and $(a, b) \cong (p, q)$ then $(r, s) \cong (p, q)$.

   (d) If (r, s) is null or straight so is (−r, −s). In the contrary case, (r, s) and (−r, −s) have the same sense because, with $\vec{r} \in [r]^+$ and $\vec{s} \in [s]^+$, $-\vec{r} \in [-r]^+$ and $-\vec{s} \in [-s]^+$ and

$$\begin{vmatrix} \vec{r}\cdot-\vec{r} & \vec{r}\cdot-\vec{s} \\ \vec{s}\cdot-\vec{r} & \vec{s}\cdot-\vec{s} \end{vmatrix} = \begin{vmatrix} -(\vec{r}\cdot\vec{r}) & -(\vec{r}\cdot\vec{s}) \\ -(\vec{s}\cdot\vec{r}) & -(\vec{s}\cdot\vec{s}) \end{vmatrix} = \begin{vmatrix} \vec{r}\cdot\vec{r} & \vec{r}\cdot\vec{s} \\ \vec{s}\cdot\vec{r} & \vec{s}\cdot\vec{s} \end{vmatrix} > 0.$$

   And, in the same case, $\cos(\vec{r}, \vec{s}) = \cos(-\vec{r}, -\vec{s})$ because $\vec{r}\cdot\vec{s} = -\vec{r}\cdot-\vec{s}$, $\|\vec{r}\| = \|-\vec{r}\|$, and $\|\vec{s}\| = \|-\vec{s}\|$. Hence, in any case, $(r, s) \cong (-r, -s)$.

   (e) By part (d), $(r, -s) \cong (-r, --s) = (-r, s)$.

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2. Consider a circle with center O and diameters $\overline{AB}$ and $\overline{CD}$.
   (a) Given that $(\overrightarrow{OA}, \overrightarrow{OD})$, $(\overrightarrow{OD}, \overrightarrow{OC})$, $(\overrightarrow{OC}, \overrightarrow{OB})$, and $(\overrightarrow{OB}, \overrightarrow{OA})$ are congruent to each other, what can you say about $\overline{AB}$ and $\overline{CD}$? About $ACBD$?
   (b) It is convenient to use '$\angle AOD$' as an abbreviation for '$(\overrightarrow{OA}, \overrightarrow{OD})$'. Similarly, '$\angle DOA$' is an abbreviation for '$(\overrightarrow{OD}, \overrightarrow{OA})$'. Given that $\angle AOD$, $\angle DOB$, $\angle BOC$, and $\angle COA$ are congruent to each other, what can you say about $\overline{AB}$ and $\overline{CD}$? About $ACBD$?
3. Suppose that (a, b) and (c, d) are coplanar sensed angles and that $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$, and $\overrightarrow{d}$ are *unit vectors* in the senses of the rays a, b, c, and d, respectively. Give arguments to support each of the following. [These should be familiar results.]
   (a) $(\overrightarrow{a} \cdot \overrightarrow{b})^2 + (\overrightarrow{a}^{\perp} \cdot \overrightarrow{b})^2 = 1$ and $(\overrightarrow{c} \cdot \overrightarrow{d})^2 + (\overrightarrow{c}^{\perp} \cdot \overrightarrow{d})^2 = 1$.
   (b) $|\overrightarrow{a}^{\perp} \cdot \overrightarrow{b}| = \sqrt{1 - \cos^2 (a, b)}$ and $|\overrightarrow{c}^{\perp} \cdot \overrightarrow{d}| = \sqrt{1 - \cos^2 (c, d)}$
   [*Note*: $\cos^2 (a, b) = [\cos (a, b)]^2$]
   (c) If (a, b) is neither null nor straight, $|\overrightarrow{a}^{\perp} \cdot \overrightarrow{b}| = \sin (a \cup b)$.
   (d) If (a, b) is positively sensed, $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b} = \sin (a \cup b)$.
   (e) If (a, b) is negatively sensed, $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b} = -\sin (a \cup b)$.
4. Given the sensed angles (a, b) and (c, d) described in Exercise 3.
   (a) If (a, b) and (c, d) are both positively sensed and are congruent, what can you say about $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b}$ and $\overrightarrow{c}^{\perp} \cdot \overrightarrow{d}$?
   (b) If (a, b) and (c, d) are both negatively sensed and are congruent, what can you say about $\overrightarrow{a}^{\perp} \cdot \overrightarrow{b}$ and $\overrightarrow{c}^{\perp} \cdot \overrightarrow{d}$?
5. Consider the converses of the conditional sentences in Exercises 3(d) and 3(e). Are they theorems or not? Explain your answer.

For a sensed angle, (r, s), of a plane π, $\cos (r, s) = \overrightarrow{r} \cdot \overrightarrow{s}$, where $\overrightarrow{r}$ and $\overrightarrow{s}$ are unit vectors in the senses of rays r and s, respectively. We know that the dot product $\overrightarrow{r} \cdot \overrightarrow{s}$ does not depend on the orientation chosen for π. And, we know that the dot product $\overrightarrow{r}^{\perp} \cdot \overrightarrow{s}$ *does* depend on the chosen orientation, for $\overrightarrow{r}^{\perp} \cdot \overrightarrow{s}$ is positive if and only if (r, s) is positively sensed and is negative if and only if (r, s) is negatively sensed. Thus, the dot product $\overrightarrow{r}^{\perp} \cdot \overrightarrow{s}$ in some respect characterizes the sense class to which (r, s) belongs under a chosen perping operation $^{\perp}$ on $[\pi]$. Since, for noncollinear rays r and s, $\overrightarrow{r}^{\perp} \cdot \overrightarrow{s}$ is related to the sine of $r \cup s$, this suggests that we define a function, like sine, on sensed angles. We do this as follows:

Definition 18–9 Choosing a perping operation, $^{\perp}$, on $[\pi]$, and given a sensed angle (r, s) of π,

$$\sin^{\perp} (r, s) = (\overrightarrow{r}^{\perp} \cdot \overrightarrow{s})/(\|\overrightarrow{r}\| \|\overrightarrow{s}\|),$$

where $\overrightarrow{r} \in [r]^+$ and $\overrightarrow{s} \in [s]^+$.

[Read '$\sin^{\perp} (r, s)$' as you would 'sine perp of r, s'.]

Answers for Part C [cont.]

2. (a) $\overline{AB} = \overline{CD}$; ACBD is a segment. [Recall that ACBD $= \overline{AC} \cup \overline{CB} \cup \overline{BD} \cup \overline{DA}$.]
   (b) $\overline{AB} \perp \overline{CD}$; ACBD is a square.

3. (a) Since $(\vec{a}, \vec{a}^{\perp})$ is a basis for the bidirection of the plane of the sensed angle, $\vec{b} = \vec{a}b_1 + \vec{a}^{\perp}b_2$ where, since $\vec{b}$ is a unit vector, $b_1^2 + b_2^2 = 1$. It follows that $\vec{a} \cdot \vec{b} = \vec{a} \cdot (\vec{a}b_1) + \vec{a} \cdot (\vec{a}^{\perp}b_2)$ $= (\vec{a} \cdot \vec{a})b_1 + (\vec{a} \cdot \vec{a}^{\perp})b_2 = b_1$. [Since $\vec{a}$ is a unit vector, $\vec{a} \cdot \vec{a} = 1$, and, in any case, $\vec{a} \cdot \vec{a}^{\perp} = 0$.] Similarly, $\vec{a}^{\perp} \cdot \vec{b} = b_2$. [Since $\vec{a}$ is a unit vector, so is $\vec{a}^{\perp}$.] Hence, $(\vec{a} \cdot \vec{b})^2 + (\vec{a}^{\perp} \cdot \vec{b})^2$ $= b_1^2 + b_2^2 = 1$. Similarly, $(\vec{c} \cdot \vec{d})^2 + (\vec{c}^{\perp} \cdot \vec{d})^2 = 1$.
   (b) These equations follow at once from the equations of part (a) and the fact that, since $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are unit vectors, $\vec{a} \cdot \vec{b} = \cos(a, b)$ and $\vec{c} \cdot \vec{d} = \cos(c, d)$.
   (c) If (a, b) is neither null nor straight then $\cos(a, b) = \cos(a \cup b)$ and $\sin(a \cup b) = \sqrt{1 - \cos^2(a \cup b)}$. So, by part (b), if (a, b) is neither null nor straight then $|\vec{a}^{\perp} \cdot \vec{b}| = \sin(a \cup b)$.
   (d) If (a, b) is positively sensed then $\vec{a}^{\perp} \cdot \vec{b} > 0$ and, so, $|\vec{a}^{\perp} \cdot \vec{b}| = \vec{a}^{\perp} \cdot \vec{b}$. Hence, by part (c), if $(\vec{a}, \vec{b})$ is positively sensed then $\vec{a}^{\perp} \cdot \vec{b} = \sin(a \cup b)$.
   (e) If (a, b) is negatively sensed then $\vec{a}^{\perp} \cdot \vec{b} < 0$ and, so, $|\vec{a}^{\perp} \cdot \vec{b}| = -\vec{a}^{\perp} \cdot \vec{b}$. Hence, by part (c), if (a, b) is negatively sensed then $-\vec{a}^{\perp} \cdot \vec{b} = \sin(a \cup b)$ and, so, $\vec{a}^{\perp} \cdot \vec{b} = -\sin(a \cup b)$.

4. (a) $\vec{a}^{\perp} \cdot \vec{b} = \vec{c}^{\perp} \cdot \vec{d}$ (b) $\vec{a}^{\perp} \cdot \vec{b} = \vec{c}^{\perp} \cdot \vec{d}$

5. If $\vec{a}^{\perp} \cdot \vec{b} = \sin(a \cup b)$ then [since it follows that $a \cup b$ is an angle], $\sin(a \cup b) > 0$ and, so, $\vec{a}^{\perp} \cdot \vec{b} > 0$. Hence, if $\vec{a}^{\perp} \cdot \vec{b} = \sin(a \cup b)$ then (a, b) is positively sensed. Consequently, the converse of (d) is a theorem. Similarly, the converse of (e) is a theorem.

*

In section 18.06 we shall define the "perp-measure", $m^{\perp}$, of a sensed angle. This will be a positive number for positively sensed angles, a negative number for negatively sensed angles. Also, in Chapter 19 we shall introduce a sine function with numerical arguments. Combining these two notions it will turn out that

$$\sin^{\perp}(r, s) = \sin[m^{\perp}(r, s)].$$

Answers for Part D

1. Suppose that $\vec{s} \in [s]^+$. Then:
   (a) $\sin^{\perp}(r, s) = (\vec{r}^{\perp} \cdot \vec{s})/(\|\vec{r}\| \|\vec{s}\|)$
   $= (\vec{r}^{\perp} \cdot \vec{s})/(\|\vec{r}^{\perp}\| \|\vec{s}\|) = \cos(r^{\perp}, s)$
   (b) $\sin^{\perp}(r, s) = (\vec{r}^{\perp} \cdot \vec{s})/(\|\vec{r}\| \|\vec{s}\|)$
   $= -(\vec{s}^{\perp} \cdot \vec{r})/(\|\vec{s}\| \|\vec{r}\|) = -\sin^{\perp}(s, r)$

Answers for Part D [cont.]

2. (a) Since $(\vec{r}, \vec{r}^{\perp})$ is an orthonormal basis and $\vec{s}$ is a unit vector, $||\vec{r}|| = ||\vec{r}^{\perp}|| = ||\vec{s}|| = 1$. Hence, $\vec{s} \cdot \vec{r} = \vec{r} \cdot \vec{s} = \cos(r, s)$ and $\vec{s} \cdot \vec{r}^{\perp} = \vec{r}^{\perp} \cdot \vec{s} = \sin^{\perp}(r, s)$. Consequently, $\vec{s} = \vec{r} \cos(r, s) + \vec{r}^{\perp} \sin^{\perp}(r, s)$. [Note that it follows that if $\vec{r}$ is a non-$\vec{0}$ vector and $\vec{s} \in [\vec{r}, \vec{r}^{\perp}]$ then

$$\vec{s} = (\vec{r} \cos(r, s) + \vec{r}^{\perp} \sin^{\perp}(r, s))(||\vec{s}||/||\vec{r}||).$$

The case $\vec{s} = \vec{0}$ requires special, but trivial, treatment.]

(b) $||\vec{s}|| = [\cos(r, s)]^2 + [\sin^{\perp}(r, s)]^2$

(c) This follows from (b) and the fact that $||\vec{s}|| = 1$.

3. By part (a) of Exercise 2, $\vec{s}^{\perp} = \vec{r} \cos(r, s^{\perp}) + \vec{r}^{\perp} \sin^{\perp}(r, s^{\perp})$. Using the results of Exercise 1 and the fact that $s^{\perp\perp} = -s$ and $\cos(r, s) = \cos(s, r)$ it follows that

$$\begin{aligned} \vec{s}^{\perp} &= \vec{r} \cos(s^{\perp}, r) - \vec{r}^{\perp} \sin^{\perp}(s^{\perp}, r) \\ &= \vec{r} \sin^{\perp}(s, r) - \vec{r}^{\perp} \cos(s^{\perp\perp}, r) \\ &= -\vec{r} \sin^{\perp}(r, s) - \vec{r}^{\perp} \cos(-s, r) \\ &= -\vec{r} \sin^{\perp}(r, s) + \vec{r}^{\perp} \cos(r, s). \end{aligned}$$

The last transformation makes use of the easily proved fact that $\cos(-s, r) = -\cos(s, r) = -\cos(r, s)$.

4. (a) $\vec{r} \cdot \vec{s} = \vec{r} \cdot \vec{r} = 1$, $\vec{r}^{\perp} \cdot \vec{s} = \vec{r}^{\perp} \cdot \vec{r} = 0$; $\cos(r, s) = 1$, $\sin^{\perp}(r, s) = 0$

(b) $\vec{r} \cdot \vec{s} = \vec{r} \cdot \vec{r}^{\perp} = 0$, $\vec{r}^{\perp} \cdot \vec{s} = \vec{r}^{\perp} \cdot \vec{r}^{\perp} = 1$; $\cos(r, s) = 0$, $\sin^{\perp}(r, s) = 1$

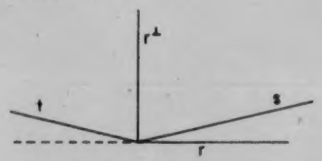
(c) $\cos(r, s) = \vec{r} \cdot \vec{s} = \vec{r} \cdot -\vec{r} = -1$, $\sin^{\perp}(r, s) = \vec{r}^{\perp} \cdot \vec{s} = \vec{r}^{\perp} \cdot -\vec{r} = -(\vec{r}^{\perp} \cdot \vec{r}) = 0$

(d) $\cos(r, s) = \vec{r} \cdot \vec{s} = \vec{r} \cdot -\vec{r}^{\perp} = -(\vec{r} \cdot \vec{r}^{\perp}) = 0$, $\sin^{\perp}(r, s) = \vec{r}^{\perp} \cdot \vec{s} = \vec{r}^{\perp} \cdot -\vec{r}^{\perp} = -(\vec{r}^{\perp} \cdot \vec{r}^{\perp}) = -1$

5. (a) 1; 0
(b) right; 0
(c) right; negatively sensed; 0
(d) right and negatively sensed
(e) 0
(f) right and positively sensed

Answers for Part E

1. (a), (b)



2. (a) $r \cup t$ is an obtuse angle.; (r, t) is positively sensed
(b) $r \cup t$ is an acute angle.; (r, t) is negatively sensed
(c) $t = r^{\perp}$
(d) $t = r$ [or $t = s$]

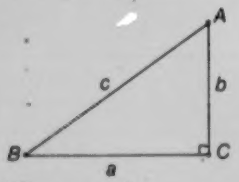
Part D

1. Given a sensed angle, (r, s), of π, let $r^{\perp}$ be the ray of π whose vertex is that of (r, s) and whose sense is that of $\vec{r}^{\perp}$, where $\vec{r} \in [r]^{+}$. Show the following.
   (a) $\sin^{\perp}(r, s) = \cos(r^{\perp}, s)$
   (b) $\sin^{\perp}(r, s) = -\sin^{\perp}(s, r)$
2. Suppose that $(\vec{r}, \vec{r}^{\perp})$ is an orthonormal basis for $[\pi]$. Then, for any unit vector $\vec{s}$ in $[\vec{r}, \vec{r}^{\perp}]$, $\vec{s} = \vec{r}(\vec{s} \cdot \vec{r}) + \vec{r}^{\perp}(\vec{s} \cdot \vec{r}^{\perp})$.
   (a) Express $\vec{s}$ in terms of '$\cos(r, s)$' and '$\sin^{\perp}(r, s)$'.
   (b) Express the norm of $\vec{s}$ in terms of '$\cos(r, s)$' and '$\sin^{\perp}(r, s)$'.
   (c) Show that $[\cos(r, s)]^2 + [\sin^{\perp}(r, s)]^2 = 1$.
3. Repeat Exercise 2(a) for the unit vector $\vec{s}^{\perp}$.
4. Suppose that sensed angle $(r, r^{\perp})$ is right and positively sensed, and that s is any coplanar ray whose vertex is that of $(r, r^{\perp})$. Let $\vec{r}$ and $\vec{s}$ be unit vectors in the senses of r and s, respectively.
   (a) Given that $s = r$, compute $\vec{r} \cdot \vec{s}$ and $\vec{r}^{\perp} \cdot \vec{s}$. What can you say about $\cos(r, s)$ and $\sin^{\perp}(r, s)$?
   (b) Given that $s = r^{\perp}$, compute $\vec{r} \cdot \vec{s}$ and $\vec{r}^{\perp} \cdot \vec{s}$. What does this tell you about $\cos(r, s)$ and $\sin^{\perp}(r, s)$?
   (c) Compute $\cos(r, s)$ and $\sin^{\perp}(r, s)$ given that $s = -r$.
   (d) Compute $\cos(r, s)$ and $\sin^{\perp}(r, s)$ given that $s = -r^{\perp}$.
5. Complete these sentences.
   (a) If (r, s) is null, $\cos(r, s) =$____ and $\sin^{\perp}(r, s) =$____.
   (b) If (r, s) is____and positively sensed, $\sin^{\perp}(r, s) = 1$ and $\cos(r, s) =$____.
   (c) If (r, s) is____and____, $\sin^{\perp}(r, s) = -1$ and $\cos(r, s) =$____.
   (d) If $\sin^{\perp}(r, s) = -1$, (r, s) is____.
   (e) If $\sin^{\perp}(r, s) =$____, (r, s) is straight or null.
   (f) If $\sin^{\perp}(r, s) = 1$, (r, s) is____.

Part E

1. On your paper, draw a picture of a positively sensed right angle, $(r, r^{\perp})$.
   (a) Draw any coplanar ray s whose vertex is that of r.
   (b) Now, draw a second coplanar ray, t, whose vertex is that of r and such that $\sin^{\perp}(r, s) = \sin^{\perp}(r, t)$.
2. Suppose that s and t are any rays such as those described in Exercise 1.
   (a) Given that $t \neq s$ and $r \cup s$ is an acute angle, what can you say about $r \cup t$? About (r, t)?
   (b) Given that $t \neq s$, $r \cup s$ is obtuse, and (r, s) is negatively sensed, what can you say about $r \cup t$? About (r, t)?
   (c) Given that $s = r^{\perp}$, what can you say about t?
   (d) Given that $s = -r$, what can you say about t?

3. Is it the case that for any acute angle $r \cup s$ of π there is an obtuse angle $r \cup t$ of π such that $\sin^{\perp}(r, s) = \sin^{\perp}(r, t)$? Explain.
4. Given an angle, $r \cup s$, of π we already have established that $\sin^{\perp}(r, s) = \cos(r^{\perp}, s)$. Suppose that $s_1 \cup s_2$ is an angle of π whose vertex is that of r and that $r^{\perp}$ bisects $s_1 \cup s_2$. Show that $\sin^{\perp}(r, s_1) = \sin^{\perp}(r, s_2)$.
5. Given that $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$ and that $\angle CBA$ is positively sensed. Assume that $AB = c$, $BC = a$, and $CA = b$, as shown in the picture at the right.



   (a) Show that $\sin^{\perp} \angle CBA = \cos \angle BAC = \cos \angle A$.
   (b) Show that $\sin^{\perp} \angle ABC = -\cos \angle A = -b/c$.
6. Make use of the information in Exercise 5 to express each of the following in terms of 'a', 'b', and 'c'.

(a) $\cos \angle BAC$	(b) $\cos \angle ACB$	(c) $\sin^{\perp} \angle BAC$
(d) $\sin^{\perp} \angle CAB$	(e) $\sin^{\perp} \angle BCA$	(f) $\sin^{\perp} \angle ACB$

*

Earlier, we saw that if (r, s) is a sensed angle of an oriented plane π, $\sin^{\perp}(r, s) = -\sin^{\perp}(s, r)$. It is not difficult to make use of what we know about dot products and perping operations to obtain many such statements. To obtain some insight as to how we can do this consider the expression '$\sin^{\perp}(r, s^{\perp})$', and let $\vec{r}$ and $\vec{s}$ be unit vectors in the senses of r and s, respectively. Then, the given expression can be transformed as:

$$\begin{aligned} \sin^{\perp}(r, s^{\perp}) &= \vec{r}^{\perp} \cdot \vec{s}^{\perp} && \text{[Why?]} \\ &= \vec{r} \cdot \vec{s} && \text{[Why?]} \\ &= \cos(r, s) && \text{[Why?]} \end{aligned}$$

Thus, we know that $\sin^{\perp}(r, s^{\perp}) = \cos(r, s)$. This is pictured as follows:

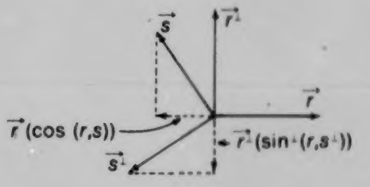


Fig. 18–12

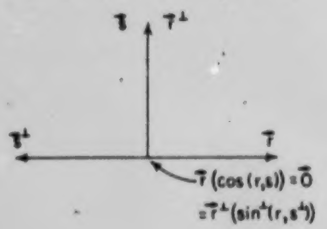
[Fig. 18–12 shows the case where $\cos(r, s) < 0$. Draw pictures for the cases where $\cos(r, s) = 0$ and $\cos(r, s) > 0$.]

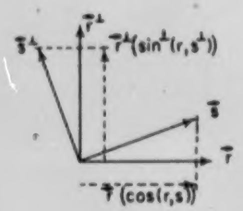
Answers for Part E [cont.]

3. Yes. Suppose that $\vec{r}$ and $\vec{s}$ are unit vectors in $[r]^{+}$ and $[s]^{+}$. Then $\vec{s} = \vec{r}s_1 + \vec{r}^{\perp}s_2$ where, since $r \cup s$ is acute $\vec{r} \cdot \vec{s} = s_1 > 0$. Let $\vec{t} = -\vec{r}s_1 + \vec{r}^{\perp}s_2$ and let t be the ray with the same vertex as r and s and whose sense is $[\vec{t}]^{+}$. Since $\|\vec{t}\| = s_1^2 + s_2^2 = \|\vec{s}\| = 1$, $\vec{t}$ is a unit vector. It follows that $\cos(r \cup t) = \vec{r} \cdot \vec{t} = -s_1 < 0$ and, so, that $r \cup t$ is obtuse. Also,
$$\sin^{\perp}(r, t) = \vec{r}^{\perp} \cdot \vec{t} = s_2 = \vec{r}^{\perp} \cdot \vec{s} = \sin^{\perp}(r, s).$$
4. Suppose that $\vec{r}$, $\vec{s}_1$, and $\vec{s}_2$ are unit vectors in the senses of r, s_1, and s_2. It follows that, since $r^{\perp}$ bisects $s_1 \cup s_2$, that $\vec{r}^{\perp} = (\vec{s}_1 + \vec{s}_2)/\sqrt{2}$. So,
$$\sin^{\perp}(r, s_1) = \vec{r}^{\perp} \cdot \vec{s}_1 = (\vec{s}_1 \cdot \vec{s}_1 + \vec{s}_2 \cdot \vec{s}_1)/\sqrt{2} = (1 + \vec{s}_2 \cdot \vec{s}_1)/\sqrt{2}, \text{ and}$$
$$\sin^{\perp}(r, s_2) = \vec{r}^{\perp} \cdot \vec{s}_2 = (\vec{s}_1 \cdot \vec{s}_2 + \vec{s}_2 \cdot \vec{s}_2)/\sqrt{2} = (\vec{s}_1 \cdot \vec{s}_2 + 1)/\sqrt{2}.$$
Hence, $\sin^{\perp}(r, s_1) = \sin^{\perp}(r, s_2)$.
5. (a) Suppose, as usual that $\vec{a} = C - B$, $\vec{b} = A - C$, and $\vec{c} = B - A$. By definition, $\sin^{\perp} \angle CBA = (\vec{a}^{\perp} \cdot -\vec{c})/(ac)$. Since $\angle CBA$ is positively sensed, $\vec{a}^{\perp} = \vec{b}(a/b)$. So, $\sin^{\perp} \angle CBA = (\vec{b} \cdot -\vec{c})/(bc) = (\vec{c} \cdot -\vec{b})/(cb) = \cos \angle BAC = \cos \angle A$.
   (b) $\sin^{\perp} \angle ABC = -\sin^{\perp} \angle CBA = -\cos \angle A$, by part (a). From earlier results, $\cos \angle A = b/c$.
6. (a) b/c (b) 0 (c) a/c (d) −a/c (e) −1 (f) 1

Answers for 'Why?'s: Definition 18-9, Theorem 18-1(c), Definition 18-7

Figures supplementing Figure 18-12:


$\vec{s}$
$\vec{r}^{\perp}$
$\vec{s}^{\perp}$
$\vec{r}$
$\vec{r}(\cos(r,s)) = \vec{0}$
$= \vec{r}^{\perp}(\sin^{\perp}(r,s^{\perp}))$



Answers for Part F

[In the following, $\vec{r}$ and $\vec{s}$ are unit vectors in the senses of r and s, respectively. Then, for example, $\vec{r}^{\perp}$ is a unit vector in the sense of $r^{\perp}$ (this is how $r^{\perp}$ was defined) and $-\vec{r}$ is a unit vector in the sense of $-r$.]

1. (a) $\sin^{\perp}(r, -s) = \vec{r}^{\perp} \cdot -\vec{s} = -(\vec{r}^{\perp} \cdot \vec{s}) = -\sin^{\perp}(r, s)$
   (b) $\cos(r^{\perp}, s) = \vec{r}^{\perp} \cdot \vec{s} = \sin^{\perp}(r, s)$
   (c) $\cos(-r^{\perp}, s) = -\vec{r}^{\perp} \cdot \vec{s} = -(\vec{r}^{\perp} \cdot \vec{s}) = -\sin^{\perp}(r, s)$
   (d) $\sin^{\perp}(r^{\perp}, s) = \vec{r}^{\perp\perp} \cdot \vec{s} = -\vec{r} \cdot \vec{s} = -(\vec{r} \cdot \vec{s}) = -\cos(r, s)$

2. (a) $\cos(r, s)$ (b) $\sin^{\perp}(r, s)$ (c) $\cos(r, s)$
   (d) $\cos(r, s)$ (e) $\sin^{\perp}(r, s)$ (f) $\sin^{\perp}(r, s)$
   (g) $-\sin^{\perp}(r, s)$ (h) $-\cos(r, s)$ (i) $\sin^{\perp}(r, s)$

   [The parts of Exercise 2 can be answered by using earlier parts and the results of Exercise 1.]

3. (a) $\cos\angle A = 1/2$, $\sin\angle A = \sqrt{3}/2$
   (b) $\cos(r, s) = 1/2$, $\sin^{\perp}(r, s) = \sqrt{3}/2$
   (c) $\cos(r, s) = 1/2$, $\sin^{\perp}(r, s) = -\sqrt{3}/2$
   (d) $\cos(r, -s) = -1/2$, $\cos(r^{\perp}, s) = \sqrt{3}/2$,
   $\sin^{\perp}(r, -s) = -\sqrt{3}/2$, $\sin^{\perp}(r^{\perp}, s) = -1/2$,

   $\cos(r, s^{\perp}) = -\sqrt{3}/2$, $\cos(r, -s^{\perp}) = \sqrt{3}/2$,
   $\sin^{\perp}(r, s^{\perp}) = 1/2$, $\sin^{\perp}(r, -s^{\perp}) = -1/2$,

4. (a) $\cos(r, -s) = -1/\sqrt{2}$, $\cos(r^{\perp}, s) = 1/\sqrt{2}$,
   $\sin^{\perp}(r, -s) = -1/\sqrt{2}$, $\sin^{\perp}(r^{\perp}, s) = -1/\sqrt{2}$,

   $\cos(r, s^{\perp}) = -1/\sqrt{2}$, $\cos(r, -s^{\perp}) = 1/\sqrt{2}$
   $\sin^{\perp}(r, s^{\perp}) = 1/\sqrt{2}$, $\sin^{\perp}(r, -s^{\perp}) = -1/\sqrt{2}$

   (b) $\cos(r, -s) = -\sqrt{3}/2$, $\cos(r^{\perp}, s) = 1/2$,
   $\sin^{\perp}(r, -s) = -1/2$, $\sin^{\perp}(r^{\perp}, s) = -\sqrt{3}/2$,

   $\cos(r, s^{\perp}) = -1/2$, $\cos(r, -s^{\perp}) = 1/2$
   $\sin^{\perp}(r, s^{\perp}) = \sqrt{3}/2$, $\sin(r, -s^{\perp}) = -\sqrt{3}/2$

   (c) $\cos(r, -s) = 1/2$, $\cos(r^{\perp}, s) = \sqrt{3}/2$,
   $\sin^{\perp}(r, -s) = -\sqrt{3}/2$, $\sin^{\perp}(r^{\perp}, s) = 1/2$,

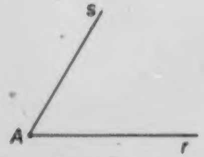
   $\cos(r, s^{\perp}) = -\sqrt{3}/2$, $\cos(r, -s^{\perp}) = \sqrt{3}/2$
   $\sin^{\perp}(r, s^{\perp}) = -1/2$, $\sin^{\perp}(r, -s^{\perp}) = 1/2$

   (d) $\cos(r, -s) = 1/\sqrt{2}$, $\cos(r^{\perp}, s) = 1/\sqrt{2}$,
   $\sin^{\perp}(r, -s) = -1/\sqrt{2}$, $\sin^{\perp}(r^{\perp}, s) = 1/\sqrt{2}$,

   $\cos(r, s^{\perp}) = -1/\sqrt{2}$, $\cos(r, -s^{\perp}) = 1/\sqrt{2}$
   $\sin^{\perp}(r, s^{\perp}) = -1/\sqrt{2}$, $\sin^{\perp}(r, -s^{\perp}) = 1/\sqrt{2}$

Part F

1. Make use of what you know about dot products and perping operations to prove each of the following
   (a) $\sin^{\perp}(r, -s) = -\sin^{\perp}(r, s)$ (b) $\cos(r^{\perp}, s) = \sin^{\perp}(r, s)$
   (c) $\cos(-r^{\perp}, s) = -\sin^{\perp}(r, s)$ (d) $\sin^{\perp}(r^{\perp}, s) = -\cos(r, s)$
2. Express each of the following in terms of either $\cos(r, s)$ or $\sin^{\perp}(r, s)$.
   (a) $\cos(r^{\perp}, s^{\perp})$ (b) $\sin^{\perp}(r^{\perp}, s^{\perp})$ (c) $\cos(-r, -s)$
   (d) $\cos(-r^{\perp}, -s^{\perp})$ (e) $\sin^{\perp}(-r, -s)$ (f) $\sin^{\perp}(-r^{\perp}, -s^{\perp})$
   (g) $\cos(r, s^{\perp})$ (h) $\cos(r, -s)$ (i) $\cos(r, -s^{\perp})$



3. Suppose that $\angle A$ is an angle of $\pi/3$ radians and that $\angle A = r \cup s$, as shown in the picture at the right. Answer the following.
   (a) What are $\cos\angle A$ and $\sin\angle A$?
   (b) Given that (r, s) is positively sensed, what are $\cos(r, s)$ and $\sin^{\perp}(r, s)$?
   (c) Given that (r, s) is negatively sensed, what are $\cos(r, s)$ and $\sin^{\perp}(r, s)$?
   (d) Evaluate each of the following, given that (r, s) is positively sensed.

   $\cos(r, -s)$ $\cos(r^{\perp}, s)$ $\cos(r, s^{\perp})$ $\cos(r, -s^{\perp})$
   $\sin^{\perp}(r, -s)$ $\sin^{\perp}(r^{\perp}, s)$ $\sin^{\perp}(r, s^{\perp})$ $\sin^{\perp}(r, -s^{\perp})$
4. Repeat Exercise 3(d) in case $\angle A$ is an angle of
   (a) $\pi/4$ radians; (b) $\pi/6$ radians;
   (c) $2\pi/3$ radians; (d) $3\pi/4$ radians.

18.06 Measures of Sensed Angles

In Chapter 17, we found it useful to assign measures to angles. It is also useful to assign *sensed measures* to sensed angles. Now, given any angle—say, $\angle ABC$—there are two sensed angles, $\angle ABC$ and $\angle CBA$, associated with it. Of course, $\angle ABC$ and $\angle CBA$ are oppositely sensed and have the same "absolute size". We wish to have the sensed measures we assign to them reflect these two properties. An obvious and reasonable way to do this is to assign the radian-measure of the related angle to the positively sensed of the two sensed angles and the opposite of that measure to the negatively sensed one. Having done this, we can complete the task by assigning measures to the "extreme" sensed angles – those which are null or straight. Natural choices for the latter are 0 [for null sensed angles] and π [for straight sensed angles].

Before we proceed to formalize the assigning of measures to sensed angles, it is worth recalling that whether a given sensed angle is positively or negatively sensed is directly linked to which of the two perping operations has been selected as the positive one. So, it should be clear that there are two "sensed measure" functions – one for each of the perping operations – which can be used to assign measures to sensed angles. In particular, if $^{\perp 1}$ and $^{\perp 2}$ are the two perping operations on $[\sigma]$ and $m^{\perp 1}$ and $m^{\perp 2}$ are the functions which assign measures to sensed angles of σ then, for sensed angle $\angle A$ of σ, we have that $m^{\perp 1}(\angle A)$ and $m^{\perp 2}(\angle A)$ are opposites.

For example, suppose that, in σ, $\angle ABC$ is an angle of $\pi/3$ radians. Let $^{\perp 1}$ be the perping operation on $[\sigma]$ for which $\angle ABC$ is positively sensed, and let $^{\perp 2}$ be the other perping operation on $[\sigma]$. By our above-mentioned "obvious and reasonable" way of assigning directed measures to sensed angles, we have that

$$m^{\perp 1}(\angle ABC) = \pi/3 \text{ and } m^{\perp 2}(\angle ABC) = -\pi/3.$$

[What is $m^{\perp 1}(\angle CBA)$? $m^{\perp 2}(\angle CBA)$?]

Recall that $\angle A$ is positively sensed if and only if $\sin^{\perp} \angle A > 0$ and is negatively sensed if and only if $\sin^{\perp} \angle A < 0$. We make use of this in assigning measure to sensed angles in:

> Definition 18–10 Given sensed angle $\angle A$ of an oriented plane,
> (a) $m^{\perp}(\angle A) = \text{sgn}(\sin^{\perp} \angle A)m(\angle A)$ if $\angle A$ is neither null nor straight;
> (b) $m^{\perp}(\angle A) = 0$ if $\angle A$ is null, and $m^{\perp}(\angle A) = \pi$ if $\angle A$ is straight.

[Read '$m^{\perp}(\angle A)$' as you would 'sensed measure of sensed angle A'.]

Exercises

Part A

1. What is $\text{sgn}(\sin^{\perp} \angle A)$ given that $\angle A$ is negatively sensed? Positively sensed?
2. What is the range of values of sensed measures of positively sensed angles? Negatively sensed angles?

Answer to questions at end of third paragraph: $m^{\perp 1}(\angle CBA) = -\pi/3$ and $m^{\perp 2}(\angle CBA) = \pi/3$.

*

Sample Quiz

Given right $\triangle PQR$ with hypotenuse $\overline{RQ}$, assume that the plane of $\triangle PQR$ is oriented so that $\angle QPR$ is positively sensed, that $\overline{PM}$ is the median and $\overline{PS}$ is the altitude from P, and that PR = 5 and PQ = 12. Compute the following.

1. RS
2. PS
3. SM
4. PM
5. $\cos \angle SPR$
6. $\sin^{\perp} \angle SPR$
7. $\cos \angle QMP$
8. $\sin^{\perp} \angle QMP$

Key to Sample Quiz

1. 25/13
2. 60/13
3. 119/26
4. 13/2
5. 12/13
6. 5/13
7. −119/169
8. −120/169

[Note: All cosine and sine-perp values are easily computed by taking ratios in appropriate right triangles and paying attention to the sense classes of the angles in question.]

Answers for Part A

1. −1; 1
2. $\{x: 0 < x < \pi\}$; $\{x: -\pi < x < 0\}$

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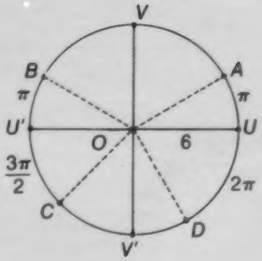
3. (a) $m^{\perp}(\angle UOA) = \pi/6$, $m^{\perp}(\angle UOD) = -\pi/3$, $m^{\perp}(\angle BOU) = -5\pi/6$, $m^{\perp}(\angle BOU') = \pi/6$, $m^{\perp}(\angle COU) = 3\pi/4$, $m^{\perp}(\angle COU') = -\pi/4$, $m^{\perp}(\angle AOB) = 2\pi/3$, $m^{\perp}(\angle DOD) = 0$
   (b) $\cos \angle UOA = \sqrt{3}/2$, $\sin^{\perp} \angle UOA = 1/2$;
   $\cos \angle UOD = \sqrt{3}/2$, $\sin^{\perp} \angle UOD = -1/2$;
   $\cos \angle BOU = -\sqrt{3}/2$, $\sin^{\perp} \angle BOU = -1/2$;
   $\cos \angle BOU' = \sqrt{3}/2$, $\sin^{\perp} \angle BOU' = 1/2$;
   $\cos \angle COU = -1/\sqrt{2}$, $\sin^{\perp} \angle COU = 1/\sqrt{2}$;
   $\cos \angle COU' = 1/\sqrt{2}$, $\sin^{\perp} \angle COU' = -1/\sqrt{2}$;
   $\cos \angle AOB = -1/2$, $\sin \angle AOB = \sqrt{3}/2$;
   $\cos \angle DOD = 1$, $\sin \angle DOD = 0$
   (c) If $\angle UOV$ is negatively sensed then each measure in (a) is replaced by its opposite and, in (b) each value of $\sin^{\perp}$ is replaced by its opposite. The values of cos remain the same.
   (d) $m^{\perp}(\angle UOU) = 0$; $m^{\perp}(\angle UOU') = \pi$; $m^{\perp}(\angle VOV') = \pi$;
   $m^{\perp}(\angle V'OV) = \pi$ [not $-\pi$]

3. Consider the circle with center O and radius 6, pictured at the right. Given that $\overline{UU'}$ and $\overline{VV'}$ are perpendicular diameters, and the arcs have measures as indicated, answer these questions.



   (a) What are the sensed measures of the following sensed angles, given that $\angle UOV$ is positively sensed?
   $\angle UOA$, $\angle UOD$, $\angle BOU$, $\angle BOU'$, $\angle COU$, $\angle COU'$, $\angle AOB$, $\angle DOD$
   (b) Give the cosine and sine-perp values for each of the sensed angles given in (a).
   (c) How are your answers for (a) affected if you are given that $\angle UOV$ is negatively sensed? How about your answers in (b)?
   (d) What is the sensed measure of $\angle UOU$? Of $\angle UOU'$? Of $\angle VOV'$? Of $\angle V'OV$?

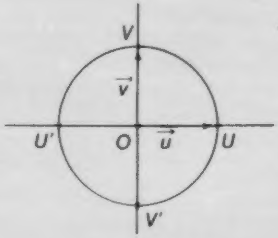
4. Suppose that, in $\triangle ABC$, $m(\angle B) = \pi/4$ and $m(\angle C) = \pi/3$. Also, let D and E be points on $\overleftrightarrow{BC}$ such that $\angle ACD$ and $\angle ABE$ are exterior angles of $\triangle ABC$. [Draw a picture.]
   (a) Given that $\overrightarrow{ABC}$ is oriented so that $\angle ABC$ is positively sensed, give the sensed measures of each of the following:

   $\angle EBA$, $\angle AEC$, $\angle ACB$, $\angle ACD$,
   $\angle EBC$, $\angle DCB$, $\angle BCA$, $\angle CBA$

   (b) Give the cosine and sine-perp values of each of the angles whose measures you gave in (a).
   (c) How are your answers in (a) affected if $\overrightarrow{ABC}$ is oriented so that $\angle ABC$ is negatively sensed? How about your answers in (b)?

Part B

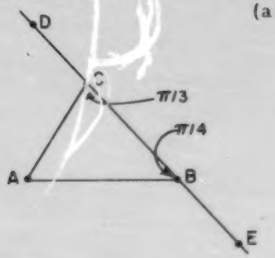
Suppose that $(\vec{u}, \vec{v})$ is a positively sensed orthonormal basis for $[\sigma]$, that O is the center of a circle of σ with radius 1, and that $U = O + \vec{u}$ and $V = O + \vec{v}$, as shown in the picture at the right. Consider the coordinate system for σ with origin O and basis $(\vec{u}, \vec{v})$.



1. On your paper, draw a picture of the given circle and coordinate system. [Squared graph paper is useful for this purpose.] In each of the following, you are given points together with the measures of

Answers for Part A [cont.]

4.



(a) $m^{\perp}(\angle EBA) = 3\pi/4$, $m^{\perp}(\angle ABC) = \pi/4$,
$m^{\perp}(\angle ACB) = -\pi/3$, $m^{\perp}(\angle ACD) = 2\pi/3$,
$m^{\perp}(\angle EBC) = \pi$, $m^{\perp}(\angle DCB) = \pi$,
$m^{\perp}(\angle BCA) = \pi/3$, $m^{\perp}(\angle CBA) = -\pi/4$

(b) $\cos \angle EBA = -1/\sqrt{2}$, $\sin^{\perp} \angle EBA = 1/\sqrt{2}$;
$\cos \angle ABC = 1/\sqrt{2}$, $\sin^{\perp} \angle ABC = 1/\sqrt{2}$;
$\cos \angle ACB = 1/2$, $\sin^{\perp} \angle ACB = -\sqrt{3}/2$;
$\cos \angle ACD = -1/2$, $\sin^{\perp} \angle ACD = \sqrt{3}/2$;
$\cos \angle EBC = -1$, $\sin^{\perp} \angle EBC = 0$;
$\cos \angle DCB = -1$, $\sin^{\perp} \angle DCB = 0$;
$\cos \angle BCA = 1/2$, $\sin^{\perp} \angle BCA = \sqrt{3}/2$;
$\cos \angle CBA = 1/\sqrt{2}$, $\sin^{\perp} \angle CBA = -1/\sqrt{2}$

(c) If $\angle ABC$ is negatively sensed then each measure in (a) is replaced by its opposite and, in (b) each value of $\sin^{\perp}$ is replaced by its opposite. The values of cos remain the same.

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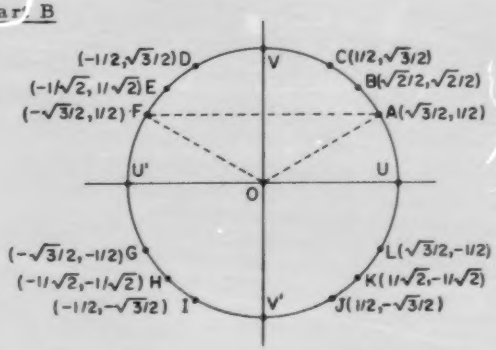
Answers for Part B

1.



2. (a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $2\pi/3$ (e) $3\pi/4$ (f) $5\pi/6$
(g) $-5\pi/6$ (h) $-3\pi/4$ (i) $-2\pi/3$ (j) $-\pi/3$ (k) $-\pi/4$ (ℓ) $-\pi/6$
3. $\underline{0,\ \pi/2,\ \pi,\ -\pi/2}$
4. $\overset{\frown}{UV'U'}$; $\overset{\frown}{UVU'}$

the counterclockwise arcs from U to those points. Plot the points and give their coordinates.

(a) $A, \pi/6$ (b) $B, \pi/4$ (c) $C, \pi/3$ (d) $D, 2\pi/3$
(e) $E, 3\pi/4$ (f) $F, 5\pi/6$ (g) $G, 7\pi/6$ (h) $H, 5\pi/4$
(i) $I, 4\pi/3$ (j) $J, 5\pi/3$ (k) $K, 7\pi/4$ (l) $L, 11\pi/6$

2. Give the sensed measures of the sensed angles $\angle UOX$, where the values of 'X' are points described in (a)–(l) of Exercise 1.
3. Give the sensed measures of $\angle UOU$, $\angle UOV$, $\angle UOU'$, and $\angle UOV'$.
4. Which of the semicircular arcs of the given circle contains just those points X such that $\angle UOX$ is negatively sensed? Positively sensed?
5. What are the cosine and sine-perp values of each of the sensed angles described in Exercises 2 and 3?
6. How are the cosine and sine-perp values of $\angle UOX$ related to the coordinates of X, where the values of 'X' are the points described in (a)–(l) of Exercise 1? Does this same relation hold for each point of the given circle? Explain your answers.
*7. How can you make use of the given information about A, B, and C together with properties of circles to obtain the corresponding information about the points $D - L$? Can the same be done with respect to any point of $\overset{\frown}{UV}$?
8. Describe the locations of two points—say, P and Q—of the given circle such that
   (a) $\cos \angle UOP = \cos \angle UOQ$ but $\sin^{\perp} \angle IOP \neq \sin^{\perp} \angle IOQ$;
   (b) $\sin^{\perp} \angle UOP = \sin^{\perp} \angle UOQ$ but $\cos \angle IOP \neq \cos \angle IOQ$.

18.07 Chapter Summary

Vocabulary Summary

sense classes
  of $[l]$
  of $[\pi]$
sensed angle
null sensed angle
straight sensed angle
sine-perp of a sensed angle

perping operations
positively sensed basis for $[\pi]$
negatively sensed basis for $[\pi]$
sense of a sensed angle
right sensed angle
cosine of a sensed angle
congruent sensed angles

Definitions

18-1. A singulary operation $^{\perp}$ on $[\pi]$ is a perping operation on $[\pi]$ if and only if, for $\vec{a}$ and $\vec{b}$ in $[\pi]$, (a) $\vec{a}^{\perp} \cdot \vec{a} = 0$, (b) $\|\vec{a}^{\perp}\| = \|\vec{a}\|$, and (c) $(\vec{a}a + \vec{b}b)^{\perp} = \vec{a}^{\perp}a + \vec{b}^{\perp}b$.

18-2. Choosing a perping operation, $^{\perp}$, on $[\pi]$, (a) a basis $(\vec{a}, \vec{b})$ for $[\pi]$ belongs to the sense class determined by $^{\perp}$ and, so, is *posi-*

Answers for Part B [cont.]

5. [See answer for Exercise 6, below.]

6. For each point X on the circle, the coordinates of X are $(\cos \angle UOX, \sin^{\perp} \angle UOX)$. This is so because the coordinates of X are the components of $X - O$ with respect to $\vec{u}$ and $\vec{v}$, and $\vec{v} = \vec{u}^{\perp}$. Hence, $\cos \angle UOX = (X - O) \cdot \vec{u}$ = first coordinate of X, and $\sin^{\perp} \angle UOX = (X - O) \cdot \vec{v}$ = second coordinate of X.

7. The points D - L can be obtained from A, B, and C by reflections in $\overleftrightarrow{OU}$ and $\overleftrightarrow{OV}$. [See below.] The reflection of a point P in $\overleftrightarrow{OU}$ has the same first coordinate as does P and its second coordinate is the opposite of that of P. [A similar statement holds with 'V' in place of 'U' and 'first' and 'second' interchanged.] To prove this, suppose that $P - O = \vec{u}p_1 + \vec{v}p_2$ and consider the point Q such that $Q - O = \vec{u}p_1 - \vec{v}p_2$. We show that Q is the reflection of P in $\overleftrightarrow{OU}$ by noting that $P - Q = \vec{v}(2p_2) \perp \vec{u}$ and that the midpoint of $\overline{PQ}$ is $Q + (P - Q)(1/2) = O + \vec{u}p_1 \in \overleftrightarrow{OU}$.

   The proof that the points D - L can be obtained from A, B, and C by reflections in $\overleftrightarrow{OU}$ and $\overleftrightarrow{OV}$ depends on the consideration of various right triangles. For example, since $m(\angle U'OF) = \pi/6 = m(\angle UOA)$ and $OF = OA$ it follows that F and A are at the same distance from $\overleftrightarrow{OU}$ and at the same distance from $\overleftrightarrow{OV}$. Since they are at the same distance from $\overleftrightarrow{OU}$, $\overleftrightarrow{FA} \parallel \overleftrightarrow{OU}$ and, so, $\overleftrightarrow{FA} \perp \overleftrightarrow{OV}$. Since they are at the same distance from $\overleftrightarrow{OV}$, the midpoint of $\overline{FA}$ belongs to $\overleftrightarrow{OV}$. Hence, F is the reflection of A in $\overleftrightarrow{OV}$. Similar arguments apply to D and E and slightly modified ones show that L, K, and J are reflections of A, B, and C in $\overleftrightarrow{OU}$. Finally, I, H, and G are shown [in the same way] to be reflections of D, E, and F in $\overleftrightarrow{OU}$.

   [Obviously, there is a lot of work involved in carrying out the details of all the arguments described above. There is no need for any but representative samples and different arguments can be farmed out to different members of the class.]

8. (a) Choose P different from U and U′ and let Q be its reflection in $\overleftrightarrow{OU}$.
   (b) Choose P different from V and V′ and let Q be its reflection in $\overleftrightarrow{OV}$.

tively sensed, if and only if $\vec{a}^{\perp} \cdot \vec{b} > 0$, and (b) a basis $(\vec{a}, \vec{b})$ for $[\pi]$ is negatively sensed if and only if $\vec{a}^{\perp} \cdot \vec{b} < 0$.

18-3. Choosing a perping operation on $[\pi]$, bases $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ for $[\pi]$ belong to the same sense class if and only if both are positively sensed or both are negatively sensed.

18-4. A sensed angle is an ordered pair of rays with the same vertex.

18-5. Given rays r and s with a common vertex. (a) (r, s) is null if and only if $s = r$. (b) (r, s) is right if and only if $s \perp r$. (c) (r, s) is straight if and only if $s = -r$.

18-6. Given that sensed angle (r, s) is neither null nor straight, the sense of (r, s) is the sense class associated with the basis $(\vec{r}, \vec{s})$, where $\vec{r} \in [r]^+$ and $\vec{s} \in [s]^+$.

18-7. Given sensed angle (r, s), $\cos(r, s) = (\vec{r} \cdot \vec{s})/(\|\vec{r}\| \, \|\vec{s}\|)$, where $\vec{r} \in [r]^+$ and $\vec{s} \in [s]^+$.

18-8. Given sensed angles (p, q) and (r, s), whose sides are contained in parallel planes, (p, q) and (r, s) are congruent if and only if both are null angles, both are straight angles, or both have the same sense and $\cos(p, q) = \cos(r, s)$.

18-9. Choosing a perping operation, $^{\perp}$, on $[\pi]$, and given a sensed angle (r, s) of π, $\sin^{\perp}(r, s) = (\vec{r}^{\perp} \cdot \vec{s})/(\|\vec{r}\| \, \|\vec{s}\|)$, where $\vec{r} \in [r]^+$ and $\vec{s} \in [s]^+$.

18-10. Given sensed angle $\angle A$ of an oriented plane, (a) $m^{\perp}(\angle A) = \operatorname{sgn}(\sin^{\perp} \angle A)\, m(\angle A)$ if $\angle A$ is neither null nor straight; (b) $m^{\perp}(\angle A) = 0$ if $\angle A$ is null, and $m^{\perp}(\angle A) = \pi$ if $\angle A$ is straight.

Other Theorems

18-1. Given that $^{\perp}$ is a perping operation on $[\pi]$, if $\vec{a}$ and $\vec{b}$ belong to $[\pi]$, then (a) $\vec{a} \cdot \vec{b}^{\perp} = -\vec{a}^{\perp} \cdot \vec{b}$, (b) $\vec{a}^{\perp\perp} = -\vec{a}$, and (c) $\vec{a}^{\perp} \cdot \vec{b}^{\perp} = \vec{a} \cdot \vec{b}$.

18-2. There are exactly two perping operations on $[\pi]$: Furthermore, if $(\vec{i}, \vec{j})$ is an orthonormal basis for $[\pi]$ and $^{\perp}$ is a perping operation on $[\pi]$ then, for any $\vec{a} \in [\pi]$, $\vec{a}^{\perp} = -\vec{i}(\vec{a} \cdot \vec{j}) + \vec{j}(\vec{a} \cdot \vec{i})$ or, for any $\vec{a} \in [\pi]$, $\vec{a}^{\perp} = \vec{i}(\vec{a} \cdot \vec{j}) - \vec{j}(\vec{a} \cdot \vec{i})$.

18-3. Given a basis $(\vec{a}, \vec{b})$ for $[\pi]$, and choosing a perping operation, $^{\perp}$, on $[\pi]$, (a) $(\vec{a}, \vec{b})$ is positively sensed if and only if $\exists_x \exists_{y>0}\, \vec{b} = \vec{a}x + \vec{a}^{\perp} y$; and (b) $(\vec{a}, \vec{b})$ is negatively sensed if and only if $\exists_x \exists_{y<0}\, \vec{b} = \vec{a}x + \vec{a}^{\perp} y$.

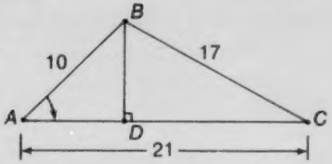
18-4. Given bases $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ for $[\pi]$, $(\vec{a}, \vec{b})$ and $(\vec{c}, \vec{d})$ belong to the same sense class if and only if $\begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix} > 0$.

18-5. Given that (r, s) and (p, q) are sensed angles, that $\vec{r} \in [r]^+$, $\vec{s} \in [s]^+$, $\vec{p} \in [p]^+$, and $\vec{q} \in [q]^+$, and that the sides of (r, s) and (p, q)

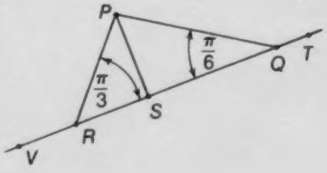
are contained in parallel planes, (r, s) and (p, q) have the same sense if and only if $\begin{vmatrix} \vec{r}\cdot\vec{p} & \vec{r}\cdot\vec{q} \\ \vec{s}\cdot\vec{p} & \vec{s}\cdot\vec{q} \end{vmatrix} > 0$. They have opposite senses if and only if this determinant is less than 0. And, one of the sensed angles is either null or straight if and only if the determinant is 0.

Chapter Test

1. Suppose that $(\vec{u}, \vec{v})$ is an orthonormal basis for $[\pi]$.
   - (a) Which of $(-\vec{u}2 + \vec{v}3, -\vec{u} + \vec{v}2)$ and $(\vec{u}2 + \vec{v}3, \vec{u}2 - \vec{v})$ belong to the same sense class as does $(\vec{u}, \vec{v})$? Explain your answers.
   - (b) Are $(-\vec{u}2 + \vec{v}3, -\vec{u} + \vec{v}2)$ and $(\vec{u}2 + \vec{v}3, \vec{u}2 - \vec{v})$ similarly sensed or not? Explain.
2. Suppose that $(\vec{a}, \vec{b})$ is a basis for $[\pi]$. Under what conditions is it the case that $(\vec{a}, \vec{b})$ and $(\vec{a}, \vec{a} + \vec{b}b)$ belong to the same sense class of $[\pi]$? Justify your answer.
3. Given $\triangle ABC$, with $AB = 10$, $BC = 17$, and $CA = 21$, assume that $\overleftrightarrow{ABC}$ is oriented so that $\angle BAC$ is positively sensed and that $\overline{BD}$ is the altitude from B, as shown in the picture at the right.



   - (a) Compute $\cos \angle BAC$ and $\cos \angle BCA$.
   - (b) What are $\sin^{\perp} \angle BAC$ and $\sin^{\perp} \angle BCA$?
   - (c) Compute BD and AD.
4. Given $\triangle PRQ$, with exterior angles, $\angle PRV$ and $\angle PQT$, as shown in the picture at the right, assume that $\angle PRQ$ is an angle of $\pi/3$ radians, that $\angle PQR$ is an angle of $\pi/6$ radians, and that $\overrightarrow{PS}$ is the angle bisector at P. Also orient $\overleftrightarrow{PQR}$ so that $\angle PRQ$ is positively sensed.



   - (a) Compute the following sensed measures:

$$m^{\perp}(\angle QPR),\ m^{\perp}(\angle QPS),\ m^{\perp}(\angle PSR),\ m^{\perp}(\angle PQT),\ m^{\perp}(\angle PRV)$$

   - (b) Evaluate each of the following:

$$\cos \angle PRS,\ \cos \angle PRV,\ \cos \angle RPS,\ \cos \angle PQT$$
$$\sin^{\perp} \angle PRS,\ \sin^{\perp} \angle PRV,\ \sin^{\perp} \angle SPQ,\ \sin^{\perp} \angle PQT$$

Answers for Chapter Test

1. (a) $\begin{vmatrix} \vec{u}\cdot(-\vec{u}2 + \vec{v}3) & \vec{u}\cdot(-\vec{u} + \vec{v}2) \\ \vec{v}\cdot(-\vec{u}2 + \vec{v}3) & \vec{v}\cdot(-\vec{u} + \vec{v}2) \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 3 & 2 \end{vmatrix} = -1$

   $\begin{vmatrix} \vec{u}\cdot(\vec{u}2 + \vec{v}3) & \vec{u}\cdot(\vec{u}2 - \vec{v}) \\ \vec{v}\cdot(\vec{u}2 + \vec{v}3) & \vec{v}\cdot(\vec{u}2 - \vec{v}) \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix} = -8$

   It follows by the determinant criterion [Theorem 18·4] that neither basis belongs to the same sense class as does $(\vec{u}, \vec{v})$.

   (b) By part (a) both bases are oppositely sensed to $(\vec{u}, \vec{v})$. Hence, they are similarly sensed.

2. They belong to the same sense class of $[\pi]$ if and only if $b > 0$. This can be shown by using the determinant criterion or by noting that $\vec{a}^{\perp}\cdot\vec{b}$ and $\vec{a}^{\perp}\cdot(\vec{a} + \vec{b}b)$ are both positive or both negative if and only if $b > 0$. $[\vec{a}^{\perp}\cdot(\vec{a} + \vec{b}b) = (\vec{a}^{\perp}\cdot\vec{b})b]$

3. (a) $\cos \angle BAC = \cos \angle A$ and $\cos \angle BCA = \cos \angle C$. By the cosine law, $\cos \angle A = 3/5$ and $\cos \angle C = 15/17$.

   (b) Since $\angle BAC$ is positively sensed and $\angle BCA$ is negatively sensed it follows that $\sin^{\perp} \angle BAC = \sin \angle BAC$ and $\sin^{\perp} \angle BCA = -\sin \angle BCA$. By part (a), $\sin \angle BAC = 4/5$ and $\sin \angle BCA = 8/17$.

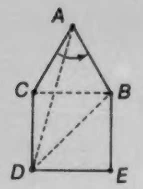
   (c) $BD = 10 \sin \angle BAC = 8$; $AD = 10 \cos \angle BAC = 6$

4. (a) $m^{\perp}(\angle QPR) = \pi/2$, $m^{\perp}(\angle QPS) = \pi/4$, $m^{\perp}(\angle PSR) = -5\pi/12$, $m^{\perp}(\angle PQT) = 5\pi/6$, $m^{\perp}(\angle PRV) = -2\pi/3$

   (b) $\cos \angle PRS = 1/2$, $\cos \angle PRV = -1/2$, $\cos \angle RPS = 1/\sqrt{2}$, $\cos \angle PQT = -\sqrt{3}/2$, $\sin^{\perp} \angle PRS = \sqrt{3}/2$, $\sin^{\perp} \angle PRV = -\sqrt{3}/2$, $\sin^{\perp} \angle SPQ = -1/\sqrt{2}$, $\sin^{\perp} \angle PQT = 1/2$

5. (a) $5\pi/6$ (b) $-7\pi/12$ (c) $\pi/12$ (d) $7\pi/12$

5. Consider the picture at the right, in which $\triangle ABC$ is equilateral and $BCDE$ is a coplanar square. Suppose that the plane of the figure is oriented so that $\angle CAB$ is positively sensed. Give the sensed measures of each of the following.



(a) $\angle DCA$ (b) $\angle ABD$ (c) $\angle CAD$ (d) $\angle ABD$

Background Topic

In Chapter 5 [pages 176–210 of Volume 1] you learned that the set $\mathcal{R} \times \mathcal{R}$ of all ordered pairs of real numbers could be made into a vector space by adopting the definitions:

(1) $(a, b) + (c, d) = (a + c, b + d)$,

(2) $\vec{0} = (0, 0)$ and $-(a, b) = (-a, -b)$, and:

(3) $(a, b)c = (ac, bc)$

With this structure [that is, with addition, '$\vec{0}$', oppositing, and multiplication by real numbers defined as above] we referred to the members of $\mathcal{R} \times \mathcal{R}$ as *measure vectors*. The resulting vector space is easily seen to be 2-dimensional. To do so, let $\vec{u} = (1, 0)$ and $\vec{v} = (0, 1)$. Then, clearly, for any real numbers a and b, $(a, b) = (a, 0) + (0, b) = \vec{u}a + \vec{v}b$. So, $(\vec{u}, \vec{v})$ spans the vector space $\mathcal{R} \times \mathcal{R}$ and, since $\vec{u}a + \vec{v}b = (a, b)$ and $(a, b) = \vec{0}$ if and only if $a = 0 = b$, $(\vec{u}, \vec{v})$ is linearly independent.

We can now go a step further and make $\mathcal{R} \times \mathcal{R}$ an inner product space by adopting:

(4) $(a, b) \cdot (c, d) = ac + bd$

This amounts to deciding that we shall take $(\vec{u}, \vec{v})$ as an orthonormal basis for $\mathcal{R} \times \mathcal{R}$ [Theorem 11–12]. In Chapter 5 you showed that if we adopt (1)–(3) then Postulates 4_1–4_8 are satisfied. It is now easy, if we adopt (4) as well, to show that Postulates 4_{11}–4_{14} are satisfied so that $\mathcal{R} \times \mathcal{R}$ is indeed an inner product space when we adopt (1)–(4). [What happens with respect to Postulates 4_9 and 4_{10}? With respect to Postulate 4_0?]

Part A

1. Show that, with definitions (1)–(4), Postulates 4_3, 4_8, and 4_{13} are satisfied.
2. Show that the measure vectors (a, b) and $(-b, a)$ are orthogonal and have the same norm.
3. Show that $(\vec{u}, \vec{v})$ is an orthonormal basis for $\mathcal{R} \times \mathcal{R}$.

The purpose of these exercises is to point out that $\mathcal{R} \times \mathcal{R}$ can be considered from two points of view — as a 2-dimensional inner product space, $\mathcal{T}_2$, and as a space of points, $\mathcal{E}_2$. When operations [and '$\vec{0}$'] are defined in $\mathcal{T}_2$ by (1) - (4), when addition of members of $\mathcal{T}_2$ to members of $\mathcal{E}_2$ is defined by (5), and when subtraction of members of $\mathcal{E}_2$ is defined by (6), all our postulates are satisfied, with '$\mathcal{T}_2$' for '$\mathcal{T}$' and '$\mathcal{E}_2$' for '$\mathcal{E}$' [except for minor changes in the dimension postulates 4_9 and 4_{10}]. As a consequence, all our theorems hold in this situation [with minor changes to take account of the fact that we are now operating in two dimensions rather than in three].

With these conventions [(1) - (6)] we shall call $\mathcal{E}_2$ [that is, $\mathcal{R} \times \mathcal{R}$] *the Euclidean number plane*. [For brevity we shall usually omit 'Euclidean' although, strictly speaking, this is required to alert the reader that we are taking into consideration the inner product defined by (5). Without this we would, properly, speak of *the affine number plane*.]

The ideas treated here will be used in Chapter 19. In fact they can be thought of as exploration exercises.

In this new situation Postulate 4_9 would be replaced by:

There are two linearly independent members of $\mathcal{R} \times \mathcal{R}$.

and Postulate 4_{10} would be replaced by:

There are not three linearly independent members of $\mathcal{R} \times \mathcal{R}$.

[That the first of these two postulates follows from (1) - (3) is shown in the text. ($\vec{u}$ and $\vec{v}$ are two linearly independent members of $\mathcal{R} \times \mathcal{R}$.) The second follows from the fact that a system of equations like:

$$\begin{cases} a_1a + b_1b + c_1c = 0 \\ a_2a + b_2b + c_2c = 0 \end{cases}$$

always has a solution (a, b, c) different from (0, 0, 0). For, if the determinant $a_1b_2 - a_2b_1 \neq 0$ then one can find a solution in which 'c' has any value one chooses — say, 1. Similar results hold if $b_1c_2 - b_2c_1 \neq 0$ and also if $c_1a_2 - c_2a_1 \neq 0$. And, if all three determinants are 0 then the left side of one equation is a multiple of the other (since (a_1, b_1, c_2) and (a_2, b_2, c_2) are linearly dependent by Theorem 10-14) so that any solution of the former is a solution of the system.] The only change required in the parts of Postulate 4_0 is the replacement of '$\mathcal{T}$' by '$\mathcal{R} \times \mathcal{R}$'. [When we introduce the notation '$\mathcal{E}_2$' and '$\mathcal{T}_2$', the '$\mathcal{R} \times \mathcal{R}$'s of this paragraph should be replaced by '$\mathcal{T}_2$'s.]

Answers for Part A

1. 4_3: $(a, b) + -(a, b) = (a, b) + (-a, -b) = (a + -a, b + -b) = (0, 0) = \vec{0}$

   4_8: $[(a, b)c]d = (ac, bc)d = (acd, bcd) = (a(cd), b(cd)) = (a, b)(cd)$

   4_{13}: $$\begin{aligned}[(a_1, a_2)c] \cdot (b_1, b_2) &= (a_1c, a_2c) \cdot (b_1, b_2) = a_1cb_1 + a_2cb_2 \\ &= a_1b_1c + a_2b_2c = (a_1b_1 + a_2b_2)c \\ &= [(a_1, a_2) \cdot (b_1, b_2)]c\end{aligned}$$

2. Since $(a, b) \cdot (-b, a) = a \cdot -b + ba = 0$, $(a, b) \perp (-b, a)$. Since $||(a, b)||^2 = a^2 + b^2$ and $||(-b, a)||^2 = (-b)^2 + a^2$, (a, b) and (−b, a) have the same norm.

3. $\vec{u} \cdot \vec{v} = (1, 0) \cdot (0, 1) = 1 \cdot 0 + 0 \cdot 1 = 0$; $\vec{u} \cdot \vec{u} = (1, 0) \cdot (1, 0) = 1$;
   $\vec{v} \cdot \vec{v} = (0, 1) \cdot (0, 1) = 1$

*

In the Background Topic at the end of Chapter 13 you learned that $\mathscr{R} \times \mathscr{R}$ can be given a different structure by using a different definition for multiplication. With this structure $\mathscr{R} \times \mathscr{R}$ is called *the complex number system.* We shall now see that $\mathscr{R} \times \mathscr{R}$ can be given still another structure—the structure of a Euclidean plane. With this structure $\mathscr{R} \times \mathscr{R}$ is called *the Euclidean number plane* and each ordered pair of real numbers is a *point* of this plane.

In introducing this new structure it will be convenient to adopt a new name, '$\mathscr{E}_2$', for $\mathscr{R} \times \mathscr{R}$. Our problem, then, is to define a set $\mathscr{T}_2$ in such a way that Postulates 1–4 are satisfied when '$\mathscr{E}$' is replaced by '$\mathscr{E}_2$' and '$\mathscr{T}$' by '$\mathscr{T}_2$'. [Of course, we shall make other changes in Postulates 4_9 and 4_{10}. Why?]

Corresponding to any measure vector $\vec{c}$ there is a mapping of $\mathscr{E}_2$ onto itself which maps any point $A \in \mathscr{E}_2$ onto the point $A + \vec{c}$, where, since points of $\mathscr{E}_2$ and measure vectors are ordered pairs, addition is defined by (1). In more detail, if $A = (a_1, a_2) \in \mathscr{E}_2$ and $\vec{c} = (c_1, c_2) \in \mathscr{T}_2$, then

$$A + \vec{c} = (a_1 + c_1, a_2 + c_2) \in \mathscr{E}_2. \tag{5}$$

Intuitively, the operation of adding a given measure vector to each point of $\mathscr{E}_2$ looks like a translation. [Draw a picture of $\mathscr{E}_2$ and show the effect of adding the measure vector (2, −3) to various points of $\mathscr{E}_2$.] Apparently, we might take for $\mathscr{T}_2$ the set of those "adding operations", one for each measure vector. It is simpler, however, to take for $\mathscr{T}_2$ just the set $\mathscr{R} \times \mathscr{R}$ itself with the inner product space structure given by (1)–(4). Note that nothing in our postulates says that the members of $\mathscr{T}$ must be translations. All that we require is that our postulates be satisfied. If they are, then any member $\vec{c}$ of $\mathscr{T}$ *determines* a translation—the translation which maps any point A on the point $A + \vec{c}$. If the members of $\mathscr{T}$ *are* translations, and '+' refers to the application of a function to its argument, then the translation associated with $\vec{c}$ is $\vec{c}$, itself. We may, however, take other objects as members of $\mathscr{T}$, and define '+' in a different way, as long as our postulates are satisfied.

We shall, then, take $\mathscr{T}_2$ to be the 2-dimensional inner product space of measure vectors—that is, $\mathscr{T}_2$ is $\mathscr{R} \times \mathscr{R}$ with the structure determined by (1)–(4). $\mathscr{E}_2$ is also $\mathscr{R} \times \mathscr{R}$, but with the structure determined by (5) and by a suitable definition of subtraction of points. This definition is easy to guess. If $A = (a_1, a_2) \in \mathscr{E}_2$ and $B = (b_1, b_2) \in \mathscr{E}_2$ then

$$B - A = (b_1 - a_1, b_2 - a_2) \in \mathscr{T}_2. \tag{6}$$

As we have seen in (5) and (6) our definitions of adding vectors to points and subtracting points from points satisfy Postulate 1 [with '$\mathscr{E}_2$' for '$\mathscr{E}$' and '$\mathscr{T}_2$' for '$\mathscr{T}$'.] In the next set of exercises you will show that they also satisfy Postulates 2 and 3. We already know that $\mathscr{T}_2$ satisfies Postulate 4 [with minor modifications]. And, of course, in any case, the real numbers satisfy Postulate 5. So, $\mathscr{E}_2$ as operated on by the members of $\mathscr{T}_2$ is a 2-dimensional Euclidean space—for short, a plane. We call it *the [Euclidean] number plane.* Since our postulates [except for 4_9 and 4_{10}] are satisfied, all our theorems hold for the number plane [with the obvious modifications required by the fact that $\mathscr{E}_2$ is 2-dimensional while our space $\mathscr{E}$ is 3-dimensional].

Part B

1. Show that, for $\mathscr{E}_2$ and $\mathscr{T}_2$ as defined above,
   (a) $(A + \vec{c}) - A = \vec{c}$, and
   (b) $A + (B - A) = B$.
   [*Hint:* Take $A = (a_1, a_2) \in \mathscr{E}_2$, $B = (b_1, b_2) \in \mathscr{E}_2$, and $\vec{c} = (c_1, c_2) \in \mathscr{T}_2$.]
2. Show that $(B - A) + (C - B) = C - A$.

*

In the next chapter we shall make use of some of the geometry of the Euclidean number plane. We shall introduce names for some of the points of $\mathscr{E}_2$:

$$O = (0, 0),\ U = (1, 0),\ V = (0, 1),\ U' = (-1, 0),\ V' = (0, -1)$$

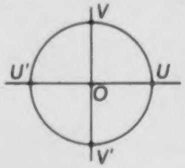


Fig. 18–13

Notice that, when $\vec{u}$ and $\vec{v}$ are the vectors (1, 0) and (0, 1) of $\mathscr{T}_2$,

$$U = O + \vec{u},\ V = O + \vec{v},\ U' = O - \vec{u}, \text{ and } V' = O - \vec{v}.$$

Since $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors each of the four points U, V, U', and V' is at unit distance from O and $\angle UOV$ is a right angle. [Name three other right angles shown in Fig. 18–13.]

According to our definition of dot multiplication in $\mathscr{T}_2$ and the related definitions of norm [for members of $\mathscr{T}_2$] and distance [for pairs of members of $\mathscr{E}_2$] it follows that, for $A = (a_1, a_2)$ and $B = (b_1, b_2)$,

$$\begin{aligned} d(A,B) &= \|B - A\| \\ &= \sqrt{(B - A) \cdot (B - A)} \\ &= \sqrt{(b_1 - a_1, b_2 - a_2) \cdot (b_1 - a_1, b_2 - a_2)} \\ &= \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}. \end{aligned}$$

Students may have difficulty at two points in the discussion of the Euclidean number plane. The first difficulty may be summarized in the question 'How do you know whether an ordered pair of real numbers is a vector or a point?'. The second difficulty may be put as 'But the members of $\mathcal{T}$ were translations. How come we can take measure vectors as the members of $\mathcal{T}_2$?'.

One answer to the first question is that whether a member of $\mathcal{R} \times \mathcal{R}$ is a measure vector or a point [or, for that matter, a complex number] depends on what structure you are assuming $\mathcal{R} \times \mathcal{R}$ to have. In the case of $\mathcal{T}_2$ and $\mathcal{E}_2$ we are considering, at the same time, $\mathcal{R} \times \mathcal{R}$ with two different structures, and an ordered pair is a measure vector or a point according to which you wish it to be. Another answer is that [assuming you are thinking and are not just writing or talking formally] you can always tell from context whether a given ordered pair referred to in that context is a measure vector or a point. One way to make quite sure is to say, as we did, that $A = (a_1, a_2) \in \mathcal{E}_2$ and $\vec{c} = (c_1, c_2) \in \mathcal{T}_2$. A final answer is that thinking of the same thing in two ways is something that you will become accustomed to. Just don't fight it!

The second question can, first, be answered by pointing out that all we have done throughout the course has been based entirely on our postulates and definitions. None of these imply that members of $\mathcal{T}$ are translations. Thinking of the members of $\mathcal{T}$ as translations has merely been an aid to our intuition. Second, given any sets $\mathcal{E}$ and $\mathcal{T}$ [and operations] which satisfy our postulates, each member of $\mathcal{T}$ does determine a mapping of $\mathcal{E}$ onto itself — the ''adding the given member of $\mathcal{T}$'' mapping. Only if the members of $\mathcal{T}$ are themselves mappings of $\mathcal{E}$ and '+' in '$A + \vec{a}$' refers to function application are we entitled to call the members of $\mathcal{T}$ themselves 'translations'. [Of course, these have been our intuitive understandings throughout the course, and motivated our choice of postulates and definitions. But, formally, they are irrelevant to the geometry we have developed.]

Answers for Part B

1. (a) Let $A = (a_1, a_2) \in \mathcal{E}_2$ and $\vec{c} = (c_1, c_2) \in \mathcal{T}_2$. Then, by (5) and (6),

$$\begin{aligned}(A + \vec{c}) - A &= [(a_1, a_2) + (c_1, c_2)] - (a_1, a_2)\\ &= (a_1 + c_1,\ a_2 + c_2) - (a_1, a_2)\\ &= (a_1 + c_1 - a_1,\ a_2 + c_2 - a_2)\\ &= (c_1, c_2) = \vec{c}.\end{aligned}$$

(b) Let $A = (a_1, a_2) \in \mathcal{E}_2$ and $B = (b_1, b_2) \in \mathcal{E}_2$. Then, by (5) and (6),

$$\begin{aligned}A + (B - A) &= (a_1, a_2) + [(b_1, b_2) - (a_1, a_2)]\\ &= (a_1, a_2) + (b_1 - a_1,\ b_2 - a_2)\\ &= (a_1 + (b_1 - a_1),\ a_2 + (b_2 - a_2))\\ &= (b_1, b_2) = B.\end{aligned}$$

2. Let $A = (a_1, a_2) \in \mathcal{E}_2$, $B = (b_1, b_2) \in \mathcal{E}_2$, and $C = (c_1, c_2) \in \mathcal{E}_2$. Then, by (5) and (6),

$$\begin{aligned}(B - A) + (C - B) &= [(b_1, b_2) - (a_1, a_2)] + [(c_1, c_2) - (b_1, b_2)]\\ &= (b_1 - a_1,\ b_2 - a_2) + (c_1 - b_1,\ c_2 - b_2)\\ &= (b_1 - a_1 + (c_1 - b_1),\ b_2 - a_2 + (c_2 - b_2))\\ &= (c_1 - a_1,\ c_2 - a_2) = (c_1, c_2) - (a_1, a_2)\\ &= C - A.\end{aligned}$$

In particular,

$$\{(x, y): x^2 + y^2 = 1\}$$

is the circle of the number plane with center O and radius 1. We shall call it *the unit circle* [of the Euclidean number plane $\mathscr{E}_2$].

Part C

1. Suppose that $P_1 = (\frac{1}{2}, \sqrt{3}/2) \in \mathscr{E}_2$. Show that P_1 belongs to the unit circle and give $\cos \angle UOP_1$ and $\sin \angle UOP_1$.
2. Repeat Exercise 1 for points P_2, P_3, and P_4 where $P_2 = (-\sqrt{3}/2, \frac{1}{2})$, $P_3 = (-\frac{1}{2}, -\sqrt{3}/2)$, and $P_4 = (\sqrt{3}/2, -\frac{1}{2})$.
3. Make a table with one line for each of the four angles in Exercises 1 and 2. In successive columns list the angle, its radian-measure, its cosine, and its sine.
4. Find $\cos \angle UOP$ and $\sin \angle UOP$ when $P = (a, b)$, where $a^2 + b^2 = 1$ and $b \neq 0$.

*5. Repeat Exercise 4 assuming that all you know about a and b is that $b \neq 0$.

*

Like any plane, $\mathscr{E}_2$ has two possible orientations each of which corresponds to one of two perping operations on $\mathscr{T}_2$. For one of these perping operations, $\vec{u}^{\perp} = \vec{v}$. What is the perp of $\vec{u}$ with respect to the other perping operation? In these exercises, and in the next chapter, we shall make use of the orientation of $\mathscr{E}_2$ corresponding to the perping operation for which $\vec{u}^{\perp} = \vec{v}$. [Intuitively, this is the counterclockwise orientation of $\mathscr{E}_2$.]

Part D

1. Repeat Exercises 1 and 2 of Part C for sensed angles—that is, replace '$\angle$' by '$\measuredangle$', and replace 'sin' by '$\sin^{\perp}$'.
2. Tabulate information, as in Exercise 3, concerning the four sensed angles of Exercise 1 and $\measuredangle UOU$, $\measuredangle UOV$, $\measuredangle UOU'$, and $\measuredangle UOV'$. For each of the eight sensed angles list its sensed radian-measure, and arrange the sensed angles in your table in increasing order of these measures. [That is, first $\measuredangle UOP_3$, next $\measuredangle UOV'$, etc.]
3. Compare comparable results in your two tables. [Pay special attention to the measures of the angles and the corresponding sensed angles and compare the values of sin for the former with the values of $\sin^{\perp}$ for the latter.]
4. Find $\cos \measuredangle UOP$ and $\sin^{\perp} \measuredangle UOP$ when $P = (a, b)$,
   (a) where $a^2 + b^2 = 1$, and
   (b) without any restriction other than that $P \neq O$.
5. Show that for any point $P \in \mathscr{E}_2$ other than O,

$$P = (r \cos \measuredangle UOP, r \sin^{\perp} \measuredangle UOP),$$

where $r = d(O, P)$.

Answers for Part C

1-3.

$\angle$	m	cos	sin
$\angle UOP_1$	$\pi/3$	$1/2$	$\sqrt{3}/2$
$\angle UOP_2$	$5\pi/6$	$-\sqrt{3}/2$	$1/2$
$\angle UOP_3$	$2\pi/3$	$-1/2$	$\sqrt{3}/2$
$\angle UOP_4$	$\pi/6$	$\sqrt{3}/2$	$1/2$

4. $\cos \angle UOP = a$, $\sin \angle UOP = |b|$ [$\sin \angle UOP = \sqrt{1 - (\cos \angle UOP)^2} = \sqrt{b^2} = |b|$]

☆5. $\cos \angle UOP = a/\sqrt{a^2 + b^2}$, $\sin \angle UOP = |b|/\sqrt{a^2 + b^2}$

Answers for Part D

1, 2.

$\measuredangle$	$m^{\perp}$	cos	$\sin^{\perp}$
$\measuredangle UOP_3$	$-2\pi/3$	$-1/2$	$-\sqrt{3}/2$
$\measuredangle UOV'$	$-\pi/2$	0	-1
$\measuredangle UOP_4$	$-\pi/6$	$\sqrt{3}/2$	$-1/2$
$\measuredangle UOU$	0	1	0
$\measuredangle UOP_1$	$\pi/3$	$1/2$	$\sqrt{3}/2$
$\measuredangle UOV$	$\pi/2$	0	1
$\measuredangle UOP_2$	$5\pi/6$	$-\sqrt{3}/2$	$1/2$
$\measuredangle UOU'$	π	-1	0

3. The sum of the measures of $\angle UOP_3$ and $\measuredangle UOP_3$ is 0, and the same holds for $\angle UOP_4$ and $\measuredangle UOP_4$. For each sensed angle the absolute value of $\sin^{\perp}$ is the value of sin for the corresponding insensed angle.
4. (a) $\cos \measuredangle UOP = a$, $\sin \measuredangle UOP = b$ [If $\vec{p} = P - O$ then $\vec{p} = (a, b)$ and is a unit vector. So, $\cos \measuredangle UOP = \vec{p} \cdot \vec{u} = a$ and $\sin^{\perp} \measuredangle UOP = \vec{p} \cdot \vec{u}^{\perp} = \vec{p} \cdot \vec{v} = b$.]
   (b) $\cos \measuredangle UOP = a/\sqrt{a^2 + b^2}$, $\sin^{\perp} \measuredangle UOP = b/\sqrt{a^2 + b^2}$ [$(a/\sqrt{a^2 + b^2}, b/\sqrt{a^2 + b^2})$ is the unit vector in the sense of $P - O$.]
5. Suppose that $P = (a, b)$, where $(a, b) \neq O$. Then $d(O, P) = \sqrt{a^2 + b^2} \neq 0$, and the desired result follows directly from Exercise 4(b).

[These last exercises, particularly Exercise 4(a), are of considerable importance since they form the basis for correlating the cosine and sine of a sensed angle with the cosine and sine of its measure as these are introduced in Chapter 19.]

*

Consider the square $UVU'V'$ of the number plane $\mathscr{E}_2$. What is its

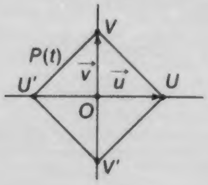


Fig. 18–14

side-measure? Imagine a particle which moves around this square, in the counterclockwise sense, at the speed of one unit of distance per second. In moving around the square the particle will pass through the point U many times. Let's choose one instant when the particle is at U and call this *the initial instant.* For $t \geq 0$, let $P(t)$ be the point where the particle is t seconds after the initial instant; for $t < 0$ let $P(t)$ be the point where the particle is $-t$ seconds before the initial instant. Our problem is to find out what we can about the function P whose domain is $\mathscr{R}$ and whose range is the square $UVU'V'$. Since, for each t, $P(t) \in \mathscr{E}_2$, there are functions c and s such that, for any t,

$$(7) \qquad P(t) = (c(t), s(t)).$$

What is the domain of the function c? What is the range of c? What are the domain and range of s?

One of the most obvious properties of the function P arises from the fact that, wherever the particle may be at a given instant, it was at the same position $4\sqrt{2}$ seconds before this instant and will be at the same position $4\sqrt{2}$ seconds after this instant. [Explain.] More generally, for any t and any integer k,

$$(8) \qquad P(t + 4\sqrt{2}k) = P(t).$$

We shall refer to this property of the function P by saying that P is *periodic* with *period* $4\sqrt{2}$.

In view of the periodicity of P it is intuitively clear that we can calculate the value of P for any value of 't' if we can calculate the value of P for all values of 't' between 0 and $4\sqrt{2}$. For these latter values, it is fairly evident that we shall need different formulas for $0 \leq t < \sqrt{2}$, for $\sqrt{2} \leq t < 2\sqrt{2}$, for $2\sqrt{2} \leq t < 3\sqrt{2}$, and for $3\sqrt{2} \leq t < 4\sqrt{2}$. For example, if $\sqrt{2} \leq t \leq 2\sqrt{2}$ then, as in Fig. 18–14, $P(t)$

In the remainder of these exercises we parody the method used in Chapter 19 to define the functions cosine and sine with numerical arguments. In Chapter 19 the square UVU′V′ is replaced by the unit circle in $\mathscr{E}_2$; the function P is replaced by the winding function W which maps $\mathscr{R}$ in a simple way onto the unit circle; the functions c and s are replaced by cos and sin; and the role of $\sqrt{2}$ is taken by $\pi/2$.

The side measure of UVU′V′ is, of course, $\sqrt{2}$.

The domain of the function c is $\mathscr{R}$ and the range of c is $\{x: |x| \leq 1\}$. The function s has the same domain and range as does c.

Formula (8) is intuitively obvious when one realizes that the perimeter of UVU′V′ is $4\sqrt{2}$.

$\in \overleftrightarrow{VU'}$. In this case, if $\vec{w}$ is the unit vector in $[U' - V]^+$ then $P(t) - V = \vec{w}(t - \sqrt{2})$. Since $\vec{w} = (-\vec{u} - \vec{v})/\sqrt{2}$ it follows that

$$P(t) - O = (V - O) + (P(t) - V) = \vec{v} + \frac{-\vec{u} - \vec{v}}{\sqrt{2}}(t - \sqrt{2})$$

$$= \vec{u}\left(1 - \frac{t}{\sqrt{2}}\right) + \vec{v}\left(2 - \frac{t}{\sqrt{2}}\right)$$

and, so, that

$$P(t) = \left(1 - \frac{t}{\sqrt{2}}, 2 - \frac{t}{\sqrt{2}}\right), \text{ for } \sqrt{2} \le t \le 2\sqrt{2}.$$

In other words, for $\sqrt{2} \le t \le 2\sqrt{2}$,

$$(*) \qquad c(t) = 1 - \frac{t}{\sqrt{2}} \text{ and } s(t) = 2 - \frac{t}{\sqrt{2}}.$$

Part E

1. From Figure 18–14, what is $P(\sqrt{2})$? $P(-\sqrt{2})$? $P(6\sqrt{2})$? $P(0)$? $P(\sqrt{2}/2)$?
2. Use (7) and (8) to show that each of the functions *c* and *s* is periodic with period $4\sqrt{2}$. [*Hint*: With regard to *c*, what you must show is that, for any *t* and any integer *k*, $c(t + 4\sqrt{2}\,k) = c(t)$.]
3. It should be intuitively obvious that, for any *t*, the points $P(t)$ and $P(-t)$ are symmetric to one another with respect to the line $\overleftrightarrow{O[\vec{u}]}$. What does this tell you about the function *c*? About the function *s*?]
4. Find formulas like (*) for computing values of *c* and *s*
   (a) in case $0 \le t \le \sqrt{2}$,
   (b) in case $2\sqrt{2} \le t \le 3\sqrt{2}$, and
   (c) in case $3\sqrt{2} \le t \le 4\sqrt{2}$.
5. Use (*) and the results of Exercise 4 to draw graphs [on the same axes] of both *c* and *s*, for $0 \le t \le 4\sqrt{2}$. [*Hint*: You should see, from (*) and the formulas from Exercise 4, that the graphs of *c* and *s* are made up of parts of straight lines and that, to draw these segments, it is enough to plot the values of these functions for the arguments 0, $\sqrt{2}$, $2\sqrt{2}$, $3\sqrt{2}$, and $4\sqrt{2}$. Use 1.4 as an approximation to $\sqrt{2}$.]
6. Use the result in Exercise 2 to extend your graphs of *c* and *s* to include arguments between $-2\sqrt{2}$ and $6\sqrt{2}$.
7. Suppose you shifted your graph of *c* a distance $\sqrt{2}$ to the right. How would the result compare with your graph of *s*? Try to state what you notice as an equation involving '*c*' and '*s*'.

*

Answers for Part E

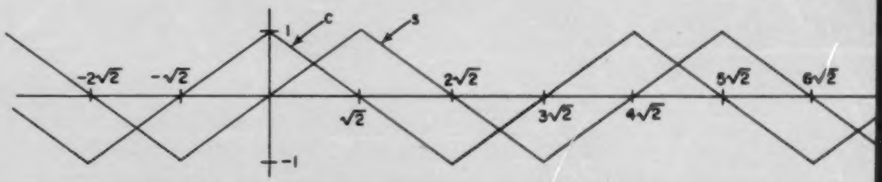
1. $P(\sqrt{2}) = (0, 1)$; $P(-\sqrt{2}) = (0, -1)$; $P(6\sqrt{2}) = (-1, 0)$; $P(0) = (1, 0)$; $P(\sqrt{2}/2) = (1/\sqrt{2}, 1/\sqrt{2})$
2. By (7), $(c(t + 4\sqrt{2}k), s(t + 4\sqrt{2}k)) = P(t + 4\sqrt{2}k)$. By (8) $P(t + 4\sqrt{2}k) = P(t)$. By (7) $P(t) = (c(t), s(t))$. So, $(c(t + 4\sqrt{2}k), s(t + 4\sqrt{2}k)) = (c(t), s(t))$ and it follows that, for any t, $c(t + 4\sqrt{2}k) = c(t)$ and $s(t + 4\sqrt{2}k) = s(t)$. Hence, c and s are periodic with period $4\sqrt{2}$.
3. Since $P(t) = (c(t), s(t))$, $P(-t) = (c(-t), s(-t))$ and $P(t)$ and $P(-t)$ are symmetric with respect to $\overleftrightarrow{O[\vec{u}]}$ it follows that $c(t) = c(-t)$ and $s(t) = -s(-t)$. In other words, c is an even function and s is an odd function.
4. (a) In case $0 \le t \le \sqrt{2}$, $P(t) \in \overleftrightarrow{UV}$. Since the unit vector in $[V - U]^+$ is $(\vec{v} - \vec{u})/\sqrt{2}$ it follows that

$$P(t) - O = (U - O) + (P(t) - U)$$
$$= \vec{u} + \frac{\vec{v} - \vec{u}}{\sqrt{2}}t = \vec{u}\left(1 - \frac{t}{\sqrt{2}}\right) + \vec{v}\,\frac{t}{\sqrt{2}}.$$

So, $P(t) = (1 - t/\sqrt{2}, t/\sqrt{2})$ and, hence, $c(t) = 1 - t/\sqrt{2}$ and $s(t) = t/\sqrt{2}$ for $0 \le t \le \sqrt{2}$.

[Parts (b) and (c) are similar. For answers, see (9) on page 408.]

5, 6.



7. If the graph of c were displaced $\sqrt{2}$ units to the right it would coincide with the graph of s. This means that, for all t, $c(t - \sqrt{2}) = s(t)$.

We can summarize the results of Exercise 4 and (*) in:

$$(9)\quad \begin{cases} P(t) = (1 - t/\sqrt{2},\ t/\sqrt{2}), \text{ for } 0 \le t \le \sqrt{2},\\ \quad\;\; = (1 - t/\sqrt{2},\ 2 - t/\sqrt{2}), \text{ for } \sqrt{2} \le t \le 2\sqrt{2},\\ \quad\;\; = (t/\sqrt{2} - 3,\ 2 - t/\sqrt{2}), \text{ for } 2\sqrt{2} \le t \le 3\sqrt{2}, \text{ and}\\ \quad\;\; = (t/\sqrt{2} - 3,\ t/\sqrt{2} - 4), \text{ for } 3\sqrt{2} \le t \le 4\sqrt{2}. \end{cases}$$

Using (8) and (9) we can obtain the result you noticed in Exercise 7. To do so, consider the case in which $0 \le t \le \sqrt{2}$. By (8), $c(t - \sqrt{2}) = c(t + 3\sqrt{2})$, where $3\sqrt{2} \le t + 3\sqrt{2} \le 4\sqrt{2}$. It follows by (9) that

$$c(t - \sqrt{2}) = \frac{t + 3\sqrt{2}}{\sqrt{2}} - 3 = \frac{t}{\sqrt{2}} = s(t).$$

In case $\sqrt{2} \le t \le 2\sqrt{2}$, $0 \le t - \sqrt{2} \le \sqrt{2}$ and, so,

$$c(t - \sqrt{2}) = 1 - \frac{t - \sqrt{2}}{\sqrt{2}} = 2 - \frac{t}{\sqrt{2}} = s(t).$$

Treating two more cases in the same way, we see that, for $0 \le t \le 4\sqrt{2}$, $c(t - \sqrt{2}) = s(t)$. To show that this holds for all t let k, for any t, be the integer such that $0 \le t - 4\sqrt{2}k < 4\sqrt{2}$. [You can compute the value of 'k' from that of 't' by using the integral part function. Try to do this.] It follows that

$$c(t - \sqrt{2}) = c((t - 4\sqrt{2}k) - \sqrt{2}) = s(t - 4\sqrt{2}k) = s(t). \quad \text{[Explain.]}$$

Part F

1. We have seen that, for any t,

$$(**)\qquad c(t - \sqrt{2}) = s(t).$$

Show that
(a) $c(t + \sqrt{2}) = -s(t)$, and
(b) $s(t + \sqrt{2}) = c(t)$.
[*Hint*: For (a) substitute '$-t$' for 't' in (**) and use what you have learned about c and s; for (b), make a different substitution in (**).]

2. Show that, for any t,

$$c(t + 2\sqrt{2}) = -c(t) \text{ and } s(t + 2\sqrt{2}) = -s(t).$$

Here are treatments of the two cases needed to complete the proof that $c(t - \sqrt{2}) = s(t)$:

In case $2\sqrt{2} \le t \le 3\sqrt{2}$, it follows that $\sqrt{2} \le t - \sqrt{2} \le 2\sqrt{2}$ and so, by the second formula in (9)

$$c(t - \sqrt{2}) = 1 - \frac{t - \sqrt{2}}{\sqrt{2}} = 2 - \frac{t}{\sqrt{2}}.$$

By the third formula in (9), $2 - t/\sqrt{2} = s(t)$. So, for $2\sqrt{2} \le t \le 3\sqrt{2}$, $c(t - \sqrt{2}) = s(t)$.

In case $3\sqrt{2} \le t \le 4\sqrt{2}$,

$$c(t - \sqrt{2}) = \frac{t - \sqrt{2}}{\sqrt{2}} - 3 = \frac{t}{\sqrt{2}} - 4 = s(t).$$

The integer k such that $0 \le t - 4\sqrt{2}k \le 4\sqrt{2}$ is $[\![t/(4\sqrt{2})]\!]$. For, $[\![t/(4\sqrt{2})]\!]$ is an integer and $[\![t/(4\sqrt{2})]\!] \le t/(4\sqrt{2}) \le [\![t/(4\sqrt{2})]\!] + 1$, whence the desired property of k follows by multiplying by $4\sqrt{2}$ and subtracting $[\![t/(4\sqrt{2})]\!]$.

The required explanation is that, since $(t - 4\sqrt{2}k) - \sqrt{2} = (t - \sqrt{2}) - 4\sqrt{2}k$ it follows by Exercise 2 of Part E that $c(t - \sqrt{2}) = c((t - 4\sqrt{2}k) - \sqrt{2})$. Now, since $0 \le t - 4\sqrt{2}k \le 4\sqrt{2}$ it follows by the earlier argument that $c((t - 4\sqrt{2}k) - \sqrt{2}) = s(t - 4\sqrt{2}k)$. Finally, by Exercise 2 of Part E, $s(t - 4\sqrt{2}k) = s(t)$.

Answers for Part F

1. (a) By (**), $c(-t - \sqrt{2}) = s(-t)$. So, since c is even and s is odd, it follows that $c(t + \sqrt{2}) = -s(t)$.
   (b) By (**), $c(t + \sqrt{2} - \sqrt{2}) = s(t + \sqrt{2})$. So, $s(t + \sqrt{2}) = c(t)$.
2. $c(t + 2\sqrt{2}) = c(t + \sqrt{2} + \sqrt{2}) = -s(t + \sqrt{2}) = -c(t)$;
   $s(t + 2\sqrt{2}) = s(t + \sqrt{2} + \sqrt{2}) = c(t + \sqrt{2}) = -s(t)$

Chapter Nineteen
The Circular Functions

19.01 The Winding Function *W*

In the preceding background topic you studied a function *P* which can be thought of as "winding" the real number line $\mathscr{R}$ around the square *UVU'V'* of the number plane $\mathscr{E}_2$. From this function *P* we obtained two functions, *c* and *s*, with domain $\mathscr{R}$ and range $\{x: |x| \leq 1\}$. In the present section we shall define a function *W* which "winds" $\mathscr{R}$ around the unit circle in $\mathscr{E}_2$. [The figure pictures both $\mathscr{E}_2$ and $\mathscr{R}$ and attempts to show part of $\mathscr{R}$ "wound around" the unit circle.] From this function *W* we shall obtain two functions whose domain is $\mathscr{R}$ but which are somewhat analogous to the cosine and sine functions for sensed angles. In fact, the cosine of a sensed angle will be the value of the first of these functions for the argument which is the measure of the sensed angle.

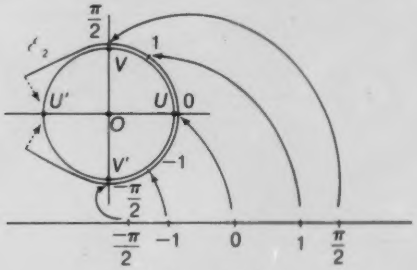


Fig. 19–1

We wish to define the mapping *W* of $\mathscr{R}$ onto the unit circle in such a way that if *a* and *b* are any two numbers such that $0 < b - a < 2\pi$ then the image of the interval $\{x: a < x < b\}$ of $\mathscr{R}$ is an arc whose measure is $b - a$—that is, is the same as the measure of the interval. We shall actually define *W* in such a way that it is intuitively obvious that

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Make sure that students clearly understand that the points of $\mathscr{E}_2$ are ordered pairs of real numbers and, so, that W is a function whose arguments are real numbers and whose values are ordered pairs of real numbers. The function P of the preceding Background Topic should have prepared them to consider functions of this kind.

The importance of W depends largely on Definition 19-2:

$$(\cos(a),\ \sin(a)) = W(a)$$

Using this definition and properties of W it is easy to arrive at the basic properties of the functions cos and sin.

Students may guess at a relation between the present cos and sin, whose arguments are real numbers and the functions cos and sin$^{\perp}$ of the preceding chapter, whose arguments are sensed angles. This relation is stated explicitly in Theorem 19-4.

The functions cos and sin introduced in Definition 19-2 are the functions you may be acquainted with by some such description as 'the cosine and sine for radian-measures of angles'. The description is justified in so far as in geometric applications of these functions their arguments are frequently numbers which have been obtained as radian-measures of angles. The description is, however, misleading in that cos and sin have many, more important applications in which their arguments are obtained in other ways — for example, as measures of duration from some initial instant.

We shall mention later functions which we shall call the degree-cosine function and the degree-sine function [°cos, °sin]. These also have numerical arguments and are useful principally in geometric applications where their arguments are obtained as the degree-measures of angles. They are related to our present cos and sin by:

$$°\cos(a) = \cos(\pi a/180),\quad °\sin(a) = \sin(\pi a/180)$$

For example, °cos(30) [or, as we shall write later, 'cos 30°'] is cos(π/6).

it has this desirable property. *Proving* that it does, belongs in a more advanced course.

We begin by noting that it is intuitively evident on the basis of our work in Chapter 17 that, for each real number a such that $0 < a < 2\pi$ there is just one arc with endpoint U, which has measure a, and either is contained in $\widehat{UV}$ [if $0 < a \leq \pi/2$] or contains $\widehat{UV}$ [if $\pi/2 \leq a < 2\pi$]. This arc we call *the counterclockwise arc from U* whose measure is a. For $0 < a < b < 2\pi$, it is also intuitively evident that of the two arcs with endpoint U and measure a, only the counterclockwise arc from U with measure a is a subset of the counterclockwise arc from U with measure b. And furthermore, the endpoint other than U of the counterclockwise arc with measure a belongs to the counterclockwise arc with measure b. So, by an earlier theorem, if $\widehat{UP}$ and $\widehat{UQ}$ are the counterclockwise arcs from U with measures a and b [$a < b$] then

$$\widehat{UQ} = \widehat{UP} \cup \{P\} \cup \widehat{PQ}, \tag{1}$$

where $\widehat{PQ}$ has no point in common with $\widehat{UP}$. [Of course, contrary to our usual convention, $\widehat{UP}$. $\widehat{UQ}$, and $\widehat{PQ}$ are not restricted to be minor arcs.]

We are now ready to define the winding function W and to establish its basic properties. As in the case of the function P of the background exercises, we shall define W first for real numbers between 0 and 2π. [Why 2π?] Then we shall define W for other arguments by requiring that it be periodic of period 2π.

Definition 19-1

(a) $W(0) = U$;

(b) for $0 < a < 2\pi$, $W(a)$ is the point X such that the measure of the counterclockwise arc $\widehat{UX}$ is a;

(c) for $a \notin \{x: 0 \leq x < 2\pi\}$, $W(a) = W(a - 2\pi\llbracket a/(2\pi)\rrbracket)$.

To show that Definition 19-1 does define W for all real number arguments it remains to be shown that $0 \leq a - 2\pi\llbracket a/(2\pi)\rrbracket < 2\pi$. This follows at once from the fact that

$$\llbracket a/(2\pi)\rrbracket \leq \frac{a}{2\pi} < \llbracket a/(2\pi)\rrbracket + 1. \qquad \text{[Explain.]}$$

[Note that the equation in part (c) of the definition is satisfied even if $a \in \{x: 0 \leq x < 2\pi\}$. Explain.]

We begin by defining W for real numbers between 0 and 2π [0 included] because the circumference of the unit circle is 2π.

By the definition of the integral part function in Part B of the Background Topic, page 313, it follows that

$$\llbracket a/(2\pi)\rrbracket \leq \frac{a}{2\pi} < \llbracket a/2\pi\rrbracket + 1.$$

So, since $2\pi > 0$,

$$2\pi\llbracket a/(2\pi)\rrbracket \leq a < 2\pi\llbracket a/2\pi\rrbracket + 2\pi$$

and, so,

$$0 \leq a - 2\pi\llbracket a/2\pi\rrbracket < 2\pi.$$

The equation in part (c) of Definition 19-1 is satisfied if $a \in \{x: 0 \leq x < 2\pi\}$ because, if $0 \leq a < 2\pi$ then $\llbracket a/2\pi\rrbracket = 0$ and the equation in question reduces to '$W(a) = W(a)$'.

It follows from the immediately preceding argument and Definition 19-1(c) that,

$$\text{for any } a,\ W(a) = W(a - 2\pi\llbracket a/2\pi\rrbracket).$$

[See Lemma 1, following.]

Note that, by definition and our knowledge that $m(\overset{\frown}{UV}) = \pi/2$, $m(\overset{\frown}{UVU'}) = \pi$, and $m(\overset{\frown}{UVV'}) = 3\pi/2$, it follows that

$$W(0) = (1, 0),\ W(\pi/2) = (0, 1),\quad W(\pi) = (-1, 0), \text{ and } W(3\pi/2) = (0, -1). \tag{2}$$

What is $W(-\pi/2)$? $W(-\pi)$? $W(-3\pi/2)$?

We have already noted the following:

Lemma 1 For any a, $W(a) = W(a - 2\pi[\![a/(2\pi)]\!])$.

Using this and a theorem we have previously proved concerning the integral part function it is easy to prove:

Theorem 19-1 W is periodic with period 2π.

For our main results concerning W we need:

Lemma 2 For $0 \leq a < b < 2\pi$, one of the arcs with endpoints $W(a)$ and $W(b)$ has the measure $b - a$.

This follows from (1) and a theorem on measures of arcs. By (1) and Definition 19-1, it follows that, for $0 < a < b < 2\pi$, the counterclockwise arc from U with measure b is $\overset{\frown}{UW(a)W(b)}$ and is the union

$$\overset{\frown}{UW(a)} \cup \{W(a)\} \cup \overset{\frown}{W(a)W(b)},$$

where $\overset{\frown}{UW(a)}$ is the counterclockwise arc from U with measure a, and $\overset{\frown}{W(a)W(b)}$ is one of the arcs with endpoints $W(a)$ and $W(b)$ and has no point in common with $\overset{\frown}{UW(a)}$. It follows by Theorem 17-16 that

$$m(\overset{\frown}{UW(a)W(b)}) = m(\overset{\frown}{UW(a)}) + m(\overset{\frown}{W(a)W(b)})$$

and, so, that

$$b = a + m(\overset{\frown}{W(a)W(b)}).$$

Thus, $m(\overset{\frown}{W(a)W(b)}) = b - a$. In case $a = 0$, which has so far been omitted, $W(a) = U$, $b - a = b$, and the arc we are looking for is merely the counterclockwise arc from U with measure b.

We can now state and easily prove:

Theorem 19-2 For $0 \leq a < b < 2\pi$,
$$d(W(a), W(b)) = d(U, W(b - a)).$$

$W(-\pi/2) = (0, -1)$; $W(-\pi) = (-1, 0)$; $W(-3\pi/2) = (0, 1)$

For a proof of Theorem 19-1, see answer for Exercise 1 of Part B on page 412.

For a proof of Theorem 19-2, see the answer for Exercise 2 of Part B.

TC 412 (1)

Answers for Part A

1. (a) π (b) π (c) $\pi/6$ (d) $7 - 2\pi$ [or: 0.72]
(e) $3400 - 2\pi \cdot 541$ [or: 0.79] (f) $-7 + 2\pi \cdot 2$ [or: 5.56]
(g) $-20 + 2\pi \cdot 4$ [or: 5.14] (h) $2\pi/3$ [i.e., $-10\pi/3 + 2\pi \cdot 2$]
(i) $2\pi/3$ [i.e., $20\pi/3 - 2\pi \cdot 3$]

[It is not necessary that students develop skill in dealing with negative arguments of W as in parts (b), (f), (g), and (h). In dealing with the functions cos and sin which give the coordinates of W (see Definition 19-2) we shall always be able to reduce problems involving negative arguments to problems involving positive arguments. This is because cos is even and sin is odd.]

2. (a) π (b) π (c) $\pi/6$ (d) $7 - 2\pi$
(e) $3400 - 3\pi \cdot 541$ (f) $-7 + 2\pi$ (g) $-20 + 2\pi \cdot 3$ (h) $2\pi/3$
(i) $2\pi/3$

[The answers for Exercise 2 are the same as those for Exercise 1 in case the latter are between 0 and π. In the contrary case one need only subtract 2π from the corresponding answer for Exercise 1. The point to be learned is that, for each x, one can find both a number a such that $0 \leq a < 2\pi$ and a number b such that $-\pi < b \leq \pi$ such that $W(a) = W(b) = W(x)$.]

Exercises

Part A

1. For each number x listed below, find a number a such that $0 \le a < 2\pi$ and $W(x) = W(a)$.
   - (a) 3π (b) -3π (c) $13\pi/6$
   - (d) 7 [Ans: $7 - 2\pi$ or, approximately, 0.72.] (e) 3400
   - (f) -7 [By Theorem 19–1, $W(-7) = W(-7 + 4\pi)$.] (g) -20
   - (h) $-10\pi/3$ (i) $20\pi/3$
2. For each number x listed in Exercise 1, find a number a such that $-\pi < a \le \pi$ and $W(x) = W(a)$.

Part B

1. Prove Theorem 19–1.
2. Prove Theorem 19–2.
3. Show that, for any $a \in \mathcal{R}$, there is a number—say, b—such that $-\pi < b \le \pi$ and $W(a) = W(b)$. [*Hint*: Let $c = a - 2\pi[\![a/(2\pi)]\!]$ so that $0 \le c < 2\pi$ and $W(a) = W(c)$. Now, find b such that $-\pi < b \le \pi$ and $W(b) = W(c)$. Consider two cases.]
4. Prove:

Theorem 19–3 For $0 < c < 2\pi$, $UW(-c) = UW(c)$.

[*Hint*: By Theorem 19–1, $W(-c) = W(2\pi - c)$. [Why?] For $0 < c < 2\pi$, $0 < 2\pi - c < 2\pi$. Why are arcs of the unit circle whose measures are c and $2\pi - c$ subtended by chords of the same length?]

5. Prove:

Corollary For $0 \le c < 2\pi$ and $0 \le d < 2\pi$,
$W(d)W(c) = UW(c - d)$.

19.02 The Circular Functions cos and sin

For any real number t, $W(t)$ is a point of the unit circle—that is, $W(t)$ is an ordered pair of real numbers. With this in mind we define two functions with domain $\mathcal{R}$,—the cosine function (cos) and the sine function (sin).

Definition 19–2 $(\cos(a), \sin(a)) = W(a)$.

For example, since $W(\pi/2) = (0, 1)$ it follows from this definition that $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$. What is $\cos \pi$? $\sin \pi$? [We shall, as here, usually omit the parenthesis indicating the application of cos or sin to an argument.]

Answers for Part B

1. For any $k \in I$,

$$\begin{aligned} W(a + 2\pi k) &= W(a + 2\pi k - 2\pi[\![(a + 2\pi k)/(2\pi)]\!]) \\ &= W(a + 2\pi k - 2\pi[\![a/(2\pi) + k]\!]) \\ &= W[a + 2\pi k - 2\pi([\![a/(2\pi)]\!] + k)] \end{aligned} \Bigg\} \text{ [(3) on page 314]}$$
$$\begin{aligned} &= W(a - 2\pi[\![a/(2\pi)]\!]) \\ &= W(a). \end{aligned}$$

2. Let $\overparen{W(a)W(b)}$ be the arc with end points $W(a)$ and $W(b)$ whose measure is $b - a$ [Lemma 2] and let $\overparen{UW(b - a)}$ be the counterclockwise arc from U with measure $b - a$. Since these arcs have the same measure they are congruent. Since they are congruent, so are their chords. The chords, being congruent, have the same measure. Hence, the theorem.

3. Following the hint, if $0 \le c \le \pi$, take $b = c$; if $\pi < c < 2\pi$, take $b = c - 2\pi$. In either case, $W(b) = W(c) = W(a)$ and $-\pi < b \le \pi$.

4. Following the hint it is sufficient to note that, for $0 < c < 2\pi$, for any arc of the unit circle of measure c there is an arc of measure $2\pi - c$ which has the same end points as the given arc and, so, has the same chord as the given arc. Since arcs of the same measure are congruent and, so, have congruent chords, the chords of any arcs of measures c and $2\pi - c$ will be congruent and, hence, will have the same measure. In particular, $UW(-c) = UW(2\pi - c) = UW(c)$ since, by definition, the counterclockwise arcs from U to $W(2\pi - c)$ and to $W(c)$ have measures $2\pi - c$ and c, respectively.

5. The case in which $d < c$ is covered by Theorem 19-2, and the case in which $d = c$ is trivial $[W(c)W(c) = 0 = UU = UW(0)]$. In case $c < d$ it follows by Theorem 19-2 that $W(c)W(d) = W(d - c)$. But, since $0 < d - c < 2\pi$ it follows from Theorem 19-3 that $W(d - c) = W(c - d)$. So, the corollary holds in all cases.

* * *

Since $W(\pi) = (-1, 0)$ it follows that $\cos \pi = -1$ and $\sin \pi = 0$.

In Chapter 15 we defined different cosine and sine functions, whose arguments are angles rather than real numbers. In Chapter 18 we have introduced another cosine function whose arguments are sensed angles and, for each orientation of a plane, a "sine perp" function whose arguments are sensed angles in that plane. These cosine and sine functions are closely related. For example, the cosine of an angle *is* the cosine of each of the "corresponding" sensed angles and the sine of an angle *is* the absolute value of $\sin^\perp$ of each of the corresponding sensed angles. We shall see that the cos and $\sin^\perp$ of a sensed angle are the cos and sin, according to Definition 19-2 of the radian-measure of that sensed angle. It may strike you as confusing to have the same name—for example, 'cos'— for each of three functions. However, as you will see, it is usually easy to tell from context [cos $\angle A$, cos $\angle A$, cos a] which function is meant. And, in this chapter, unless the contrary is made quite clear, we shall always mean the functions of Definition 19-2 when we use 'cos' and 'sin'.

Because of their close connection with the unit circle of $\mathscr{E}_2$, cos and sin are called *circular functions*. In the course of this chapter you will become acquainted with several other circular functions.

To see the relation between the functions cos and sin of Definition 19-2 and the [different] functions cos and $\sin^\perp$ of Chapter 18, consider a sensed angle, $\angle A$, in an oriented plane π. Recall that we have chosen for $\mathscr{E}_2$ the orientation for which $\vec{u}^{\perp} = \vec{v}$. [We shall use '$\perp$' in referring to both of the chosen perping operations—one in [π] and the other in [$\mathscr{E}_2$].] Let $\vec{i}$ be the unit vector in the sense of the initial side of $\angle A$ and let Q be the point of the terminal side of $\angle A$ such that $AQ = 1$.

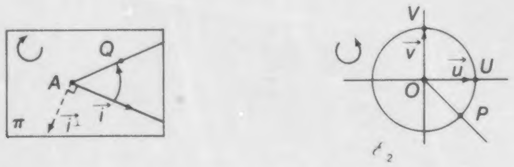

Fig. 19-2

There is an isometry f which maps π onto $\mathscr{E}_2$ in such a way that A is mapped on O, $A + \vec{i}$ on U, and $A + \vec{i}^{\perp}$ on V. In fact, if we restrict our attention to points of π and $\mathscr{E}_2$ then there is just one such isometry. [Explain.] This isometry maps Q on a point P of the unit circle in $\mathscr{E}_2$ and maps each side of $\angle A$ onto the corresponding—initial or terminal—side of $\angle UOP$. Since f is an isometry, $m(\angle A) = m(\angle UOP)$. Since f maps the initial [terminal] side of $\angle A$ onto the initial [terminal] side

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Suggestions for the exercises of section 19.02:

(i) Part A should be used in class to illustrate the discussion.
(ii) Parts B and C may be assigned for homework.

The isometry referred to is the mapping f which maps the point $A + \vec{i}p + \vec{i}^{\perp}q$ of π on the point (p, q) of $\mathscr{E}_2$. If g is any mapping of π onto $\mathscr{E}_2$ such that A is mapped on O, $A + \vec{i}$ on U and $A + \vec{i}^{\perp}$ on V then $g^{-1} \circ f$ is an isometry of π onto itself which leaves A, $A + \vec{i}$, and $A + \vec{i}^{\perp}$ fixed. We know by Theorem 14-30 that, as far as π is concerned, the only such isometry is the identity mapping. Since $g^{-1} \circ f$ is the identity mapping, g = f.

TC 414 (1)

Proof of the Corollary to Theorem 19-4:

Given ∠A, let $\measuredangle A$ be either of the sensed angles the union of whose sides is ∠A. Choose that perping operation $\perp$ in the bidirection of the plane of ∠A for which $\measuredangle A$ is positively sensed. It follows that $\cos \measuredangle A = \cos \angle A$, $\sin^{\perp} \measuredangle A = \sin \angle A$, and $m^{\perp}(\measuredangle A) = m(\angle A)$. So [by substitution into Theorem 19-4] the corollary follows.

The corollary shows us how to find the cosine and sine of an angle whose measure is known if we know how to find values of the functions cos and sin of the present chapter — that is, those introduced by Definition 19-2.

of $\angle UOP$, it follows that $m^{\perp}(\angle A) = m^{\perp}(\angle UOP)$. If $m^{\perp}(\angle UOP) = a$ then $-\pi < a \leq \pi$ and

$$P = W(a) = (\cos a, \sin a) = (\cos m^{\perp}(\angle A), \sin m^{\perp}(\angle A)).$$

But, by Exercise 5 of Part D on page 405,

$$P = (\cos \angle UOP, \sin^{\perp} \angle UOP) = (\cos \angle A, \sin^{\perp} \angle A).$$

So, we have proved:

Theorem 19-4 $\cos \angle A = \cos m^{\perp}(\angle A)$ and $\sin^{\perp} \angle A = \sin m^{\perp}(\angle A)$.

Corollary $\cos \angle A = \cos m(\angle A)$ and $\sin \angle A = \sin m(\angle A)$

Theorem 19-4 can be used in either of two ways. When we learn how to find values of cos and sin for numerical arguments we can use Theorem 19-4 to find the value of cos or $\sin^{\perp}$ of a sensed angle whose measure is known. [And, from this, we can also find cos and sin of an "ordinary" angle whose measure is known. How?] On the other hand, if, for some number a, we know the values of cos and $\sin^{\perp}$ for a sensed angle whose measure is a then we can use Theorem 19-4 to find $\cos a$ and $\sin a$.

Exercises

Part A

1. Use Theorem 19-4 in the second of the ways mentioned above, together with facts learned in Chapter 18, to compute $\cos a$ and $\sin a$ for each of the values of 'a' given below. [Tabulate your results in a table with three columns – one for the value of 'a', and the others for corresponding values of cos and sin. Save your table and save room for four more columns.]

$$-\pi, -5\pi/6, -3\pi/4, -2\pi/3, -\pi/2, -\pi/3, -\pi/4, -\pi/6,$$
$$0, \pi/6, \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4, 5\pi/6$$

2. Use the preceding results and Theorem 19-1 to find the values of cos and sin for some arguments less than $-\pi$ and for some greater than or equal to π.
3. On the same axes, draw graphs of cos and sin for arguments in $\{x: -\pi \leq x \leq 3\pi\}$. [*Hint*: Except for not having "corners" your graphs will be much like those you drew for the functions c and s in Exercises 5 and 6 of Part E on page 407.]

Answers for Part A

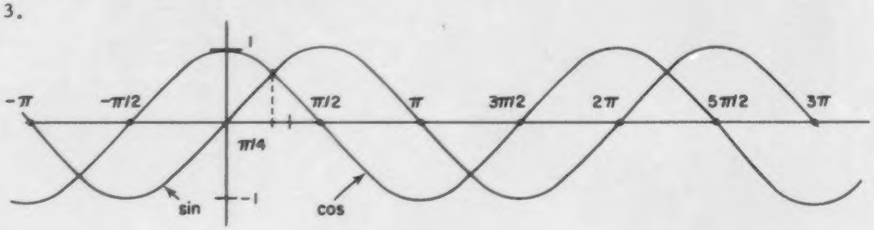
1.

a	cos a	sin a			
$-\pi$	-1	0			
$-5\pi/6$	$-\sqrt{3}/2$	$-1/2$			
$-3\pi/4$	$-1/\sqrt{2}$	$-1/\sqrt{2}$			
$-2\pi/3$	$-1/2$	$-\sqrt{3}/2$			
$-\pi/2$	0	-1			
$-\pi/3$	$1/2$	$-\sqrt{3}/2$			
$-\pi/4$	$1/\sqrt{2}$	$-1/\sqrt{2}$			
$-\pi/6$	$\sqrt{3}/2$	$-1/2$			
0	1	0			
$\pi/6$	$\sqrt{3}/2$	$1/2$			
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$			
$\pi/3$	$1/2$	$\sqrt{3}/2$			
$\pi/2$	0	1			
$2\pi/3$	$-1/2$	$\sqrt{3}/2$			
$3\pi/4$	$-1/\sqrt{2}$	$1/\sqrt{2}$			
$5\pi/6$	$-\sqrt{3}/2$	$1/2$			

[The empty columns are for the other circular functions tan, cot, sec, and csc which are introduced later in this chapter.]

2. [Students may choose any integral multiples of $\pi/6$ or $\pi/4$ and, by adding or subtracting the appropriate multiple of 2π find an argument, which is listed in the table and at which cos and sin have the same values as for the chosen argument.]

3.



[Students should use cross section paper and use the same scale on both axes. They should note that the graphs appear to cross the horizontal axis at angles of 45° (or 135°). Encourage them to do a good job and to keep their graphs for future reference.]

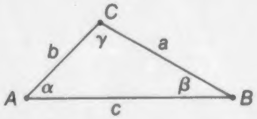
Part B

1. We have seen that the domain of cos and sin is $\mathscr{R}$. What is the range of cos? Of sin? Justify your answers.
2. What does the fact that $(\cos a, \sin a)$ belongs to the unit circle of $\mathscr{E}_2$ tell you about the relation between the numbers $\cos a$ and $\sin a$?
3. What do the graphs you drew in answer to Exercise 3 of Part A suggest as to the possible evenness or oddness of cos? Of sin?
4. Are cos and sin periodic?
*5. Your graphs should suggest to you how to complete:

$$\cos(a + \pi) = \_\_\_\_\_\_ \text{ and: } \sin(a + \pi) = \_\_\_\_\_\_$$

Part C

It is customary to use Greek letters 'α' [alpha], 'β' [beta], 'γ' [gamma], and 'δ' [delta] in referring to measures of angles. So, for example, in $\triangle ABC$, we shall let $\alpha = m(\angle A)$, $\beta = m(\angle B)$, and $\gamma = m(\angle C)$.



1. The cosine law and the sine law are stated on pages 215 and 241 in terms of cosines and sines of angles. Restate them in terms of the functions cos and sin of Definition 19–2, using the notation introduced above.
2. Restate the following in terms of cos and sin of Definition 19–2.
   (a) Both parts of Theorem 16–1 [The Projection Theorem].
   (b) Both parts of Theorem 16–6.
3. Given $\triangle ABC$ described above together with the following information, find γ, $\cos\gamma$, $\sin\gamma$, a, and b.
   (a) $\alpha = \pi/6, \beta = \pi/6, c = 6$ (b) $\alpha = \pi/3, \beta = \pi/6, c = 6$
   (c) $\alpha = \pi/3, \beta = \pi/3, c = 6$ (d) $\alpha = 2\pi/3, \beta = \pi/6, c = 6$
   (e) $\alpha = \pi/2, \beta = \pi/4, c = 6$ (f) $\alpha = \pi/4, \beta = \pi/4, c = 6$

*

From Exercises 2 and 4 of Part B and Exercise 1 of Part C we obtain three noteworthy theorems and a corollary:

Theorem 19–5 $\cos^2 a + \sin^2 a = 1$

[As illustrated, it is customary to abbreviate '$(\cos a)^2$' to '$\cos^2 a$' and '$(\sin a)^2$' to '$\sin^2 a$'.]

Theorem 19–6 cos and sin are periodic with period 2π—that is, $\cos(a + 2\pi) = \cos a$ and $\sin(a + 2\pi) = \sin a$.

Answers for Part B

1. Both cos and sin have as range $\{x: -1 \leq x \leq 1\}$. An argument to support this is that, for each x with $|x| \leq 1$ there is a point of the unit circle, $(x, \sqrt{1 - x^2})$, which has this number as its first component and a point $(\sqrt{1 - x^2}, x)$ which has this number as its second component. Furthermore, there are no points on the unit circle which have either component greater than 1 or less than -1.
2. $(\cos a)^2 + (\sin a)^2 = 1$
3. It appears likely from the graphs that cos is an even function and sin is an odd function.
4. Yes. Since W is periodic, cos and sin must be periodic. $[(\cos(a + 2\pi), \sin(a + 2\pi)) = W(a + 2\pi) = W(a) = (\cos a, \sin a)]$
5. $\cos(a + \pi) = -\cos a$, $\sin(a + \pi) = -\sin a$ [Note that these relations also show the periodicity of cos and sin. For example, $\cos(a + 2\pi) = -\cos(a + \pi) = --\cos a = \cos a$.]

Answers for Part C

1. $c^2 = a^2 + b^2 - 2ab\cos\alpha$; $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$
2. (a) $a\cos\beta + b\cos\alpha = c$; $\cos\alpha = \dfrac{c - a\cos\beta}{b}$
   (b) $\cos\gamma = -(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$;
   $\sin\gamma = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
3. (a) $\gamma = 2\pi/3$, $\cos\gamma = -1/2$, $\sin\gamma = \sqrt{3}/2$, $a = 2\sqrt{3}$, $b = 2\sqrt{3}$
   (b) $\gamma = \pi/2$, $\cos\gamma = 0$, $\sin\gamma = 1$, $a = 3\sqrt{3}$, $b = 3$
   (c) $\gamma = \pi/3$, $\cos\gamma = 1/2$, $\sin\gamma = \sqrt{3}/2$, $a = 6$, $b = 6$
   (d) $\gamma = \pi/6$, $\cos\gamma = \sqrt{3}/2$, $\sin\gamma = 1/2$, $a = 6\sqrt{3}$, $b = 6$
   (e) $\gamma = \pi/4$, $\cos\gamma = 1/\sqrt{2}$, $\sin\gamma = 1/\sqrt{2}$, $a = 6\sqrt{2}$, $b = 6$
   (f) $\gamma = \pi/2$, $\cos\gamma = 0$, $\sin\gamma = 1$, $a = 3\sqrt{2}$, $b = 3\sqrt{2}$

*

For a proof of Theorem 19-6, see the answer for Exercise 4 of Part B. The same argument establishes the corollary. Alternatively, the corollary can be derived from the theorem by mathematical induction. As a preliminary to such a proof it is convenient to note that from '$\cos(a + 2\pi) = \cos a$' one can infer '$\cos a = \cos(a - 2\pi)$' by substituting '$a - 2\pi$' in the former for 'a'.

Sample Quiz

1. Make use of your knowledge of values of cos and sin for certain special arguments to help you complete the following table.

x	$\frac{3\pi}{4}$	$\frac{\pi}{6}$	$\frac{-7\pi}{6}$	$\frac{-3\pi}{4}$	$-\pi$	$\frac{4\pi}{3}$	$\frac{28\pi}{3}$	$\frac{-27\pi}{4}$	$\frac{-37\pi}{6}$	$\frac{29\pi}{2}$
$\sin x$										
$\cos x$										

2. Prove by mathematical induction: For each $k \in I$, $\sin(p + 2k\pi) = \sin p$.

Key to Sample Quiz

1. $\sin x$: $\sqrt{2}/2$, $1/2$, $1/2$, $-\sqrt{2}/2$, 0, $-\sqrt{3}/2$, $-\sqrt{3}/2$, $-\sqrt{2}/2$, $-1/2$, 1

   $\cos x$: $-\sqrt{2}/2$, $\sqrt{3}/2$, $-\sqrt{3}/2$, $-\sqrt{2}/2$, -1, $-1/2$, $-1/2$, $-\sqrt{2}/2$, $\sqrt{3}/2$, 0

2. Case for $k = 1$: $\sin(p + 2 \cdot 1 \cdot \pi) = \sin(p + 2\pi) = \sin p$

   Case for k positive: Suppose that $\sin(p + 2n\pi) = \sin p$, for some $n \geq 1$. Then, $\sin[p + 2(n + 1)\pi] = \sin[(p + 2\pi) + 2n\pi] = \sin(p + 2\pi) = \sin p$. Thus, if $\sin(p + 2n\pi) = \sin p$ then $\sin[p + 2(n + 1)\pi] = \sin p$.

   Case for k nonpositive: Suppose that $\sin(p + 2n\pi) = \sin p$, for some $n \leq 1$. Then, $\sin[p + 2(n - 1)\pi] = \sin[(p - 2\pi) + 2n\pi] = \sin(p - 2\pi) = \sin p$. Thus, if $\sin(p + 2n\pi) = \sin p$ then $\sin[p + 2(n - 1)\pi] = \sin p$.

   Hence, by mathematical induction, $\sin(p + 2k\pi) = \sin p$, for each $k \in I$.

TC 416

The subtraction law for cos states one of the most basic properties of cos and sin. That this is so is evidenced by the fact that one can take as a definition of cos and sin [from which all their properties can be derived] the subtraction law for cos and the limit law:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

[This latter expresses the fact, noted in connection with the answer for Exercise 3 of Part A on page 414, that the graph of sin crosses the axis of arguments at an angle of 45°.] The proof that this is the case belongs in a much later course than this one. We shall, however, derive most of the usual theorems concerning cos and sin from the subtraction law for cos together with our knowledge of the values of cos and sin at $-\pi/2$, 0, $\pi/2$, and π.

A proof of the sort given here for the subtraction law for cos was first given by A. L. Cauchy, a French mathematician who lived between 1789 and 1857.

Corollary For any $k \in I$,
$\cos(a + 2k\pi) = \cos a$ and $\sin(a + 2k\pi) = \sin a$.

Theorem 19–7 If, in $\triangle ABC$, α, β, and γ are the radian-measures of $\angle A$, $\angle B$, and $\angle C$, and a, b, and c are the measures of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, then

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \text{ [cosine law] and}$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \text{ [sine law].}$$

19.03 The Subtraction Law for cos

There are many theorems concerning the functions cos and sin and the other circular functions which we shall study later. Among the most important is *the subtraction law for cos*:

$$\cos(a - b) = \cos a \cos b + \sin a \sin b \tag{1}$$

As we shall see, most of what we need to know about cos and sin follows from (1) and minor bits of information such as that $\cos\pi = -1$.

In spite of its importance, (1) is an easy consequence of results we have already proved, including the distance formula for $\mathscr{E}_2$ which was obtained in the background exercises at the end of Chapter 18. The latter is:

$$d((a_1, a_2), (b_1, b_2)) = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$$

[Recall that this is an immediate consequence of our definition of distance [$d(A, B) = \sqrt{(B - A) \cdot (B - A)}$] and the definitions (4) and (6) on pages 402 and 403.]

We begin the proof by letting

$$c = a - 2\pi[\![a/(2\pi)]\!] \text{ and } d = b - 2\pi[\![b/(2\pi)]\!]. \tag{2}$$

It follows that $0 \leq c < 2\pi$, $0 \leq d < 2\pi$, $W(c) = W(a)$, and $W(d) = W(b)$. Also, since W has period 2π and

$$c - d = (a - b) - 2\pi([\![a/(2\pi)]\!] - [\![b/(2\pi)]\!]) \tag{3}$$

it follows that $W(c - d) = W(a - b)$. From what has been said so far and the corollary to Theorem 19–3 it follows that

$$UW(a - b) = UW(c - d) = W(d)W(c) = W(b)W(a). \tag{4}$$

We now make use of Definition 19-2 and the distance formula. Since $U = (1, 0)$ and $W(a - b) = (\cos(a - b), \sin(a - b))$,

$$\begin{aligned}(5)\qquad [UW(a-b)]^2 &= [\cos(a-b) - 1]^2 + [\sin(a-b) - 0]^2\\ &= \cos^2(a-b) - 2\cos(a-b) + 1 + \sin^2(a-b)\\ &= 2[1 - \cos(a-b)]. \qquad \text{[Explain.]}\end{aligned}$$

On the other hand, since $W(a) = (\cos a, \sin a)$ and $W(b) = (\cos b, \sin b)$,

$$\begin{aligned}(6)\qquad [W(b)W(a)]^2 &= (\cos a - \cos b)^2 + (\sin a - \sin b)^2\\ &= [\cos^2 a - 2\cos a\cos b + \cos^2 b]\\ &\quad + [\sin^2 a - 2\sin a\sin b + \sin^2 b]\\ &= 2[1 - (\cos a\cos b + \sin a\sin b)] \qquad \text{[Explain.]}\end{aligned}$$

Comparing (5) and (6) we at once obtain statement (1).

Exercises

Part A

1. Use the subtraction law for cos together with data from Exercise 1 of Part A on page 414 to show that $\cos(\pi/12) = (\sqrt{6} + \sqrt{2})/4$. [*Hint*: $\frac{1}{12} = \frac{1}{4} - \frac{1}{6}$]
2. As in Exercise 1, compute $\cos(5\pi/12)$.
3. Show that cos is an even function. [*Hint*: Show that $\cos(0 - a) = \cos(a - 0)$.]
4. Show that sin is an odd function. [*Hint*: $\cos(-a - \pi/2) = \cos(-\pi/2 - a)$.]
5. **(a)** Prove *the addition law for cos*:

   $$\cos(a + b) = \cos a\cos b - \sin a\sin b$$

   [*Hint*: $a + b = a - -b$.]

   (b) Where have you seen a theorem like the addition law for cos? Explain.
6. Prove that

   (a) $\cos(\pi/2 - a) = \sin a$, and **(b)** $\sin(\pi/2 - a) = \cos a$.

   [*Hint*: Part (b) follows at once from part (a).]
7. Prove *the subtraction and addition laws for sin*:

   (a) $\sin(a - b) = \sin a\cos b - \cos a\sin b$

   (b) $\sin(a + b) = \sin a\cos b + \cos a\sin b$

   [*Hint for (a)*: $\sin(a - b) = \cos[\pi/2 - (a - b)] = \cos[(\pi/2 - a) + b]$].

   (c) Where have you seen a theorem like the addition law for sin? Explain.

The explanation asked for in connection with (5) is that, as we have seen in Exercise 2 of Part B on page 415, $\cos^2(a - b) + \sin^2(a - b) = 1$.

The explanation asked for in connection with (6) is that $\cos^2 a + \sin^2 a = 1$.

* * *

Suggestions for the exercises of section 19.03:

(i) Part A should be used to illustrate the discussion preceding and following it.
(ii) After appropriate examples, Parts B and C may be assigned as homework.
(iii) The discussion on pages 420-421, and Exercises 1-3 of Part D should be teacher directed.
(iv) Exercises 4-6 of Part D, and Part E may be used for homework. Be sure to discuss the examples for Part E before making this assignment.
(v) Part F may be developed in class.

Answers for Part A

1. $$\begin{aligned}\cos(\pi/12) &= \cos(\pi/4 - \pi/6)\\ &= \cos(\pi/4)\cos(\pi/6) + \sin(\pi/4)\sin\pi/6\\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{2}}{4}(\sqrt{3} + 1)\\ &= (\sqrt{6} + \sqrt{2})/4 \quad [\doteq 0.973]\end{aligned}$$

2. $$\begin{aligned}\cos(5\pi/12) &= \cos(3\pi/4 - \pi/3)\\ &= \cos(3\pi/4)\cos(\pi/3) + \sin(3\pi/4)\sin(\pi/3)\\ &= -\frac{1}{\sqrt{2}}\cdot\frac{1}{2} + \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\\ &= \frac{\sqrt{2}}{4}(-1 + \sqrt{3}) = (\sqrt{6} - \sqrt{2})/4 \quad [\doteq 0.259]\end{aligned}$$

3. $$\begin{aligned}\cos(-a) &= \cos(0 - a)\\ &= \cos 0\cos a + \sin 0\sin a\\ &= \cos a\cos 0 + \sin a\sin 0\\ &= \cos(a - 0)\\ &= \cos a \qquad \text{So, cos is an even function.}\end{aligned}$$

Answers for Part A [cont.]

4. $\cos(-a - \pi/2) = \cos(-a)\cos(\pi/2) + \sin(-a)\sin(\pi/2) = \sin(-a)$, since $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$. $\cos(-\pi/2 - a) = \cos(-\pi/2)\cos a + \sin(-\pi/2)\sin a = -\sin a$, since $\cos(-\pi/2) = 0$ and $\sin(-\pi/2) = -1$. So, since $-a - \pi/2 = -\pi/2 - a$ it follows that $\sin(-a) = -\sin a$ — that is, that sin is an odd function.

5. (a) $\cos(a + b) = \cos(a - -b)$

$$= \cos a \cos(-b) + \sin a \sin(-b)$$

$$= \cos a \cos b - \sin a \sin b,$$

since cos is even and sin is odd.

(b) By Exercise 2(b) of Part C on page 519,

$$\cos(\pi - (a + b)) = -(\cos a \cos b - \sin a \sin b),$$

at least if a and b are positive and $a + b < \pi$. Moreover, since cosines of supplementary angles are opposites, the corollary to Theorem 19-4 leads us to expect that $\cos(\pi - (a + b)) = -\cos(a + b)$.

6. (a) $\cos(\pi/2 - a) = \cos(\pi/2)\cos a + \sin(\pi/2)\sin a = \sin a$, since $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$.

(b) Substituting '$\pi/2 - a$' for 'a' in part (a) we see that, since $\pi/2 - (\pi/2 - a) = a$, $\cos a = \sin(\pi/2 - a)$.

7. (a) $\sin(a - b) = \cos[\pi/2 - (a - b)] = \cos[(\pi/2 - a) + b]$

$$= \cos(\pi/2 - a)\cos b - \sin(\pi/2 - a)\sin b$$

$= \sin a \cos b - \cos a \sin b$, by Exercise 6.

(b) $\sin(a + b) = \sin(a - -b)$

$$= \sin a \cos(-b) - \cos a \sin(-b)$$

$$= \sin a \cos b + \cos a \sin b,$$

since cos is even and sin is odd.

(c) By Exercise 2(b) of Part C on page 519,

$$\sin(\pi - (a + b)) = \sin a \cos b + \cos a \sin b,$$

at least for a and b positive and $a + b < \pi$. Moreover, since sines of supplementary angles are the same, the corollary to Theorem 19-4 leads us to expect that $\sin(\pi - (a + b) = \sin(a + b)$.

*

In the preceding exercises we have taken some pains to show that the addition and subtraction laws for cos and sin, and the evenness of cos and the oddness of sin all follow from the subtraction law for cos and knowledge of the values of cos and sin at $-\pi/2$, 0, and $\pi/2$. Students should not be expected to reproduce all of these arguments on demand. What they should do is memorize the subtraction formulas for cos and sin and the fact that cos is even and sin is odd. Then they should be able easily to derive the addition laws for cos and sin as in Exercises 5(a) and 7(b). Of course, in practicing doing this they will automatically memorize the addition laws as well.

Answers for Part A [cont.]

8. (a) $\cos(2a) = \cos(a + a) = \cos a \cos a - \sin a \sin a = \cos^2 a - \sin^2 a$

(b) $\sin(2a) = \sin(a + a) = \sin a \cos a + \cos a \sin a = 2 \sin a \cos a$

(c) $\cos 2a = \cos^2 a - \sin^2 a = \cos^2 a - (1 - \cos^2 a) = 2\cos^2 a - 1$

(d) $\cos 2a = \cos^2 a - \sin^2 a = (1 - \sin^2 a) - \sin^2 a = 1 - 2\sin^2 a$

9. (a) $\cos(\pi/3) = \cos(2\pi/6) = \cos^2(\pi/6) - \sin^2(\pi/6) = 3/4 - 1/4 = 1/2$

(b) $\sin(\pi/3) = \sin(2\pi/6) = 2\sin(\pi/6)\cos(\pi/6) = 2\sqrt{3}/2 \cdot 1/2 = \sqrt{3}/2$

(c) $\cos(\pi/3) = \cos(2\pi/6) = 2\cos^2(\pi/6) - 1 = 3/2 - 1 = 1/2$

(d) $\cos(\pi/3) = \cos(2\pi/6) = 1 - 2\sin^2(\pi/6) = 1 - 2/4 = 1/2$

10. (a) $\cos(a + \pi) = \cos a \cos\pi - \sin a \sin\pi = -\cos a$

(b) $\sin(a + \pi) = \sin a \cos\pi + \cos a \sin\pi = -\sin a$

(c) $\cos(a - \pi) = \cos a \cos\pi + \sin a \sin\pi = -\cos a$

(d) $\sin(a - \pi) = \sin a \cos\pi - \cos a \sin\pi = -\sin a$

[Notice that (c) and (d) can be obtained from (a) and (b) by substituting '$a - \pi$' for 'a'.]

11. $\cos(a + 2\pi) = \cos((a + \pi) + \pi) = -\cos(a + \pi) = --\cos a = \cos a$;

$\sin(a + 2\pi) = \sin((a + \pi) + \pi) = -\sin(a + \pi) = --\sin a = \sin a$

8. Make use of the results in Exercises 5 and 7 to show the following. Parts (a) and (b) are sometimes called *the doubling laws for cos and sin.*
   (a) $\cos (2a) = \cos^2 a - \sin^2 a$ (b) $\sin (2a) = 2 \sin a \cos a$
   (c) $\cos 2a = 2 \cos^2 a - 1$ (d) $\cos 2a = 1 - 2 \sin^2 a$
9. Check the formulas in Exercise 8 by using them to compute $\cos (\pi/3)$ and $\sin (\pi/3)$.
10. Express in terms of '$\cos a$' or '$\sin a$'.
    (a) $\cos (a + \pi)$ (b) $\sin (a + \pi)$
    (c) $\cos (a - \pi)$ (d) $\sin (a - \pi)$
11. Use parts (a) and (b) of Exercise 10 to show that cos and sin have period 2π.

*

The results so far obtained are worth collecting into numbered theorems. We have:

Theorem 19–8 [The Subtraction and Addition Laws for cos and sin]
(a) $\cos (a - b) = \cos a \cos b + \sin a \sin b$
(b) $\cos (a + b) = \cos a \cos b - \sin a \sin b$
(c) $\sin (a - b) = \sin a \cos b - \cos a \sin b$
(d) $\sin (a + b) = \sin a \cos b + \cos a \sin b$

Theorem 19–9 cos is even and sin is odd.

Theorem 19–10 (a) $\cos (\pi/2 - a) = \sin a$
(b) $\sin (\pi/2 - a) = \cos a$

Corollary $a + b = \pi/2 \longrightarrow \cos b = \sin a$

Theorem 19–11 (a) $\cos 2a = \cos^2 a - \sin^2 a$
(b) $\sin 2a = 2 \sin a \cos a$

Corollary (a) $\cos 2a = 2 \cos^2 a - 1$
(b) $\cos 2a = 1 - 2 \sin^2 a$

Theorem 19–12 (a) $\cos (a + \pi) = -\cos a = \cos (a - \pi)$
(b) $\sin (a + \pi) = -\sin a = \sin (a - \pi)$

[Incidentally, the corollary to Theorem 19–10 is just another way of stating Theorem 19-10(a). Explain.]

As you have seen, all of these except Theorem 19–8(a) might reasonably be listed as corollaries of the latter. Of more practical value is

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Theorems 19-8 — 19-12 are important consequences of the subtraction and addition laws for cos and sin. Students often complain of the number of such consequences they are expected to memorize. The best procedure for memorizing is to practice deriving these formulas from the subtraction and addition laws.

The corollary to Theorem 19-10 is equivalent to:

$$b = \pi/2 - a \Longrightarrow \cos b = \sin a$$

and to:

$$a = \pi/2 - b \Longrightarrow \cos b = \sin a$$

The first of these is logically equivalent to '$\cos(\pi/2 - a) = \sin a$' and the second is logically equivalent to '$\cos b = \cos(\pi/2 - b)$'. To prove the first logical equivalence one uses the derivations:

$$\dfrac{\pi/2 - a \overset{*}{=} \pi/2 - a \qquad \dfrac{b = \pi/2 - a \Longrightarrow \cos b = \sin a}{\pi/2 - a = \pi/2 - a \Longrightarrow \cos(\pi/2 - a) = \sin a}}{\cos(\pi/2 - a) = \sin a}$$

and:

$$\dfrac{\dfrac{b = \overset{*}{\pi}/2 - a \qquad \cos(\pi/2 - a) = \sin a}{\cos b = \sin a}}{b = \pi/2 - a \Longrightarrow \cos b = \sin a}\,{*}$$

Theorem 19-10(b) constitutes the reason for the name 'cosine'. The cosine of [the measure of] an angle is the sine of [the measure of] the complement of the angle. As this suggests, sines are historically prior to cosines. Nevertheless, cosines are more basic than sines because of their relation to projection and of the importance of the subtraction law for cos.

the remark that if you remember Theorem 19–8(a) and (c) and Theorem 19–9, as well as some of the data from Exercise 1 of Part A on page 414 then it is *easy* to derive the others whenever you need them. As to the two subtraction laws [(a) and (c) of Theorem 19–8], it is easy to check whether one has the correct operator—'+' or '−'—in the middle by computing $\cos(a - a)$ and recalling that $\cos 0 = 1$ and $\sin 0 = 0$. [Explain.]

The following exercises deal with some other immediate consequences of the subtraction and addition laws.

Part B

1. Prove:

Theorem 19–13

(a) $\cos a \cos b = [\cos(a - b) + \cos(a + b)]/2$
(b) $\sin a \sin b = [\cos(a - b) - \cos(a + b)]/2$
(c) $\sin a \cos b = [\sin(a - b) + \sin(a + b)]/2$
(d) $\cos a \sin b = -[\sin(a - b) - \sin(a + b)]/2$

2. Prove:

Theorem 19–14

(a) $\cos d + \cos c = 2\cos[(c + d)/2]\cos[(c - d)/2]$
(b) $\cos d - \cos c = 2\sin[(c + d)/2]\sin[(c - d)/2]$
(c) $\sin d + \sin c = 2\sin[(c + d)/2]\cos[(c - d)/2]$
(d) $\sin d - \sin c = -2\cos[(c + d)/2]\sin[(c - d)/2]$

[*Hint*: In Theorem 19–13, let $c = a + b$ and $d = a - b$. (What, then, are 'a' and 'b' in terms of 'c' and 'd'?)]

*

You will find Theorems 19–13 and 19–14 very useful in later mathematics courses, and we shall have uses for Theorem 19–14 in the present course. Fortunately, they are not as difficult to remember as it may seem. The trick is not to attempt to remember them but to recall how to derive them. What you must have well in mind are the subtraction and addition laws. Suppose, for example, that you need to express '$\sin 2x \sin 3x$' as a sum or difference. Instead of substituting in Theorem 19–13(b), recall that this term occurs in instances of both the subtraction law and the addition law for cos:

$$\cos(2x - 3x) = \cos 2x \cos 3x + \sin 2x \sin 3x$$
$$\cos(2x + 3x) = \cos 2x \cos 3x - \sin 2x \sin 3x$$

The explanation of how to remember which sign to use is:

$$1 = \cos 0 = \cos(a - a) = \cos a \cos a + \sin a \sin a,$$
checking with '$\cos^2 a + \sin^2 a = 1$';
$$0 = \sin 0 = \sin(a - a) = \sin a \cos a - \cos a \sin a$$

*

Answers for Part B

1. (a)
$$\begin{array}{ll} \cos a \cos b + \sin a \sin b & = \cos(a - b) \\ \underline{\cos a \cos b - \sin a \sin b} & \underline{= \cos(a + b)} \\ 2\cos a \cos b & = \cos(a - b) + \cos(a + b) \end{array}$$
Therefore, $\cos a \cos b = [\cos(a - b) + \cos(a + b)]/2$.

[Parts (b), (c), and (d) are similar.]

2. (a) In Theorem 19-14(a) substitute '$(c + d)/2$' for 'a' and '$(c - d)/2$' for 'b'.

[Parts (b), (c), and (d) are similar.]

*

The parts of Theorem 19-14 may be thought of as factoring formulas analogous, for example, to '$a^2 - b^2 = (a - b)(a + b)$'. Like the latter they are useful in solving equations and in reducing fractions to lowest terms. The parts of Theorem 19-13 are useful in calculus where it is sometimes an advantage to replace products by sums. Our principal use for Theorem 19-13 is as a step in obtaining Theorem 19-14.

*

Sample Quiz

1. Complete in terms of '$\cos p$' or '$\sin p$'.
   (a) $\sin(3\pi/2 - p) =$ _____ (b) $\cos(3\pi/2 - p) =$ _____
   (c) $\sin(-5\pi/4 + p) =$ _____ (d) $\sin(p - \pi/6) =$ _____
   (e) $\cos(7\pi/6 + p) =$ _____ (f) $\cos(-7\pi/6 - p) =$ _____
2. (a) Complete in terms of '$\cos p$': $\cos 2p =$ _____
   (b) Make use of your result from (a) to complete this sentence in terms of '$\cos p$':
   $|\cos(p/2)| =$ _____

Key to Sample Quiz

1. (a) $-\cos p$ (b) $-\sin p$
   (c) $(\cos p - \sin p)\sqrt{2}/2$ (d) $(\sqrt{3}\sin p - \cos p)/2$
   (e) $(\sin p - \sqrt{3}\cos p)/2$ (f) [Same as (e).]
2. (a) $2\cos^2 p - 1$
   (b) By (a), $\cos p = 2\cos^2(p/2) - 1$. So, $|\cos(p/2)| = \sqrt{(1 + \cos p)/2}$.

and, so, that

$$\begin{aligned} 2 \sin 2x \sin 3x &= \cos(2x - 3x) - \cos(2x + 3x) \\ &= \cos(-x) - \cos 5x \\ &= \cos x - \cos 5x. \end{aligned}$$

So, $\sin 2x \sin 3x = [\cos x - \cos 5x]/2$. With a little practice you will learn to write down the proper expression without writing the instances of the subtraction and addition laws from which it comes.

Similarly, in case you wish to express, say, '$\sin 3x - \sin 2x$' as a product, recall that you can find numbers c and d such that $3x = c + d$ and $2x = c - d$. In fact, with a little scratch-work you find that $c = 5x/2$ and $d = x/2$. So, recalling the addition and subtraction laws for sin,

$$\sin 3x = \sin(5x/2 + x/2) = \sin(5x/2)\cos(x/2) + \cos(5x/2)\sin(x/2)$$
$$\sin 2x = \sin(5x/2 - x/2) = \sin(5x/2)\cos(x/2) - \cos(5x/2)\sin(x/2)$$

and

$$\sin 3x - \sin 2x = 2\cos(5x/2)\sin(x/2).$$

Again, with a little practice you will not need to write down the instances of the addition and subtraction laws.

Part C

1. Express each indicated product as a sum or difference and each indicated sum or difference as a product.
   (a) $\cos 2x \cos x$ (b) $\sin 5a + \sin 3a$
   (c) $\sin 2a \cos 3a$ (d) $\cos 2b - \cos b$
   (e) $\sin(\pi/4 + b)\sin(\pi/4 - b)$ (f) $\cos(5a/3) + \cos(5a/6)$
   (g) $4 \sin 2a \cos 3a \cos 4a$ (h) $\sin 6x - \sin 7x$
2. Show that $\cos(x - y)\cos(x + y) = \cos^2 x - \sin^2 y$. [*Hint*: Use Theorem 19-13(a) and the corollary to Theorem 19-11.]
3. Show that $\sin(x - y)\sin(x + y) = \sin^2 x - \sin^2 y$.
4. Show that $\cos(x + y)\cos x + \sin(x + y)\sin x = \cos y$.

*

Theorems 19-10 and 19-12 contain examples of so-called *reduction formulas*. Using these it is not difficult to establish two quite general reduction formulas:

> Theorem 19-15 For $k \in I$,
> (a) $\cos(a + k\pi) = (-1)^k \cos a$, and
> (b) $\sin(a + k\pi) = (-1)^k \sin a$.

Answers for Part C

1. (a) $[\cos x + \cos 3x]/2$ (b) $2 \sin 4a \cos a$
   (c) $[-\sin a + \sin 5a]/2$ (d) $-2 \sin(3b/2)\sin(b/2)$
   (e) $[\cos 2b]/2$ (f) $2 \cos(5a/4)\cos(5a/12)$
   (g) $\sin a + \sin 3a - \sin 5a + \sin 9a$ (h) $-2\cos(13x/2)\sin(x/2)$
2. $$\begin{aligned} \cos(x - y)\cos(x + y) &= [\cos(-2y) + \cos 2x]/2 \\ &= [\cos 2y + \cos 2x]/2 \\ &= [1 - 2\sin^2 y + 2\cos^2 x - 1]/2 \\ &= \cos^2 x - \sin^2 y \end{aligned}$$
   [Note that, for $y = x$, this reduces to the doubling law for cos.]
3. $$\begin{aligned} \sin(x - y)\sin(x + y) &= [\cos(-2y) - \cos 2x]/2 \\ &= [\cos 2y - \cos 2x]/2 \\ &= [1 - 2\sin^2 y - 1 + 2\sin^2 x]/2 \\ &= \sin^2 x - \sin^2 y \end{aligned}$$
   [An alternate solution involves use of the subtraction and addition laws for sin and Theorem 19-5.]
4. $\cos(x + y)\cos x + \sin(x + y)\sin x = \cos(x + y - x) = \cos y$

[Check these formulas against earlier theorems for $k = -1, 1$, and 2.]

Theorem 19–16 For $k \in I$,
(a) $\cos [a + (2k + 1)\pi/2] = -(-1)^k \sin a$, and
(b) $\sin [a + (2k + 1)\pi/2] = (-1)^k \cos a$.

[Use earlier theorems to check these formulas in case $k = -1$.]

As an example we shall use mathematical induction to prove Theorem 19–15(a). Recall that for such a proof it is sufficient to establish these three things:

(i) $\cos (a + 0\pi) = (-1)^0 \cos a$
(ii) if $\cos (a + k\pi) = (-1)^k \cos a$
then $\cos [a + (k + 1)\pi] = (-1)^{k+1} \cos a$
(iii) if $\cos (a + k\pi) = (-1)^k \cos a$
then $\cos [a + (k - 1)\pi] = (-1)^{k-1} \cos a$

Statement (i) is obviously correct. [Why?] To prove statements (ii) and (iii) we shall assume that, for a given k, $\cos (a + k\pi) = (-1)^k \cos a$. Then for (ii),

$$\begin{aligned} \cos [a + (k + 1)\pi] &= \cos [(a + k\pi) + \pi] = -\cos (a + k\pi) \\ &= -(-1)^k \cos a = (-1)^{k+1} \cos a \end{aligned} \tag{7}$$

and, for (iii),

$$\begin{aligned} \cos [a + (k - 1)\pi] &= \cos [(a + k\pi) - \pi] = -\cos (a + k\pi) \\ &= -(-1)^k \cos a = (-1)^{k-1} \cos a. \end{aligned} \tag{8}$$

[Explain the steps indicated in (7) and (8).] Statement (ii) follows at once [by the deduction rule] from the work done in establishing (7) and statement (iii) follows, similarly, by (8).

As a second example we shall show how Theorem 19–15(b) can be used in proving Theorem 19–16(a). In the proof we shall also use a consequence of Theorems 19–10(a) and 19–9. To begin with we note that

$$\cos [a + (2k + 1)\pi/2] = \cos [(a + k\pi) + \pi/2]. \qquad \text{[Why?]} \tag{9}$$

Now, since sin is odd, it follows from Theorem 19–10(a) that

$$\cos (b + \pi/2) = \cos (\pi/2 - -b) = \sin (-b) = -\sin b.$$

So, by (9),

$$\cos [a + (2k + 1)\pi/2] = -\sin (a + k\pi) = -(-1)^k \sin a. \qquad \text{[Why?]}$$

In the proof of Theorem 19-15(a), (i) is correct because $a + 0\pi = a$ and $(-1)^0 = 1$. The steps indicated in (7) are justified, first, by the distributive principle and the associative principle for addition, second, by Theorem 19-12(a), third, by the inductive hypothesis, and, fourth, by the fact that $-(-1)^k = -1(-1)^k = (-1)^{k+1}$. The steps indicated in (8) are justified in a similar manner. Note that $(-1)^k = -1(-1)^{k+1} = -(-1)^{k+1}$ and, so, $-(-1)^k = (-1)^{k+1}$.

As to (9), $(2k + 1)\pi/2 = k\pi + \pi/2$.

The answer for the 'Why?' is: Theorem 19-15(b).

Part D

1. Prove Theorem 19–15(b).
2. Prove Theorem 19–16(b).
3. Prove the following corollary of Theorems 19–15 and 19–16.

> **Corollary** For $k \in I$,
> (a) $\cos k\pi = (-1)^k$ (b) $\cos (2k + 1)\pi/2 = 0$
> (c) $\sin k\pi = 0$ (d) $\sin (2k + 1)\pi/2 = (-1)^k$.

4. Use Theorems 19–15 and 19–16 [and earlier theorems] to complete the following, where $k \in I$.
   (a) $\cos (a - k\pi) =$ ______ (b) $\cos (k\pi - a) =$ ______
   (c) $\sin (a - k\pi) =$ ______ (d) $\sin (k\pi - a) =$ ______
   (e) $\cos [(2k + 1)\pi/2 - a] =$ ______ (f) $\sin [(2k + 1)\pi/2 - a] =$ ______
   (g) $\cos [a + (2k - 1)\pi/2] =$ ______ (h) $\sin [a + (2k - 1)\pi/2] =$ ______
   [*Hint*: For (g) and (h), $2k - 1 = 2 \cdot ? + 1$.]
5. Evaluate each of the following. [Your answers should be '–1', '0' or '1'.]
   (a) $\cos 3\pi$ (b) $\sin (-3\pi/2)$ (c) $\sin 205\,\pi$
   (d) $\cos (-7\pi)$ (e) $\cos 137\,\pi$ (f) $\sin (-15\pi/2)$
6. Simplify.
   (a) $\cos (a + 5\pi/2)$ (b) $\cos (a + 7\pi)$ (c) $\sin (a + 3\pi)$
   (d) $\sin (b - 9\pi)$ (e) $\cos (c - 11\pi/2)$ (f) $\sin (b - 3\pi/2)$

*

The corollary to Theorems 19–15 and 19–16 may be reformulated, in part, by saying that the cosine of an odd multiple of $\pi/2$ is 0 and the sine of an even multiple of $\pi/2$ is 0. In addition to this we need to know:

> **Theorem 19–17**
> (a) $\cos a = 0 \Longrightarrow \exists_{k \in I}\, a = (2k + 1)\pi/2$
> (b) $\sin a = 0 \Longrightarrow \exists_{k \in I}\, a = k\pi$

It is convenient to begin by proving (b). We first notice that, for $0 \le a < 2\pi$, if $\sin a = 0$ then $a = 0$ or $a = \pi$. This follows from the fact that the unit circle intersects $\overleftrightarrow{OU}$ only at U and at U' and that, by the definition of W, for $0 \le a < 2\pi$, $W(a) = U$ if and only if $a = 0$ and $W(a) = U'$ if and only if $a = \pi$. Suppose, now, for any $a \in \mathscr{R}$, that $\sin a = 0$. It follows from Lemma 1 on page 411 and Definition 19–2 that $\sin (a - 2\pi[\![a/(2\pi)]\!]) = 0$. Since $0 \le a - 2\pi[\![a/(2\pi)]\!] < 2\pi$ it follows that $a - 2\pi[\![a/(2\pi)]\!] = 0$ or $a - 2\pi[\![a/(2\pi)]\!] = \pi$. Since $[\![a/(2\pi)]\!] \in I$ it follows that there is a $j \in I$ such that $a = 2j\pi$ or $a = (2j + 1)\pi$. Since, for $j \in I$, $2j \in I$, and $2j + 1 \in I$ it follows that, in either case, there is a $k \in I$ such that $a = k\pi$. This proves Theorem 19–17(b).

Answers for Part D

1. To begin with, $\sin(a + 0\pi) = \sin a = (-1)^0 \sin a$. Suppose, now, that, for a given k, $\sin(a + k\pi) = (-1)^k \sin a$. It follows that
$$\begin{aligned}\sin[a + (k + 1)\pi] &= \sin[(a + k\pi) + \pi] = -\sin(a + k\pi)\\ &= -(-1)^k \sin a = (-1)^{k+1}\sin a\end{aligned}$$
and that
$$\begin{aligned}\sin[a + (k - 1)\pi] &= \sin[(a + k\pi) - \pi] = -\sin(a + k\pi)\\ &= -(-1)^k \sin a = (-1)^{k+1} \sin a.\end{aligned}$$
Hence, by mathematical induction, Theorem 19-15(b).
2. $$\begin{aligned}\sin[a + (2k + 1)\pi/2] &= \sin[(a + k\pi) + \pi/2]\\ &= \sin[\pi/2 - -(a + k\pi)]\\ &= \cos[-(a + k\pi)]\\ &= \cos(a + k\pi)\\ &= (-1)^k \cos a\end{aligned}$$
3. Parts (a) and (c) of the corollary follow from Theorem 19-15 for a = 0 [since cos 0 = 1 and sin 0 = 0]. Parts (b) and (d) of the corollary follow similarly from Theorem 19-16. [Note that we might have begun by using mathematical induction to prove the corollary and then derived Theorem 19-15 from the corollary and the addition laws. It is perhaps easier to remember the corollary and do exercises like those in Exercise 4 by using the subtraction and addition laws.]
4. (a) $(-1)^k \cos a$ (b) $(-1)^k \cos a$ (c) $(-1)^k \sin a$ (d) $-(-1)^k \sin a$
   (e) $(-1)^k \sin a$ (f) $(-1)^k \cos a$ (g) $(-1)^k \sin a$ (h) $(-1)^{k-1} \cos a$
5. (a) -1 (b) 1 (c) 0 (d) -1 (e) -1 (f) 1
6. (a) $-\sin a$ (b) $-\cos a$ (c) $-\sin a$
   (d) $-\sin b$ (e) $-\sin c$ (f) $\cos b$

To prove Theorem 19–17(a) we note that, by Theorems 19–9 and 19–10(a), $\cos a = -\sin (a - \pi/2)$. It follows that if $\cos a = 0$ then $\sin (a - \pi/2) = 0$ and so, by Theorem 19–17(a), there is a $k \in I$ such that $a = k\pi + \pi/2$. Since $k\pi + \pi/2 = (2k + 1)\pi/2$, this proves Theorem 19–17(a).

For convenience we combine Theorem 19–17 and parts of the preceding corollary into:

> Corollary
> (a) $\cos a = 0 \longleftrightarrow \exists_{k \in I}\, a = (2k + 1)\pi/2$
> (b) $\sin a = 0 \longleftrightarrow \exists_{k \in I}\, a = k\pi$

This corollary, sometimes together with Theorem 19–14, is useful in solving equations involving 'cos' and 'sin'. We give some examples.

Example 1. Solve '$\cos (x/3) = 0$'.

Solution By the corollary to Theorem 19–17, $\cos (x/3) = 0$ if and only if, for some $k \in I$, $x/3 = (2k + 1)(\pi/2)$. So, the solution set of the equation is $\{x: \exists_{k \in I}\, x = (2k + 1)\, 3\pi/2\}$.

Example 2 Solve '$\sin 2x = 0$'.

Solution By the corollary, $\sin 2x = 0$ if and only if, for some $k \in I$, $2x = k\pi$. So, the solution set of the equation is $\{x: \exists_{k \in I}\, x = k\pi/2\}$.

Example 3. Solve '$\cos (x/3) \sin 2x = 0$'.

Solution From the preceding examples, the solution set of this equation is

$$\{x: \exists_{k \in I}\, x = (2k + 1)3\pi/2\} \cup \{x: \exists_{k \in I}\, x = k\pi/2\}.$$

Since each odd multiple of $3\pi/2$ is a multiple of $\pi/2$, the solution set is $\{x: \exists_{k \in I}\, x = k\pi/2\}$.

Example 4. Solve '$\cos 5x = \cos x$'.

Solution. By Theorem 19–14(b), $\cos 5x - \cos x = -2 \sin 3x \sin 2x$. So, the given equation is equivalent to '$\sin 3x \sin 2x = 0$'. By the corollary to Theorem 19–17, $\sin 3x = 0$ if and only if, for some $k \in I$, $3x = k\pi$; and $\sin 2x = 0$ if and only if, for some $k \in I$, $2x = k\pi$. So, the solution set of the given equation is $\{x: \exists_{k \in I}\, (x = k\pi/3 \text{ or } x = k\pi/2)\}$.

Example 5. Solve '$\cos(2x - 1) = 0$'.

Solution. By the corollary, $\cos(2x - 1) = 0$ if and only if, for some $k \in I$, $2x - 1 = (2k + 1)\pi/2$. So, the solution set is $\{x: \exists_{k \in I}\, x = (2k + 1)\pi/4 + \frac{1}{2}\}$.

Part E

Solve the following equations.

1. $\cos(x/2) = 0$
2. $\cos 3x = 0$
3. $\cos 5x + \cos 3x = 0$
4. $\cos(x + \pi/4) = 0$

*

From the definition of W and the fact that $W(\pi/2) = (1, 0)$ it follows that

$$0 < a < \pi/2 \longrightarrow (\cos a > 0 \text{ and } \sin a > 0).$$

Since cos is even and $\cos 0 = 1$, $\cos a > 0$ for $-\pi/2 < a < \pi/2$. Since $\sin(\pi - a) = \sin a$ [Why?] and $\sin(\pi/2) = 1$ it follows that $\sin a > 0$ for $0 < a < \pi$. Using Theorem 19–15 we can extend these results to determine all arguments for which $\cos a > 0$ and all arguments for which $\sin a > 0$. For example, it follows from the first of these results and Theorem 19–15(a) that

$$(10) \qquad -\pi/2 < a < \pi/2 \longrightarrow \operatorname{sgn}[\cos(a + k\pi)] = (-1)^k.$$

[Recall that the signum function, sgn, is such that $\operatorname{sgn}(b) = 1$ or -1 according as $b > 0$ or $b < 0$.] As an instance of (10) we have:

$$-\pi/2 < a - k\pi < \pi/2 \longrightarrow \operatorname{sgn}[\cos a] = (-1)^k$$

From this we easily obtain part (a) of:

Theorem 19–18

(a) $(2k - 1)\pi/2 < a < (2k + 1)\pi/2 \longrightarrow \operatorname{sgn}(\cos a) = (-1)^k$

(b) $k\pi < a < (k + 1)\pi \longrightarrow \operatorname{sgn}(\sin a) = (-1)^k$

Part (b) is proved in a similar manner. So, as your graphs of cos and sin should suggest, each of these functions is alternately positive and negative in stretches of measure π.

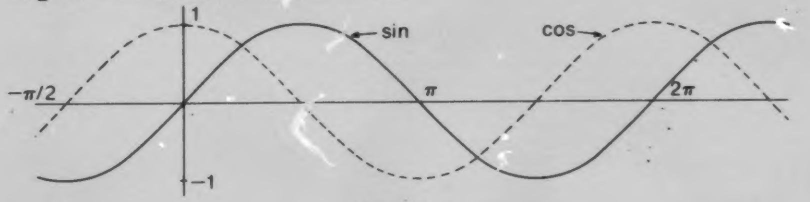


Fig. 19–3

Answers for Part E

[As in the examples, we give the solution sets of the equations.]

1. $\{x: \exists_{k \in I}\; x = (2k + 1)\pi\}$
2. $\{x: \exists_{k \in I}\; x = (2k + 1)\pi/6\}$
3. $\{x: \exists_{k \in I}\; (x = (2k + 1)\pi/8 \text{ or } x = (2k + 1)\pi/2)\}$
4. $\{x: \exists_{k \in I}\; x = (4k + 1)\pi/4\}$

*

For a proof of Theorem 19-18(b), see the answer for Exercise 2 of Part F on page 425.

TC 425 (1)

The explanation asked for in the proof of Theorem 19-19(a) is that, for any c and d such that $c < d$,

$$c < \frac{c + d}{2} < d.$$

A proof of Theorem 19-19(b) is given in the answer for Exercise 3 of Part F on page 425. Here are more general results than those given in Theorem 19-19:

By Theorem 19-14 and 19-18, for $d > c$ and $0 < d - c < \pi$,

$$\operatorname{sgn}(\cos d - \cos c) = -\operatorname{sgn}(\sin[(c + d)/2]),$$

$$\operatorname{sgn}(\sin d - \sin c) = \operatorname{sgn}(\cos[(c + d)/2]).$$

So,

$$\operatorname{sgn}(\cos d - \cos c) = -(-1)^k \text{ for } k\pi < c < d < (k + 1)\pi,$$

$$\operatorname{sgn}(\sin d - \sin c) = (-1)^k \text{ for } (2k - 1)\pi/2 < c < d < (2k + 1)\pi/2.$$

In particular, cos is increasing in $\overline{k\pi, (k + 1)\pi}$ if k is odd and is decreasing if k is even.

It is also apparent from graphs of cos and sin that each of these functions is alternately decreasing—that is, has smaller values for larger arguments—and increasing—that is, has larger values for larger arguments—in stretches of measure π. More specifically, it appears that cos is decreasing where sin is positive and is increasing where sin is negative. And sin appears to be increasing where cos is positive and to be decreasing where cos is negative. These results can be established by using Theorem 19–14(b) and (d) and Theorem 19–18. We shall content ourselves with the proof of:

> **Theorem 19–19**
> (a) cos is decreasing for $0 \le a \le \pi$, and
> (b) sin is increasing for $-\pi/2 \le a \le \pi/2$.

To prove part (a) we need to show that if $0 \le c < d \le \pi$ then $\cos d - \cos c < 0$. Assuming that $0 \le c < d \le \pi$ it follows that $0 < (c + d)/2 < \pi$. [Explain.] So, by Theorem 19–18(b), $\sin [(c + d)/2] > 0$. On the other hand, $-\pi/2 \le (c - d)/2 < 0$ and so, by Theorem 19–18(b), $\sin [(c - d)/2] < 0$. Hence, by Theorem 19–14(b), $\cos d - \cos c < 0$. Consequently, if $0 \le c < d \le \pi$ then $\cos d < \cos c$—that is, cos is decreasing between 0 and π. The proof of Theorem 19–19(b) is similar.

Part F

1. Complete the following table to indicate whether cos and sin are positive [> 0] or negative [< 0] and whether they are decreasing [↓] or increasing [↑] in the specified intervals.

	$\overline{0, \pi/2}$	$\overline{\pi/2, \pi}$	$\overline{\pi, 3\pi/2}$	$\overline{3\pi/2, 2\pi}$
cos	> 0, ↓			
sin				

2. Prove Theorem 19–18(b).
3. Prove Theorem 19–19(b).

19.04 The Functions °cos and °sin

Due to Theorem 19–4 and the relation between cos ∠A and cos ∠A and between sin ∠A and sin ∠A, it follows that

$$\text{(1)} \qquad \cos \angle A = \cos (m(\angle A)) \text{ and } \sin \angle A = \sin (m(\angle A)).$$

In other words, the cosine and sine of an angle [in the sense of Chapter 14] are the cosine and sine, respectively of the radian-measure of the given angle. So, the cos and sin functions of Definition 19–2 are well-suited for use in geometrical problems dealing with angles whose

TC 425 (2)

Answers for Part F

1.

	$\overline{0, \pi/2}$	$\overline{\pi/2, \pi}$	$\overline{\pi, 3\pi/2}$	$\overline{3\pi/2, 2\pi}$
cos	> 0, ↓	< 0, ↓	< 0, ↑	> 0, ↑
sin	> 0, ↑	> 0, ↓	< 0, ↓	< 0, ↑

2. As shown on page 531, $\sin a > 0$ for $0 < a < \pi$. So, by Theorem 19-15(b)

$$0 < a < \pi \Longrightarrow \text{sgn}[\sin(a + k\pi)] = (-1)^k.$$

It follows from this that

$$0 < a - k\pi < \pi \Longrightarrow \text{sgn}(\sin a) = (-1)^k$$

and, so, that

$$k\pi < a < (k + 1)\pi \Longrightarrow \text{sgn}(\sin a) = (-1)^k.$$

3. We need to show that if $-\pi/2 \le c < d \le \pi/2$ then $\sin d - \sin c > 0$. For $-\pi/2 \le c < d \le \pi/2$, $-\pi/2 < (c + d)/2 < \pi/2$ and $-\pi/2 \le (c - d)/2 < 0$. [For the latter, note that $0 < d - c \le \pi$.] Now, by Theorem 19-14,

$$\sin d - \sin c = -2 \cos[(c + d)/2] \sin[(c - d)/2].$$

Since $-\pi/2 < (c + d)/2 < \pi/2$, $\cos[(c + d)/2] > 0$. Since $-\pi/2 \le (c - d)/2 < 0$, $\sin[(c - d)/2] < 0$. Hence, for $-\pi/2 \le c < d \le \pi/2$, $\sin d - \sin c > 0$. In short, sin is increasing in $\overline{-\pi/2, \pi/2}$.

TC 426 (1)

The degree-cosine and degree-sine have been mentioned briefly in TC 409.

If the degree-measure of ∠A is a then the radian-measure of ∠A is $\pi a/180$.

Note that the '°' both in '°cos 30' and, later, in 'cos 30°' is part of the name of the degree-cosine function.

To emphasize the difference between cos and °cos, for example, it is helpful to draw graphs of both functions on the same set of axes, using the same scale for both axes. Such a graph of cos is given in Figure 19-3 on page 424. A corresponding graph of °cos on these axes is essentially a horizontal line through (0, 1). For, for example, $°\cos 3\pi \doteq \cos(12.4)° \doteq 0.98$.

* * *

Suggestions for the exercises of section 19.04:

(i) Part A and the discussion should be teacher directed.
(ii) After appropriate examples, Parts B, C, and D may be assigned for homework.
(iii) Part E may be used for supervised practice and individual help.
(iv) Exploration Exercises may be assigned as homework, but should be discussed carefully with the class.

radian-measures are known. In many cases, however, it is the degree-measure of an angle which is known and, while it is easy enough to find the corresponding radian-measure [How?], it is more convenient to have circular functions which are analogous to cos and sin but are better suited to problems involving degree-measures. These functions we shall call *the degree-cosine* ['°cos'] and *the degree-sine* ['°sin'] *functions* and define them by:

Definition 19–3 $^\circ\cos a = \cos(\pi a/180)$
$^\circ\sin a = \sin(\pi a/180)$

Since, for $0 < a < 180$, an angle whose degree-measure is a has radian-measure $\pi a/180$, it follows from (1) and Definition 19–3 that, for any $\angle A$,

$$\cos \angle A = {}^\circ\cos({}^\circ m(\angle A)) \text{ and } {}^\circ\sin \angle A = {}^\circ\sin({}^\circ m(\angle A)). \tag{2}$$

It is customary to modify the preceding notation by using '$\cos a^\circ$' and '$\sin a^\circ$' to refer to the degree-cosine and degree-sine of an angle of a°, and to rewrite Definition 19–3:

$$\cos a^\circ = \cos(\pi a/180) \text{ and } \sin a^\circ = \sin(\pi a/180) \tag{3}$$

The convention (3) is used even when a is a number which is not between 0 and 180 [and, so, is not the degree-measure of any angle]. As an immediate consequence of the definition [as modified in (3)] we have:

Theorem 19–20 $\cos a = \cos(180a/\pi)^\circ$
$\sin a = \sin(180a/\pi)^\circ$

The functions °cos and °sin have properties much like those of cos and sin which are mentioned in Theorems 19–1 through 19–17 and their corollaries. All one need do to transform one of these theorems into a theorem about °cos and °sin is to replace 'cos' by '°cos', 'sin' by '°sin', and 'π' by '180'. [In Theorem 19–7, α, β, and γ must be taken to be the degree-measures of the angles.] For example, °cos is periodic with period $2 \cdot 180$ [—that is, with period 360]. This follows at once from Theorem 19–6 and Definition 19–3:

$$\begin{aligned} {}^\circ\cos(a + 360) &= \cos[\pi(a + 360)/180] \\ &= \cos[\pi a/180 + 2\pi] \\ &= \cos(\pi a/180) = {}^\circ\cos a \end{aligned}$$

TC 426 (2)

Sample Quiz

1. Given the winding function W on the unit circle, consider $W(-5\pi/6)$ and $W(7\pi/4)$.
   (a) Give the coordinates of $W(-5\pi/6)$ and of $W(7\pi/4)$.
   (b) What is $\cos t$, where $t = -5\pi/6 + 7\pi/4$? What is $\sin t$?
   (c) What is $\cos s$, where $s = -5\pi/6 - 7\pi/4$?
2. Given that $-\pi/2 < t < \pi/2$, determine the values of 't' which satisfy the following equations.
   (a) $\cos(5\pi/4) = \sin t$ (b) $\sin(-2\pi/3) = -\cos t$
3. Prove: For each $k \in I$, $\sin[p + (2k + 1)\pi] = -\sin p$.

Key to Sample Quiz

1. (a) $(-\sqrt{3}/2, -1/2)$; $(\sqrt{2}/2, -\sqrt{2}/2)$
   (b) $-(\sqrt{6} + \sqrt{2})/4$; $(\sqrt{6} - \sqrt{2})/4$
   (c) $(\sqrt{2} - \sqrt{6})/4$
2. (a) $t = -\pi/4$ (b) $t = \pi/6$ or $t = -\pi/6$
3. $$\begin{aligned} \sin[p + (2k + 1)\pi] &= \sin[(p + \pi) + 2k\pi] \\ &= \sin(p + \pi)\cos 2k\pi + \cos(p + \pi)\sin 2k\pi \\ &= \sin(p + \pi)\cdot 1 + \cos(p + \pi)\cdot 0 \\ &= \sin(p + \pi) \\ &= -\sin p \end{aligned}$$

TC 427 (1)

Answers for Part A

1. (a) $\cos(2\pi/3)$ (b) $\sin(5\pi/4)$ (c) $\cos(-3\pi/4)$
   (d) $\sin(\pi/6)$ (e) $\cos(-\pi/3)$ (f) $\sin(4\pi/3)$
2. (a) $\cos 30^\circ$ (b) $\sin(-120)^\circ$ (c) $\cos 135^\circ$
   (d) $\sin(-45)^\circ$ (e) $\cos 180^\circ$ (f) $\sin(180/\pi)^\circ$
3. 19-5: $\cos^2 a^\circ + \sin^2 a^\circ = 1$
   19-6: $\cos(a + 360)^\circ = \cos a^\circ$, $\sin(a + 360)^\circ = \sin a^\circ$
   Corollary: For $k \in I$, $\cos(a + 360k)^\circ = \cos a^\circ$ and $\sin(a + 360k)^\circ = \sin a^\circ$.
   19-7: If, in $\triangle ABC$, α, β, and γ are the degree-measures of $\angle A$, $\angle B$, and $\angle C$ and a, b, and c are the measures of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, then
   $$c^2 = a^2 + b^2 - 2ab\cos\gamma^\circ \text{ and}$$
   $$\frac{\sin\alpha^\circ}{a} = \frac{\sin\beta^\circ}{b} = \frac{\sin\gamma^\circ}{c}.$$
   19-8: (a) $\cos(a - b)^\circ = \cos a^\circ \cos b^\circ + \sin a^\circ \sin b^\circ$
   (b) $\cos(a + b)^\circ = \cos a^\circ \cos b^\circ - \sin a^\circ \sin b^\circ$
   (c) $\sin(a - b)^\circ = \sin a^\circ \cos b^\circ - \cos a^\circ \sin b^\circ$
   (d) $\sin(a + b)^\circ = \sin a^\circ \cos b^\circ + \cos a^\circ \sin b^\circ$
   19-9: °cos is even and °sin is odd.

For another example we shall derive the subtraction law for °cos from Theorem 19–8(a):

$$\begin{aligned} {}^{\circ}\cos(a-b) &= \cos[\pi(a-b)/180] = \cos(\pi a/180 - \pi b/180) \\ &= \cos(\pi a/180)\cos(\pi b/180) + \sin(\pi a/180)\sin(\pi b/180) \\ &= {}^{\circ}\cos a\, {}^{\circ}\cos b + {}^{\circ}\sin a\, {}^{\circ}\sin b \end{aligned}$$

Analogues of the other theorems are derivable in an equally straightforward manner.

Exercises

Part A

1. Express each of the following in terms of 'cos' or 'sin'.
   (a) $\cos 120°$ (b) $\sin 225°$ (c) $\cos(-135)°$
   (d) $\sin 30°$ (e) $\cos(-60)°$ (f) $\sin 240°$
2. Express each of the following in terms of degree-cosine or degree-sine.
   (a) $\cos(\pi/6)$ (b) $\sin(-2\pi/3)$ (c) $\cos(3\pi/4)$
   (d) $\sin(-\pi/4)$ (e) $\cos \pi$ (f) $\sin 1$
3. State the analogues for °cos and °sin of Theorem 19–5 through Theorem 19–19 and their corollaries.
4. Use one of the results of Exercise 3 to show that

$$\cos(45+a)° = \sin(45-a)° \text{ and } \sin(45+a)° = \cos(45-a)°.$$

5. Justify the steps in each of the following computations.
   (a) $\cos 530° = \cos(180 \cdot 2 + 170)° = \cos 170° = -\sin 80°$
   $= -\cos 10°$
   (b) $\sin(-240)° = -\sin 240° = \sin 60° = \cos 30°$

*

As illustrated in Exercise 5 we can reduce the problem of finding the value of °cos or °sin for any argument to that of finding the value of one of these functions for an argument between 0 and 45, inclusive. To do so one needs the following theorems, each of which we have already verified:

(4) °cos is even and °sin is odd.
(5) $\cos(180k+a)° = (-1)^k \cos a°$, $\sin(180k+a)° = (-1)^k \sin a°$
(6) $\cos(90+a)° = -\sin a°$, $\sin(90+a)° = \cos a°$
(7) $\cos(45+a)° = \sin(45-a)°$, $\sin(45+a)° = \cos(45-a)°$

[To recall where the '−' sign belongs in (6), remember that, while cosines and sines of acute angles are both positive, cosines of obtuse angles are negative and sines of obtuse angles are positive.] Not all of

Answers for Part A [cont.

3. 19-10: (a) $\cos(90-a)° = \sin a°$
   (b) $\sin(90-a)° = \cos a°$

   Corollary: $a + b = 90 \Longrightarrow \cos b° = \sin a°$

   19-11: (a) $\cos(2a)° = \cos^2 a° - \sin^2 a°$
   (b) $\sin(2a)° = 2 \sin a° \cos a°$

   Corollary: (a) $\cos(2a)° = 2\cos^2 a° - 1$
   (b) $\cos(2a)° = 1 - 2\sin^2 a°$

   19-12: (a) $\cos(a+180)° = -\cos a° = \cos(a-180)°$
   (b) $\sin(a+180)° = -\sin a° = \sin(a-180)°$

   19-13: (a) $\cos a° \cos b° = [\cos(a-b)° + \cos(a+b)°]/2$
   (b) $\sin a° \sin b° = [\cos(a-b)° - \cos(a+b)°]/2$
   (c) $\sin a° \cos b° = [\sin(a-b)° + \sin(a+b)°]/2$
   (d) $\cos a° \sin b° = -[(a-b)° - \sin(a+b)°]/2$

   19-14: (a) $\cos d° + \cos c° = 2\cos[(c+d)/2]° \cos[(c-d)/2]°$
   (b) $\cos d° - \cos c° = 2\sin[(c+d)/2]° \sin[(c-d)/2]°$
   (c) $\sin d° + \sin c° = 2\sin[(c+d)/2]° \cos[(c-d)/2]°$
   (d) $\sin d° - \sin c° = -2\cos[(c+d)/2]° \sin[(c-d)/2]°$

   19-15: For $k \in I$, $\cos(a+180k)° = (-1)^k \cos a°$,
   $\sin(a+180k)° = (-1)^k \sin a°$.

   19-16: For $k \in I$, $\cos[a+90(2k+1)]° = -(-1)^k \sin a°$ and
   $\sin[a+90(2k+1)]° = (-1)^k \cos a°$.

   Corollary: For $k \in I$,
   (a) $\cos(180k)° = (-1)^k$ (b) $\cos[90(2k+1)]° = 0$
   (c) $\sin(180k)° = 0$ (d) $\sin[90(2k+1)]° = (-1)^k$.

   19-17: (a) $\cos a° = 0 \Longrightarrow \exists_{k \in I}\ a = 90(2k+1)$
   (b) $\sin a° = 0 \Longrightarrow \exists_{k \in I}\ a = 180k$

   Corollary: (a) $\cos a° = 0 \Longleftrightarrow \exists_{k \in I}\ a = 90(2k+1)$
   (b) $\sin a° = 0 \Longleftrightarrow \exists_{k \in I}\ a = 180k$

   19-18: (a) $90(2k-1) < a < 90(2k+1) \Longrightarrow \operatorname{sgn}(\cos a°) = (-1)^k$
   (b) $180k < a < 180(k+1) \Longrightarrow \operatorname{sgn}(\sin a°) = (-1)^k$

   19-19: (a) °cos is decreasing for $0 \le a \le 180$, and
   (b) °sin is increasing for $-90 \le a \le 90$.

4. Since $(45+a) + (45-a) = 90$, these follow from the analogue of the corollary to Theorem 19-10.

5. (a) $530 = 180 \cdot 2 + 170$; °cos has period 360; Theorem 19-16 for $k = 0$; corollary to Theorem 19-10

   (b) °sin is odd; Theorem 19-12; corollary to Theorem 19-10 these results are needed for each problem, but the procedure can be described as follows:

(i) If the given argument is negative, use (4) to reduce the problem to the case of a nonnegative argument. [See Exercise 5(b).]

(ii) If the argument is greater than or equal to 180, use (5) to reduce the problem to the case of a nonnegative argument which is less than 180. [Divide the given argument by 180, and take the quotient for k and the remainder for a.]

(iii) If the argument is greater than or equal to 90, use (6) to reduce the problem to the case of a nonnegative argument which is less than 90.

(iv) If the argument is greater than 45, use (7) to reduce the problem to the case of a nonnegative argument which is less than or equal to 45.

Of course, if at some stage you obtain 0, 90, or 180 as argument, you should be able to complete the evaluation at once.

As an example we shall reduce the problem of finding cos (−470)°.

$$\begin{aligned}
\cos(-470)^\circ &= \cos 470^\circ && \text{[by (4)]}\\
&= \cos 110^\circ && \text{[by (5), since } 470 = 180 \cdot 2 + 110\text{]}\\
&= -\sin 20^\circ && \text{[by (6), since } 110 = 90 + 20\text{]}
\end{aligned}$$

In this example there is no need to carry out step (iv) since $0 \le 20 \le 45$.

Part B

Apply the reduction procedure (i)–(iv) in each of the following cases.

1. cos (−283)°	2. sin 619°	3. cos 415°
4. sin (−117)°	5. cos 191.3°	6. sin 92.8°
7. cos 1800°	8. sin (−731.2)°	9. cos (−520)°
10. sin 47.2°	11. cos 947°	12. sin 540°

*

To complete the procedure of evaluating the degree-functions we make use of a table which lists approximations to the values of these functions for chosen arguments between 0 and 45, inclusive. Such a table is given on page 514 and a portion of it is reproduced here.

°m	°sin	°cos	……	
0	0.0000	1.0000		90
⋮	⋮	⋮		⋮
20	0.3420	0.9397		70
21	0.3584	0.9336		69
⋮	⋮	⋮		⋮
45	0.7071	0.7071	……	45
	°cos	°sin		°m

Suppose, for example, that we wish to compute the number cos 699°. Applying the reduction procedure we find that

$$\cos 699^\circ = -\cos 159^\circ = -\sin 69^\circ = -\cos 21^\circ.$$

Reading down the °cos column of the table and noting the list of arguments in the left-hand column, we see that $\cos 21^\circ \doteq 0.9336$. [The sign '≐' is read as 'is approximately equal to'.] So, we know that

$$(*) \qquad \cos 699^\circ \doteq -0.9336.$$

Notice that, by taking advantage of (7) on page 427, the table is arranged so that step (iv) of the reduction process may be omitted. Using the lower captions in the table and reading up the right-hand column of arguments, we find that $\sin 69^\circ \doteq 0.9336$. So, by steps (i)–(iii) of the reduction procedure above, we again obtain (*).

An extra step is needed if we wish to obtain a reasonable approximation to, say, sin (−429.3)°. Using the reduction procedure we find that

$$\sin(-429.3)^\circ = -\sin(429.3)^\circ = -\sin 69.3^\circ = -\cos 20.7^\circ.$$

Now, cos 20.7° should be between cos 20° and cos 21°. In fact, since 20.7 is 0.7 of the way from 20 to 21 it is reasonable to expect that cos 20.7° is approximately 0.7 of the way from cos 20° to cos 21°. To use the tables to determine a reasonable approximation for cos 20.7°, it is convenient to arrange the work as follows:

$$\begin{array}{llll}
\cos 21^\circ & \doteq & 0.9336 & \\
\cos 20.7^\circ & \doteq & \xrightarrow{\hspace{4em}} & \underline{0.9354} \\
\cos \underline{20^\circ\quad} & \doteq & \underline{0.9397} = & 0.9397 \\
\quad\;\; \underbrace{0.7 & \times & -0.0061}_{} \doteq & -0.0043
\end{array}$$

The procedure is to find the differences in the arguments and in the values of °cos, as indicated, then multiply them together and add the result to cos 20°. The result is that $\cos 20.7^\circ \doteq 0.9354$ and, so, that $\sin(-429.3)^\circ \doteq -0.9354$.

Answers for Part B

1. $\cos(-283)^\circ = \cos 283^\circ = -\cos 103^\circ = \sin 13^\circ$
2. $\sin 619^\circ = -\sin 79^\circ = -\cos 11^\circ$
3. $\cos 415^\circ = \cos 55^\circ = \sin 35^\circ$
4. $\sin(-117)^\circ = -\sin 117^\circ = \cos 27^\circ$
5. $\cos 191.3^\circ = -\cos 11.3^\circ$
6. $\sin 92.8^\circ = \cos 2.8^\circ$
7. $\cos 1800^\circ = \cos 0^\circ = 1$
8. $\sin(-173.2)^\circ = -\sin 173.2^\circ = -\cos 83.2^\circ = -\sin 6.8^\circ$
9. $\cos(-520)^\circ = \cos 520^\circ = \cos 160^\circ = -\sin 70^\circ = -\cos 20^\circ$
10. $\sin 47.2^\circ = \cos 42.8^\circ$
11. $\cos 947^\circ = -\cos 47^\circ = -\sin 43^\circ$
12. $\sin 540^\circ = -\sin 0^\circ = 0$

Answers for Part C

1. $\cos(-283)^\circ \doteq 0.2250$; $\sin 619^\circ \doteq -0.9816$; $\cos 415^\circ \doteq 0.5736$; $\sin(-117)^\circ \doteq 0.8910$; $\cos 191.3^\circ \doteq -0.9805$; $\sin 92.8^\circ \doteq 0.9988$; $\cos 1800^\circ = 1$; $\sin(-173.2)^\circ \doteq -0.1184$; $\cos(-520)^\circ \doteq 0.9397$; $\sin 47.2^\circ \doteq 0.7333$; $\cos 947^\circ \doteq 0.6820$; $\sin 540^\circ = 0$
2. (a) 0.6691 (b) 0.3420 (c) 0.3746
   (d) 0.0523 (e) −0.8192 (f) 0.8387

Angles have the same cosine if and only if they are congruent, and they are congruent if and only if they have the same degree-measure.

*

Part C

1. Complete the computation of approximations to the numbers listed in Part B by using answers to Part B and the table on page 514.
2. Compute each of the following by using Theorem 19–20 and then proceeding as in Exercise 1.
   (a) $\cos(219\pi/45)$ (b) $\sin(-17\pi/9)$ (c) $\cos(29\pi/90)$
   (d) $\sin(59\pi/60)$ (e) $\cos(-29\pi/36)$ (f) $\sin 1$
   [*Hint*: When necessary use 3.14 as an approximation for π.]

*

The table on page 514 can also be used to find an approximation to the degree-measure of an angle whose cosine or sine is given. For example, suppose that $\sin \angle A = 0.9300$. Recalling that $\sin \angle A = {}^\circ\sin\, {}^\circ m(\angle A)$, we look in the table and note that, since $\sin 68^\circ \doteq 0.9272$ and $\sin 69^\circ \doteq 0.9336$, the degree-measure of $\angle A$ [at least if $\angle A$ is acute] is between 68 and 69.

To obtain a closer estimate we compute differences as indicated:

$$\begin{array}{lll} \sin 69^\circ \doteq 0.9336 & & \\ & & \Big] \; 0.0064 \\ \sin \angle A \doteq 0.9300 & & \\ & \Big] \; 0.0028 & \\ \sin 68^\circ \doteq 0.9272 & & \end{array}$$

Since $\frac{28}{64} \doteq 0.4$ [and $69 - 68 = 1$] it is reasonable to estimate ${}^\circ m(\angle A)$ to be about 68.4. This is, however, under the assumption that $\angle A$ is acute. Recalling that angles have the same sine if and only if they are congruent or supplementary, we see that $180 - 68.4$ [that is, 111.6] is another possible approximation to ${}^\circ m(\angle A)$. So, from the assumption that $\sin \angle A = 0.9300$, we may infer that $\angle A$ is an angle of about 68.4° or of about 111.6°. In order to decide which it is we would need more information about $\angle A$.

The ambiguity illustrated in the example does not arise if we are given the cosine of an angle and seek the degree-measure of the angle. For, as we know, angles have the same cosine if and only if they have the same degree-measure. [Explain.] There is a slight complication in case we know that the cosine of an angle is a given negative number. Suppose, for example, that $\cos \angle B = -0.7321$. Since the table does not list any negative values of °cos we use the fact that the cosine of the supplement of $\angle B$ is 0.7321 [Explain.] and use the table to compute the degree-measure of this supplement. Finally, by subtracting from 180 we obtain ${}^\circ m(\angle B)$.

Once we have found the degree-measure of an angle we can find its radian-measure by multiplying by $\pi/180$. [$\pi/180 \doteq 0.0175$]

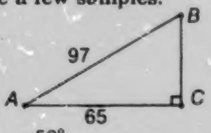
Part D

1. Find approximations to the degree-measures of the angles whose cosine or sine are given below.
   (a) cos ∠A = 0.8347 (b) sin ∠B = 0.6931
   (c) sin ∠C = 0.6494 (d) cos ∠D = −0.8740
2. Find approximations to the radian-measures of the angles referred to in Exercise 1.

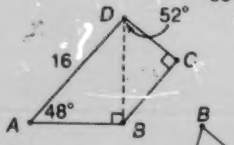
Part E

The techniques developed in the preceding exercises can be used, in conjunction with the cosine law and the sine law, to solve a wide variety of numerical exercises. The following are a few samples.

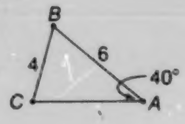
1. Given, in right △ABC, that AB = 65 and CA = 97, as shown, find °m(∠A), °m(∠B), and, without using the Pythagorean Theorem, find BC.



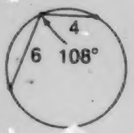
2. Given quadrilateral $ABCD$ find AB, BD, BC, and CD.



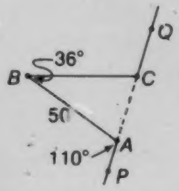
3. Given △ABC with °m∠A = 40, AB = 6, and BC = 4, find °m(∠C), °m(∠B), and AC assuming that ∠C is (i) acute and (ii) obtuse.



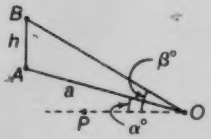
4. Find the radius of a circle in which chords of measures 4 and 6 determine an inscribed angle of 108°.



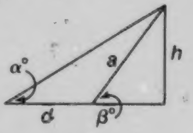
5. (a) Given °m(∠BAP), °m(∠B) and AB, as shown, find AC, BC, and °m(∠BCQ).
   (b) Give formulas for finding AC, BC, and °m(∠BCQ) given that °m(∠BAP) = α, °m(∠B) = β, and AB = a.



6. (a) Give a formula for finding h, given a, α, and β.
   (b) Find h if α = 20, β = 30, and a = 100.
   [∠POA and ∠POB are *the angles of elevation* of A and B, respectively, from O.]



7. (a) Give a formula for finding h, given d, α, and β.
   (b) Find h if d = 25, α = 30, and β = 45.



Answers for Part D

1. (a) °m(∠A) ≐ 33.4 (b) °m(∠B) ≐ 43.9 or 136.1
   (c) °m(∠C) ≐ 40.5 or 139.5 (d) °m(∠D) ≐ 150.9
2. (a) m(∠A) ≐ 0.585 (b) m(∠B) ≐ 0.768 or 2.38
   (c) m(∠C) ≐ 0.709 or 2.44 (d) m(∠D) ≐ 2.64

Answers for Part E

1. °m(∠A) ≐ 47.9, °m(∠B) ≐ 42.1, BC ≐ 71.9
2. AB ≐ 10.71, BD ≐ 11.89, BC ≐ 9.37, CD ≐ 7.32
3. (i) °m(∠C) ≐ 74.6, °m(∠B) ≐ 65.4, AC ≐ 5.7
   (ii) °m(∠C) ≐ 105.4, °m(∠B) ≐ 34.6, AC ≐ 3.5
4. 4.30 [approximately]
5. (a) AC ≐ 30.6, BC ≐ 48.9, °m(∠BCQ) = 106
   (b) AC = a sinβ°/sin(α - β)°, BC = a sin(180 - α)°/sin(α - β)°, °m(∠BCQ) = 180 - α + β
6. (a) h = a sin(β - α)°/sin(90 - β)°
   (b) 34.7 [approximately]
7. (a) h = d sinα° sinβ°/sin(β - α)° [h = a sinβ°, where a = d sinα°/sin(β - α)°
   (b) 34.2 [approximately]

Answers for Exploration Exercises

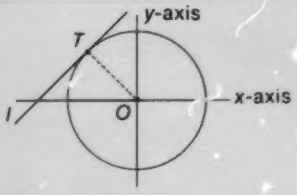
1. The coordinates of T are (0, 1) or (0, −1).
2. The coordinates of T are (1, 0) or (−1, 0).
3. (a) The slope of ℓ is −cos t/sin t.
   (b) (/cos t, 0), (0, /sin t)
   (c) PT = $\sqrt{/\cos^2 t - 1}$ = |sin t/cos t|
   QT = $\sqrt{/\sin^2 t - 1}$ = |cos t/sin t|
   PQ = $\sqrt{/\cos^2 t + /\sin^2 t}$ = |/(cos t sin t)|
4. x cos t + y sin t = 1

These exercises are intended to draw attention to the functions whose values at t are given by '/cos t', '/sin t', 'sin t/cos t' and 'cos t/sin t'. These are, of course, the functions sec, csc, tan, and cot, respectively.

Remark concerning the definitions of 'tan', 'cot', 'sec', and 'csc':

Due to our treatment of /0, the expressions 'sin a/cos a' and '/cos a' do have real values when the value of a is an odd multiple of π/2. We choose, however, to ignore this and to leave tan and sec undefined for such values. One reason for doing so is that we wish to make use of the usual definition of 'identity' which is given in Exercise 1 of Part B. For similar reasons we leave cot and csc undefined at even multiples of π/2.

Exploration Exercises



Given any point T of the unit circle, we know that, for some t, the coordinates of T are $(\cos t, \sin t)$. Let l be the line which is tangent at T to the unit circle.

1. Suppose that l is parallel to the x-axis. What can you say about the coordinates of T?
2. Suppose that l is parallel to the y-axis. What can you say about the coordinates of T?
3. Suppose that l is not parallel to either coordinate axis.
   (a) Give the slope of l in terms of 'cos t' and 'sin t'.
   (b) What are the coordinates of the points of intersection of l with the coordinate axes?
   (c) Given that l intersects the x-axis and y-axis in points P and Q, respectively, compute PT, QT, and PQ in terms of 'sin t' and 'cos t'.
4. Given that l is not parallel to either axis, what is an equation for the line $\overleftrightarrow{OT}$?

19.05 Some Other Circular Functions

From the exploration exercises just completed, we see that any point T of the unit circle which is not a point of one of the coordinate axes has nonzero coordinates $(\cos t, \sin t)$, for some t, and gives rise to

(a) lines whose slopes are $\sin t/\cos t$ and $-\cos t/\sin t$,
(b) points whose coordinates are $(/\cos t, 0)$ and $(0, /\sin t)$, and
(c) segments whose measures are $|\cos t/\sin t|$ and $|\sin t/\cos t|$.

This suggests that we we look into some functions which can be defined in terms of 'cos' and 'sin'. Any such functions are called *circular functions.*

There are four other circular functions which are usually treated together with cos and sin. They are *the tangent function* [for short 'tan'], the *secant function* ['sec'], *the cotangent function* ['cot'], and *the cosecant function* ['csc'].

The first two are defined only for real numbers which are not odd multiples of $\pi/2$ by:

$$\text{(1)}\quad \tan a = \frac{\sin a}{\cos a} \qquad \sec a = \frac{1}{\cos a} \qquad \text{[Definition 19-4(a), (b)]}$$

The last two are defined only for real numbers which are not even multiples of $\pi/2$ by:

$$\text{(2)}\quad \cot a = \frac{\cos a}{\sin a} \qquad \csc a = \frac{1}{\sin a} \qquad \text{[Definition 19-4(c), (d)]}$$

The restrictions on the domains of these functions are to ensure that in (1), $\cos a \neq 0$ and that, in (2), $\sin a \neq 0$.

Recalling the graphs of cos and sin as they are shown here:

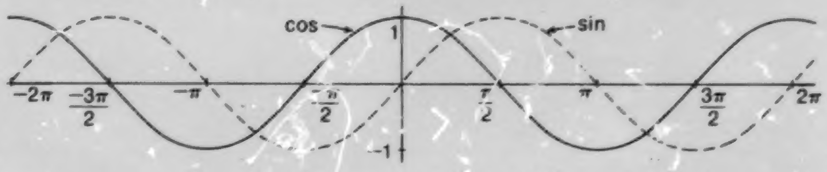
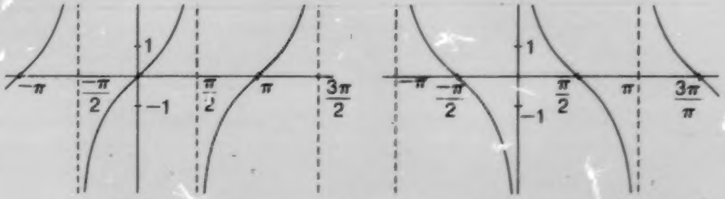


Fig. 19–4

you can easily verify the general shape of the graphs of tan and cot shown below:



(a) Graph of tan (b) Graph of cot

Fig. 19–5

[The vertical dashed lines are not parts of the graphs but show how parts of these graphs fit together.] Similarly, it is easy to verify the general shape of the graphs of sec and csc as they are shown in Fig. 19–6.

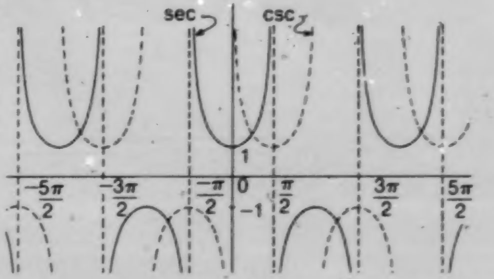


Fig. 19–6

Exercises

Part A

1. For Exercise 1 of Part A on page 414 you made a table of values of cos and sin for certain arguments. Extend this table to include columns for tan, cot, sec, and csc and complete the extended table. [When a function is not defined for a given argument draw a cross [×] in the corresponding box of your table.]
2. **(a)** Show that tan and cot are periodic with period π.
   (b) Show that sec and csc are periodic with period 2π.
3. Show that tan is an odd function. [A function f whose domain does not include all of $\mathscr{R}$ is odd if, for each $x \in \mathscr{D}f$, $-x \in \mathscr{D}f$ and $f(-x) = -f(x)$. Similarly, g is even if, for each $x \in \mathscr{D}g$, $-x \in \mathscr{D}g$ and $g(-x) = g(x)$.]
4. **(a)** Are any of the functions cot, sec, and csc odd functions? If so, tell which.
   (b) Are any of the functions cot, sec, and csc even functions? If so, tell which.
5. Use the results in Exercises 1–4 in drawing your own graphs of tan and cot. Of sec and csc. [Use arguments in $\{x: -\pi \le x \le 3\pi\}$.]

Part B

Prove:

Theorem 19–21

(a) $\sec^2 a - \tan^2 a = 1$ [a not an odd multiple of $\pi/2$]
(b) $\csc^2 a - \cot^2 a = 1$ [a not an even multiple of $\pi/2$]

[The equations in Theorem 19–21, together with '$\cos^2 a + \sin^2 a = 1$', are called *the Pythagorean identities.* An *identity* is an open sentence which has no false instances. The equation of Theorem 19–21(a) has no false instances but, since sec and tan are not defined for odd multiples of $\pi/2$, this equation has meaningless instances such as, for example, '$\sec^2(\pi/2) - \tan^2(\pi/2) = 1$'.]

2. Prove *the subtraction and addition laws for tan*:

   (a) $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$, for neither a, b, nor $a - b$ an odd multiple of $\pi/2$

   (b) $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$, for neither a, b, nor $a + b$ an odd multiple of $\pi/2$

   [*Hint*: Use the definition (1) and the subtraction [addition] laws for cos and sin.]

3. Use the results from Exercise 2 to show that

   $\tan 2a = \dfrac{2 \tan a}{1 - \tan^2 a}$, for a not an odd multiple of $\pi/4$ or of $\pi/2$.

   This is sometimes called *the doubling law for tan.*

Suggestions for the exercises of section 19.05:
(i) Part A may be used for class discussion.
(ii) After one or two examples, Part B may be assigned for homework.
(iii) The discussion on pages 435-438 should be teacher directed.
(iv) Parts C and D may be used for homework.

Answers for Part A

1.

a	cos a	sin a	cot a	tan a	sec a	csc a
$-\pi$	-1	0	×	0	-1	×
$-5\pi/6$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$1/\sqrt{3}$	$-2/\sqrt{3}$	-2
$-3\pi/4$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
$-2\pi/3$	$-1/2$	$-\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	-2	$-2/\sqrt{3}$
$-\pi/2$	0	-1	0	×	×	-1
$-\pi/3$	$1/2$	$-\sqrt{3}/2$	$-1/\sqrt{3}$	$-\sqrt{3}$	2	$-2/\sqrt{3}$
$-\pi/4$	$1/\sqrt{2}$	$-1/\sqrt{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
$-\pi/6$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$-1/\sqrt{3}$	$2/\sqrt{3}$	-2
0	1	0	×	0	1	×
$\pi/6$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	$2/\sqrt{3}$	2
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	2	$2/\sqrt{3}$
$\pi/2$	0	1	0	×	×	1
$2\pi/3$	$-1/2$	$\sqrt{3}/2$	$-1/\sqrt{3}$	$-\sqrt{3}$	-2	$2/\sqrt{3}$
$3\pi/4$	$-1/\sqrt{2}$	$1/\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
$5\pi/6$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-1/\sqrt{3}$	$-2/\sqrt{3}$	2

2. (a) $\tan(a + \pi) = \dfrac{\sin(a + \pi)}{\cos(a + \pi)} = \dfrac{-\sin a}{-\cos a} = \dfrac{\sin a}{\cos a} = \tan a$;

   $\cot(a + \pi) = \dfrac{\cos(a + \pi)}{\sin(a + \pi)} = \dfrac{-\cos a}{-\sin a} = \dfrac{\cos a}{\sin a} = \cot a$

   (b) $\sec(a + 2\pi) = /\cos(a + 2\pi) = /\cos a = \sec a$;

   $\csc(a + 2\pi) = /\sin(a + 2\pi) = /\sin a = \csc a$

3. $\tan(-a) = \sin(-a)/\cos(-a) = -\sin a/\cos a = -\tan a$

4. (a) csc and cot are odd functions.

   (b) sec is an even function.

5. [See Figures 19-5 and 19-6.]

Answers for Part B

1. (a) $\sec^2 a - \tan^2 a = \dfrac{1}{\cos^2 a} - \dfrac{\sin^2 a}{\cos^2 a} = \dfrac{1 - \sin^2 a}{\cos^2 a} = \dfrac{\cos^2 a}{\cos^2 a} = 1$, for a not an odd multiple of $\pi/2$.

   (b) $\csc^2 a - \cot^2 a = \dfrac{1}{\sin^2 a} - \dfrac{\cos^2 a}{\sin^2 a} = \dfrac{1 - \cos^2 a}{\sin^2 a} = \dfrac{\sin^2 a}{\sin^2 a} = 1$, for a not an even multiple of $\pi/2$.

Answers for Part B [cont.]

2. (a) $$\tan(a - b) = \frac{\sin(a - b)}{\cos(a - b)}$$

$$= \frac{\sin a \cos b - \cos a \sin b}{\cos a \cos b + \sin a \sin b}$$

$$= \frac{\dfrac{\sin a \cos b}{\cos a \cos b} - \dfrac{\cos a \sin b}{\cos a \cos b}}{\dfrac{\cos a \cos b}{\cos a \cos b} + \dfrac{\sin a \sin b}{\cos a \cos b}} = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

(b) [Similar, with '+' and '-' interchanged.]

By Exercise 2(b), $\tan 2a = \dfrac{2 \tan a}{1 - \tan^2 a}$

for neither a nor $2a$ an odd multiple of $\pi/2$ — or, equivalently, for a neither an odd multiple of $\pi/2$ nor an odd multiple of $\pi/4$.

[Note that we cannot use the addition formula for tan with $b = (2k + 1)\pi/2$.]

$$\tan[a + (2k + 1)\pi/2] = \frac{\sin[a + (2k + 1)\pi/2]}{\cos[a + (2k + 1)\pi/2]}$$

$$= \frac{(-1)^k \cos a}{-(-1)^k \sin a} = -\frac{\cos a}{\sin a} = -\cot a,$$

for a not an even multiple of $\pi/2$.

(a) $\tan(\pi/2 - a) = \dfrac{\sin(\pi/2 - a)}{\cos(\pi/2 - a)} = \dfrac{\cos a}{\sin a} = \cot a$, for a not an even multiple of $\pi/2$.

(b) $\cot(\pi/2 - a) = \dfrac{\cos(\pi/2 - a)}{\sin(\pi/2 - a)} = \dfrac{\sin a}{\cos a} = \tan a$, for a not an odd multiple of $\pi/2$.

Since $\tan a = \sin a/\cos a$ for $\cos a \neq 0$ and is otherwise undefined it follows that $\tan a = 0$ if and only if $\sin a = 0$, and $\sin a = 0$ if and only if a is an even multiple of $\pi/2$. Similarly, $\cot a = \cos a/\sin a$ for $\sin a \neq 0$ and is otherwise undefined. So, $\cot a = 0$ if and only if $\cos a = 0$, and $\cos a = 0$ if and only if a is an odd multiple of $\pi/2$.

By Theorem 19-14,

$$\frac{\sin a - \sin b}{\sin a + \sin b} = \frac{-2 \cos[(b + a)/2] \sin[(b - a)/2]}{2 \sin[(b + a)/2] \cos[(b - a)/2]}$$

$$= \frac{-\sin[(b - a)/2]/\cos[(b - a)/2]}{\sin[(b + a)/2]/\cos[(b + a)/2]}$$

$$= \frac{-\tan[(b - a)/2]}{\tan[(b + a)/2]} = \frac{\tan[(a - b)/2]}{\tan[(a + b)/2]}.$$

The first step requires that $(b + a)/2$ not be an even multiple of $\pi/2$ and that $(b - a)/2$ not be an odd multiple of $\pi/2$. The second step requires, in addition, that $(b + a)/2$ not be an odd multiple of $\pi/2$. These restrictions are easily seen to be equivalent to those given in the exercise. [The result of this exercise, when applied to a triangle with angle measures α and β, leads to the tangent law of Theorem 19-28. This law is of some use in "solving triangles".]

4. Prove *the reduction formula for tan*:

$$\tan[a + (2k + 1)\pi/2] = \cot a \quad [a \text{ not an even multiple of } \pi/2]$$

5. Prove that
(a) $\tan(\pi/2 - a) = \cot a$ [a not an even multiple of $\pi/2$]
(b) $\cot(\pi/2 - a) = \tan a$ [a not an odd multiple of $\pi/2$]

6. Prove that $\tan a = 0$ if and only if a is an even multiple of $\pi/2$ and that $\cot a = 0$ if and only if a is an odd multiple of $\pi/2$.

7. Prove that

$$\frac{\sin a - \sin b}{\sin a + \sin b} = \frac{\tan[(a - b)/2]}{\tan[(a + b)/2]} \quad [a + b \text{ not a multiple of } \pi \text{ and } a - b \text{ not an odd multiple of } \pi].$$

[*Hint*: Recall Theorem 19-14.]

*

In the preceding exercises you have proved a number of theorems concerning tan, sec, cot and csc. All of them, as you should have found, come quite easily from corresponding formulas concerning cos and sin.

Theorem 19-22 tan and cot are periodic with period π; sec and csc are periodic with period 2π.

Theorem 19-23 tan, cot, and csc are odd functions; sec is an even function.

Theorem 19-24

(a) $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$ [$a, b, a - b$ not odd multiples of $\pi/2$]

(b) $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$ [$a, b, a + b$ not odd multiples of $\pi/2$]

Corollary For a not an odd multiple of $\pi/4$ or of $\pi/2$,

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}.$$

Theorem 19-25 For $k \in I$ and a not an even multiple of $\pi/2$

$$\tan[a + (2k + 1)\pi/2] = -\cot a.$$

Theorem 19-26

(a) $\tan a = 0 \longleftrightarrow a$ is an even multiple of $\pi/2$
(b) $\cot a = 0 \longleftrightarrow a$ is an odd multiple of $\pi/2$

The emphasis on tan in the preceding theorems suggests that sec, csc, and cot are of less importance than cos, sin, and tan. This is the case. For most purposes, as regards sec, csc, and cot it is sufficient to recall that they are reciprocals of cos, sin, and tan, respectively.

Because of the importance of tan in geometric applications it is customary to introduce a tangent function for angles:

Definition 19–5 For $\angle A$ not a right angle,

$$\tan \angle A = \sin \angle A / \cos \angle A.$$

from which we have:

Theorem 19–27 For $\angle A$ not a right angle,

$$\tan \angle A = \tan m(\angle A).$$

in which the 'tan' on the left refers to the function introduced in Definition 19–5 and the 'tan' on the right refers to the tangent function we have been studying in this section. [It would be easy to define '$\tan^{\perp} \angle A$', but we shall not need this function.]

Exercise 7 of Part B, together with the sine law, yields a relation, called *the tangent law*, between the sides and angles of a triangle. To obtain this, suppose, as usual that, in $\triangle ABC$, α, β, and γ are the radian-measures of $\angle A$, $\angle B$, and $\angle C$, respectively, and that a, b, and c are the measures of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$. Since α, β, and γ are positive and $\alpha + \beta + \gamma = \pi$ it follows that

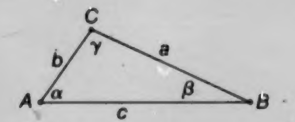


Fig. 19–7

$$0 < \alpha + \beta < \pi \text{ and } -\pi < \alpha - \beta < \pi. \qquad \text{[Explain.]}$$

In particular, $\alpha + \beta$ is not a multiple of π and $\alpha - \beta$ is not an odd multiple of π. It follows from Exercise 7 that

$$\frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta} = \frac{\tan [(\alpha - \beta)/2]}{\tan [(\alpha + \beta)/2]}. \tag{3}$$

Since, by the sine law, $\sin \alpha$ and $\sin \beta$ are proportional to a and b it follows from (3) that

$$\frac{a - b}{a + b} = \frac{\tan [(\alpha - \beta)/2]}{\tan [(\alpha + \beta)/2]}.$$

[Explain. Note that, by the sine law, there is a number k such that $\sin \alpha = ka$ and $\sin \beta = kb$.]

As a consequence, we have

Theorem 19–28 If α and β are the radian-measures of two angles of a triangle, and a and b are the measures of the sides opposite these angles, then

$$\frac{a - b}{a + b} = \frac{\tan [(\alpha - \beta)/2]}{\tan [(\alpha + \beta)/2]}.$$

As in the case of cos and sin there are degree-functions °tan, °cot, °sec, and °csc corresponding with the functions tan, cot, sec, and csc. For example,

$${}^\circ\tan a = \tan (\pi a/180) \qquad [a \text{ not an odd multiple of } 90]$$

[See Definition 19–3.] and

$$\tan a = {}^\circ\tan (180a/\pi) \qquad [a \text{ not an odd multiple of } \pi/2].$$

[See Theorem 19–20.] Theorems like Theorems 19–21 through Theorem 19–26 and their corollaries, as well as a theorem like Theorem 19–28, hold for the degree-functions, and are obtained in the same way as are those for °cos and °sin. The table on page 514 lists approximations to the values of these four new functions as well as of °sin and °cos. The procedure for computing values of these functions by reduction and use of the table is much like that for °cos and °sin as given on pages 427–430. In the case of °sec and °csc the reduction process is the same as for °cos and °sin. The reduction process for °tan and °cot is based on the theorems:

(4) °tan and °cot are odd

(5) $\tan (180k + a)^\circ = \tan a^\circ$, $\cot (180k + a)^\circ = \cot a^\circ$

(6) $\tan (90 + a)^\circ = -\cot a^\circ$, $\cot (90 + a)^\circ = -\tan a^\circ$

(7) $\tan (45 + a)^\circ = \cot (45 - a)^\circ$, $\cot (45 + a)^\circ = \tan (45 - a)^\circ$

The explanation of how the tangent law is obtained from (3) and the sine law is very nearly given in the hint. By the sine law,

$$\frac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \frac{ka - kb}{ka + kb} = \frac{k(a - b)}{k(a + b)} = \frac{a - b}{a + b}.$$

*

Answers for Part C

1. (a) $\tan 470^\circ = \tan 110^\circ = -\cot 20^\circ \doteq -2.7475$
   (b) $\cot(-240.3)^\circ = -\cot 240.3^\circ = -\cot 60.3^\circ = -\tan 29.7^\circ \doteq -0.5705$
   (c) $\sec(-405)^\circ = \sec 405^\circ = \sec 45^\circ = \sqrt{2} \doteq 1.414$
2. (a) $^\circ m(\angle A) \doteq 66.5$ (b) $^\circ m(\angle B) \doteq 147.5$
   (c) $^\circ m(\angle C) \doteq 67.8$ or 112.2
3. $^\circ m(\angle A) \doteq 37.9$, $^\circ m(\angle B) \doteq 52.1$, $CA \doteq 11.4$
4. (a) $h = d/[\cot\alpha - \cot\beta]$
   (b) 64 [approximately]
5. With $a = 15$, $b = 10$, $\alpha + \beta = 180 - 64$, the tangent law yields:

$$\frac{5}{25} = \frac{\tan[(\alpha - \beta)/2]^\circ}{\tan 58^\circ}$$

So, $\tan[(\alpha - \beta)/2]^\circ = \tan 58^\circ/5 \doteq 1.6003/5 \doteq 0.3201$. It follows that $(\alpha - \beta)/2 \doteq 17.8^\circ$. Since $(\alpha + \beta)/2 = 58^\circ$ it follows that $\alpha \doteq 75.8$ and $\beta \doteq 40.2$. So, $^\circ m(\angle A) \doteq 75.8$ and $^\circ m(\angle B) = 40.2$. [The method for finding the degree-measures of $\angle A$ and $\angle B$ is somewhat shorter than using the cosine law to find AB and the sine law to find the degree-measures of the angles.]

The theorems (4)–(7) are, of course, subject to easily formulated restrictions. Subject to the appropriate restrictions (4)–(7) follow from (4)–(7) on page 427 and the facts that

$$\tan a^\circ = \frac{\sin a^\circ}{\cos a^\circ} \qquad [a \text{ not an odd multiple of } 90]$$

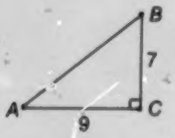
and

$$\cot a^\circ = \frac{\cos a^\circ}{\sin a^\circ} \qquad [a \text{ not an even multiple of } 90].$$

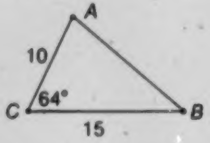
The interpolation procedures for finding values or degree-measures between those listed in the table are similar to those for °cos and °sin. If we are given the cosecant of an angle there are two [supplementary] possibilities for the degree-measure of the angle. If we are given a negative number as the tangent, cotangent, or secant of an angle then we use the tables to find the degree-measure of the supplement of the given angle and subtract the result from 180.

Part C

1. Evaluate each of the following.
   (a) $\tan 470^\circ$ (b) $\cot(-240 \cdot 3)^\circ$ (c) $\sec(-405)^\circ$
2. Find approximations to the degree-measures of the given angles.
   (a) $\tan \angle A = 2.3009$ (b) $\cot \angle B = -1.5699$ (c) $\csc \angle C = 1.0800$
3. Given, in right $\triangle ABC$, that $AB = 9$ and $BC = 7$, as shown, find $^\circ m(\angle A)$, $^\circ m(\angle B)$, and, without using the Pythagorean theorem, find AB.



4. Solve Exercise 7 of Part E on page 431 by expressing 'h' in terms of 'd', 'cot α°' and 'cot β°'.
5. Given $\triangle ABC$ with $^\circ m(\angle C) = 64$, $BC = 15$, and $CA = 10$, use the tangent law [Theorem 19–26] for degree-measures to find $^\circ m(\angle A)$ and $^\circ m(\angle B)$. [*Hint*: If $^\circ m(\angle A) = \alpha$ and $^\circ m(\angle B) = \beta$ then $\alpha + \beta = 180 - 64$. Use this and the tangent law to find $\alpha - \beta$.]



Part D

1. Enlarge the table you made for Exercise 1 of Part F on page 425 by adding a line for tan and one for sec.
2. Since, for a not an odd multiple of $\pi/2$, $\sec^2 a = 1 + \tan^2 a$ [Why?] it follows that $|\sec a| = \sqrt{1 + \tan^2 a}$ and, so, that

$$\cos a = \frac{\operatorname{sgn}(\cos a)}{\sqrt{1 + \tan^2 a}}, \text{ for } a \text{ not an odd multiple of } \pi/2.$$

[Recall that sgn has the value 1 for positive arguments and the value -1 for negative arguments, and that $\operatorname{sgn}(0) = 0$.] With the same restriction,

$$\sin a = \cos a \tan a = \frac{\operatorname{sgn}(\cos a) \tan a}{\sqrt{1 + \tan^2 a}}.$$

In each exercise, find the cosine and sine of the indicated argument.

(a) $\tan a = \sqrt{3}$, $0 < a < \pi/2$ (b) $\tan a = \sqrt{3}$, $\pi < a < 3\pi/2$
(c) $\tan a = 4/3$, $\cos a < 0$ (d) $\tan a = -4/3$, $\sin a < 0$
(e) $\sec a = 2$, $\sin a < 0$ (f) $\sec a = 1/2$, $0 < a < \pi$
(g) $\cot a = -\sqrt{3}$, $0 \le a \le \pi$ (h) $\cot a = 1$, $\cos a < 0$

19.06 Halving Formulas

The three doubling formulas:

$$\cos 2a = 2\cos^2 a - 1, \quad \cos 2a = 1 - 2\sin^2 a, \quad \sin 2a = 2 \sin a \cos a$$

from Theorem 19–11 and its corollary can be used as the basis of halving formulas. From the first two we obtain:

$$\cos^2 a = \frac{1 + \cos 2a}{2} \text{ and: } \sin^2 a = \frac{1 - \cos 2a}{2}$$

which have as instances:

$$(1) \qquad \cos^2 \frac{a}{2} = \frac{1 + \cos a}{2} \text{ and: } \sin^2 \frac{a}{2} = \frac{1 - \cos a}{2}$$

By division we obtain, for a not an odd multiple of π:

$$(2) \qquad \tan^2 \frac{a}{2} = \frac{1 - \cos a}{1 + \cos a}$$

Answers for Part D

1.

	$\overline{0, \pi/2}$	$\overline{\pi/2, \pi}$	$\overline{\pi, 3\pi/2}$	$\overline{3\pi/2, 2\pi}$
cos	> 0, ↓	< 0, ↓	< 0, ↑	> 0, ↑
sin	> 0, ↑	> 0, ↓	< 0, ↓	< 0, ↑
tan	> 0, ↑	< 0, ↑	> 0, ↑	< 0, ↑
sec	> 0, ↑	< 0, ↑	< 0, ↓	> 0, ↓

Notice that although tan is not an increasing function it is increasing on each interval of measure π on which it is defined. The secant behaves in the way opposite to that in which its reciprocal, the cosine, does.

2. (a) $\cos a = 1/2$, $\sin a = \sqrt{3}/2$
   (b) $\cos a = -1/2$, $\sin a = -\sqrt{3}/2$
   (c) $\cos a = -3/5$, $\sin a = -4/5$
   (d) $\cos a = 3/5$, $\sin a = -4/5$
   (e) $\cos a = 1/2$, $\sin a = -\sqrt{3}/2$
   (f) Impossible. [$|\sec a| \ge 1$]
   (g) $\cos a = -\sqrt{3}/2$, $\sin a = 1/2$
   (h) $\cos a = -1/\sqrt{2}$, $\sin a = -1/\sqrt{2}$

Sample Quiz

1. On the same picture of the number plane, make sketches of cos and sec for arguments in $\overline{-3\pi/4, \pi}$.
2. On a second picture of the number plane, make sketches of tan and 1/tan for arguments in $\overline{-3\pi/4, \pi}$.
3. On a third picture of the number plane, make sketches of sin and 1/sin for arguments in $\overline{-3\pi/4, \pi}$.
4. (a) What is the domain of tan?

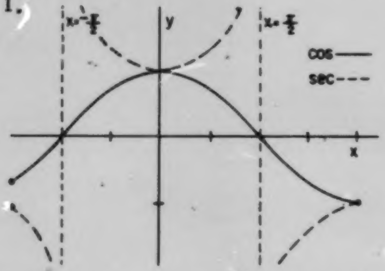
   (b) What is the range of sec?

   (c) What is the domain of 1/tan?

   ☆(d) Do the functions 1/tan and sec have any points in common? Explain your answer.

Key to Sample Quiz

1.



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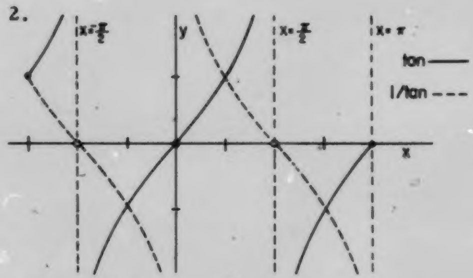

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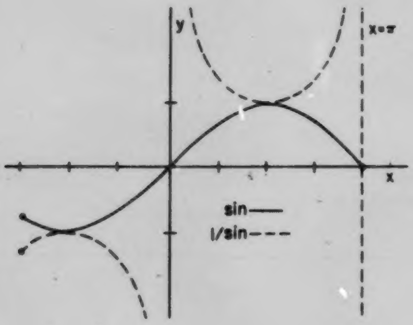

4. (a) $\{x\colon x$ is not an odd multiple of $\pi/2\}$

   (b) $\{x\colon x \geq 1 \text{ or } x \leq -1\}$

   (c) $\{x\colon x$ is not a multiple of $\pi/2\}$

   (d) Yes. There is a number t such that $1/\tan t = \sec t$, for there is a number t such that $\cos t/\sin t = 1/\cos t$, that is, such that $\sin^2 t + \sin t - 1 = 0$. That there is such a number t follows from the facts that the latter equation is equivalent to '$\sin t = (\sqrt{5} - 1)/2$', $-1 \leq (\sqrt{5} - 1)/2 \leq 1$, and $(\sqrt{5} - 1)/2$ is not a multiple of $\pi/2$.

We must *not* assume from, say, the first formula in (1) that

$$(*)\qquad \cos\frac{a}{2} = \sqrt{\frac{1+\cos a}{2}},$$

for the values of the expression on the right side of (*) are all nonnegative whereas cos $(a/2)$ sometimes has negative values. For example, with $a = 4\pi/3$, $\cos(a/2) = \cos(2\pi/3) = -\frac{1}{2}$ while $\sqrt{(1+\cos a)/2} = \sqrt{[1+\cos(2\pi/3)]/2} = \sqrt{(1+-1/2)/2} = \frac{1}{2}$. Evidently, (*) holds only for values of 'a' for which $\cos(a/2) \ge 0$. We can, however, obtain formulas which hold for all values of 'a' by using the signum function (sgn) which, as you should recall, has the value 1 for positive arguments. This leads us to:

Theorem 19-29

(a) $\cos\frac{a}{2} = \operatorname{sgn}\left(\cos\frac{a}{2}\right)\sqrt{\frac{1+\cos a}{2}}$

(b) $\sin\frac{a}{2} = \operatorname{sgn}\left(\sin\frac{a}{2}\right)\sqrt{\frac{1-\cos a}{2}}$

(c) $\tan\frac{a}{2} = \operatorname{sgn}\left(\tan\frac{a}{2}\right)\sqrt{\frac{1-\cos a}{1+\cos a}}$

[Formula (c) holds for a not an odd multiple of π.]

It is not difficult to see that Theorem 19-29 has the corollary:

Corollary

(a) $\cos\frac{a}{2} = \sqrt{\frac{1+\cos a}{2}}$ $\quad[-\pi \le a \le \pi]$

(b) $\sin\frac{a}{2} = \sqrt{\frac{1-\cos a}{2}}$ $\quad[0 \le a \le 2\pi]$

(c) $\tan\frac{a}{2} = \sqrt{\frac{1-\cos a}{1+\cos a}}$ $\quad[0 \le a < \pi]$

[Where have you seen a formula like formula (a) of the corollary?] Of course, the formulas of the corollary are valid for other ranges of arguments besides those specified in the corollary.

We can obtain a halving formula for tan that is sometimes more useful than (c) by using the first and third of the doubling formulas given at the beginning of this section:

$$\tan\frac{a}{2} = \frac{\sin(a/2)}{\cos(a/2)} = \frac{2\sin(a/2)\cos(a/2)}{2\cos^2(a/2)} = \frac{\sin a}{1+\cos a},$$

0

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The expression '$\operatorname{sgn}(\cos\frac{a}{2})$' in Theorem 19-29(a) replaces, and is more specific than the more customary '±'. You might call students' attention to the customary forms.

Formula (a) of the corollary is related to the formula for the cosine of half an angle developed in Exercise 5 of Part C on page 227.

* * *

Suggestions for the exercises of section 19.06:

(i) Part A may be assigned for homework.

(ii) Part B should be teacher directed.

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The fundamental reason for the existence of a formula like that of Theorem 19-30 is that the function whose value is given by '$\tan\frac{a}{2}$' has, like cos and sin, the period 2π. There are no such formulas for $\cos\frac{a}{2}$ and $\sin\frac{a}{2}$ because the corresponding functions have period 4π but do not have period 2π.

for a not an odd multiple of π.

Theorem 19–30

$$\tan\frac{a}{2} = \frac{\sin a}{1 + \cos a}\ [a \text{ not an odd multiple of } \pi]$$

Exercises

Part A

1. Check the halving formulas for cos and sin by using them to compute cos ($\pi/4$) and sin ($\pi/4$). [*Hint*: $\pi/4 = (\pi/2)/2$, and you know the value of cos at $\pi/2$.]
2. Check the halving formulas for cos, sin, and tan by using them to compute the values of these functions at $\pi/6$. At $\pi/3$. [You should know the values of cos and sin at $\pi/6$, $\pi/3$, and $2\pi/3$.]
3. Use the halving formulas to compute cos, sin, and tan at $\pi/12$, $\pi/8$, $-5\pi/12$, and $7\pi/8$.
4. Find the values of cos and sin for an argument a such that $\tan 2a = -5/12$ and $0 \le 2a < \pi$.
5. Repeat Exercise 4 for tan $2a$ equal to
   (a) $3/4$; (b) $-4/3$; (c) $12/5$.

*Part B

For some purposes it is convenient to introduce a seventh circular function, crc [pronounced 'sirk'] by:

Definition 19–6 $\operatorname{crc} a = \tan\dfrac{a}{2}$ [a not an odd multiple of π]

1. What is the period of crc?
2. Prove:

Theorem 19–31 For a not an odd multiple of π,

(a) $\operatorname{crc} a = \dfrac{\sin a}{1 + \cos a}$

(b) $\cos a = \dfrac{1 - \operatorname{crc}^2 a}{1 + \operatorname{crc}^2 a}$

(c) $\sin a = \dfrac{2 \operatorname{crc} a}{1 + \operatorname{crc}^2 a}$

[*Hint for (b)*: By Theorem 19–30,

$$\operatorname{crc}^2 a = \frac{\sin^2 a}{(1 + \cos a)^2} = \frac{1 - \cos^2 a}{(1 + \cos a)^2}.]$$

Answers for Part A

1. $\cos(\pi/4) = \sqrt{[1 + \cos(\pi/2)]/2} = \sqrt{1/2} = 1/\sqrt{2}$;
   $\sin(\pi/4) = \sqrt{[1 - \cos(\pi/2)]/2} = \sqrt{1/2} = 1/\sqrt{2}$
2. $\cos(\pi/6) = \sqrt{[1 + \cos(\pi/3)]/2} = \sqrt{[1 + 1/2]/2} = \sqrt{3/4} = \sqrt{3}/2$;
   $\sin(\pi/6) = \sqrt{[1 - \cos(\pi/3)]/2} = \sqrt{[1 - 1/2]/2} = \sqrt{1/4} = 1/2$;
   $\tan(\pi/6) = \sqrt{\dfrac{1 - \cos(\pi/3)}{1 + \cos(\pi/3)}} = \sqrt{\dfrac{1 - 1/2}{1 + 1/2}} = \sqrt{1/3} = 1/\sqrt{3}$
3. $\cos(\pi/12) = \sqrt{2 + \sqrt{3}}/2$, $\sin(\pi/12) = \sqrt{2 - \sqrt{3}}/2$, $\tan(\pi/12) = 2 - \sqrt{3}$;
   $\cos(\pi/8) = \sqrt{2 + \sqrt{2}}/2$, $\sin(\pi/8) = \sqrt{2 - \sqrt{2}}/2$, $\tan(\pi/8) = \sqrt{2} - 1$;
   $\cos(-5\pi/12) = \sqrt{2 - \sqrt{3}}/2$, $\sin(-5\pi/12) = -\sqrt{2 + \sqrt{3}}/2$,
   $\tan(-5\pi/12) = -(2 + \sqrt{3})$;
   $\cos(7\pi/8) = -\sqrt{2 + \sqrt{2}}/2$, $\sin(7\pi/8) = \sqrt{2 - \sqrt{2}}/2$, $\tan(7\pi/8) = 1 - \sqrt{2}$
4. $\cos a = 1/\sqrt{26}$, $\sin a = 5/\sqrt{26}$
5. (a) $\cos a = 3/\sqrt{10}$, $\sin a = 1/\sqrt{10}$
   (b) $\cos a = 1/\sqrt{5}$, $\sin a = 2/\sqrt{5}$
   (c) $\cos a = 3/\sqrt{13}$, $\sin a = 2/\sqrt{13}$

Answers for Part B

1. crc has period 2π.
2. (a) This is by definition and Theorem 19-30.
   (b) By Definition 19-6 and Theorem 19-29(c),

$$\operatorname{crc}^2 a = \frac{1 - \cos a}{1 + \cos a}.$$

So, $(1 + \cos a)\operatorname{crc}^2 a = 1 - \cos a$ and, hence, $\cos a(1 + \operatorname{crc}^2 a) = 1 - \operatorname{crc}^2 a$ and

$$\cos a = \frac{1 - \operatorname{crc}^2 a}{1 + \operatorname{crc}^2 a}.$$

[Alternatively, by Definition 19-6 and Theorem 19-30,

$$\operatorname{crc}^2 a = \frac{\sin^2 a}{(1 + \cos a)^2} = \frac{1 - \cos^2 a}{(1 + \cos a)^2}$$

$$= \frac{(1 - \cos a)(1 + \cos a)}{(1 + \cos a)^2} = \frac{1 - \cos a}{1 + \cos a}.$$

Then, proceed as in the preceding derivation.]

   (c) By the doubling formula for tan and Definition 19-6,

$$\tan a = \frac{2 \operatorname{crc} a}{1 - \operatorname{crc}^2 a}.$$

So, since $\sin a = \tan a \cos a$,

$$\sin a = \frac{2 \operatorname{crc} a}{1 + \operatorname{crc}^2 a}.$$

19.07 Summary of Main Results

The preceding sections contain a large number of theorems concerning the circular functions. In the present section we shall summarize the results which you need to remember and point out how to obtain from these the other results which you need to be able to use. The numbered results are the ones which you should commit to memory; the lettered results are easy consequences of the numbered ones. You will find that you will come to remember the latter, also, if you spend some time deriving each of them a few times from the former.

To begin with, you need to be thoroughly familiar with the intuitive description of the winding function W as illustrated in Figure 19–1 on page 409. In particular you should be aware that

$$(1)\quad \begin{cases} W(a) = U \text{ or } W(a) = U' \text{ if and only if } a \text{ is an even multiple} \\ \qquad \text{of } \pi/2, \text{ and} \\ W(a) = V \text{ or } W(a) = V' \text{ if and only if } a \text{ is an odd multiple} \\ \qquad \text{of } \pi/2. \end{cases}$$

You then need the definitions of the circular functions:

$$(2)\quad \begin{cases} (\cos a, \sin a) = W(a), \\ \tan a = \dfrac{\sin a}{\cos a} \text{ and } \sec a = \dfrac{1}{\cos a} \\ \qquad\qquad [a \text{ not an odd multiple of } \pi/2] \\ \cot a = \dfrac{\cos a}{\sin a} \text{ and } \csc a = \dfrac{1}{\sin a} \\ \qquad\qquad [a \text{ not an even multiple of } \pi/2] \end{cases}$$

and you should have a good mental image of, and be able to make quick sketches of, graphs of these functions. Many of the properties of the circular functions are easily guessed – or remembered – when one glances at their graphs. Incidentally, the restrictions in (2) are easily recalled if you remember (1) and that these restrictions are intended to rule out consideration of multiplication by /0. In fact, it follows from (1) and (2) that

$$(a)\quad \begin{cases} \cos a \text{ [and } \cot a] \text{ is } 0 \text{ if and only if } a \text{ is an odd multiple} \\ \qquad \text{of } \pi/2, \text{ and} \\ \sin a \text{ [and } \tan a] \text{ is } 0 \text{ if and only if } a \text{ is an even multiple} \\ \qquad \text{of } \pi/2. \end{cases}$$

Also, sec and csc are never zero. [Why?]

The main point of this section is to stress that most of the numerous theorems concerning the circular functions are easy consequences of a few of them — including, of course, definitions. So, it is more sensible to familiarize one's self with ways of deriving the former from the latter rather than to attempt to memorize all theorems. And, after deriving a theorem a few times one reaches the point at which either this derivation is an automatic process or one has memorized the theorem.

The functions sec and csc are never 0 because, wherever they are defined they are reciprocals of nonzero values of cos and sin; and reciprocals of nonzero real numbers are never 0.

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The unit circle is $\{(x, y): x^2 + y^2 = 1\}$. So, since, for any a, (cos a, sin a) is a point of the unit circle it follows that, for any a, $\cos^2 a + \sin^2 a = 1$. The other two Pythagorean identities follow from the first and the definitions in (2). For example,

$$\csc^2 a - \cot^2 a = \frac{1}{\sin^2 a} - \frac{\cos^2 a}{\sin^2 a} = \frac{1 - \cos^2 a}{\sin^2 a} = \frac{\sin^2 a}{\sin^2 a} = 1$$

for values of 'a' for which $\sin a \neq 0$. But, it is just for such values that csc and cot are defined.

Using (6) we see that, for example,

$$\cos(a + 2\pi) = (-1)^2 \cos a = \cos a$$

and that

$$\tan(a + \pi) = \frac{\sin(a + \pi)}{\cos(a + \pi)} = \frac{-1 \cdot \sin(a + \pi)}{-1 \cdot \cos(a + \pi)}$$

$$= \frac{\sin(a + \pi)}{\cos(a + \pi)} = \tan(a + \pi).$$

So, the periodicity properties in (5) follow from (6) and definitions.

The Pythagorean identities:

$$(3)\qquad \begin{cases} \cos^2 a + \sin^2 a = 1 \\ \sec^2 a - \tan^2 a = 1 \\ \csc^2 a - \cot^2 a = 1 \end{cases}$$

are used so frequently that it is well to memorize them. [On the other hand, the first follows immediately from the fact that the range of W is contained in the unit circle, and the others follow from the first and the definitions given in (2). Explain.]

You also need to have well in mind that

(4) cos and sec are even; sin, tan, cot, and csc are odd,

and that

(5) cos, sin, sec, and csc have period 2π;
tan and cot have period π.

Actually, as to (4), it is sufficient to remember that cos is even and sin is odd—the rest follows at once from this and the definitions in (2). [Explain.] As to (5), it follows from the result (6), below. [Explain.]

As to reduction formulas it is sufficient to remember

$$(6)\qquad \cos(a + k\pi) = (-1)^k \cos a,\ \sin(a + k\pi) = (-1)^k \sin a$$

$$(7)\qquad a + b = \pi/2 \longrightarrow \cos b = \sin a$$
$$\longrightarrow \cot b = \tan a\ [a \text{ not an odd multiple of } \pi/2$$
$$(\text{or, equivalently, } b \text{ not an even multiple of } \pi/2)]$$

and, perhaps,

$$(8)\qquad \begin{cases} \cos(a + \pi/2) = -\sin a,\ \sin(a + \pi/2) = \cos a, \\ \cot(a + \pi/2) = -\tan a,\ \tan(a + \pi/2) = -\cot a \end{cases}$$

[with appropriate restrictions on the last two]. Note that the last two results in (8) follow immediately from the first two and definitions, and that the first two are easy to derive from (4) and (7). [By (7), $\cos(\pi/2 - a) = \sin a$ and, so, $\cos(a + \pi/2) = \sin(-a) = -\sin a$, by (4).] As to (7), itself, to say that $a + b = \pi/2$ amounts to saying that the midpoint of $\overline{a, b}$ is $\pi/4$. With this in mind, the intersection of graphs of cos and sin at $(\pi/4, \sqrt{2}/2)$, and of tan and cot at $(\pi/4, 1)$, makes it easy to recall (7). [Sketch the graphs in question and verify the claim just made.]

The more general reduction formulas:

$$(b)\qquad \begin{cases} \cos[a + (2k + 1)\pi/2] = -(-1)^k \sin a, \\ \sin[a + (2k + 1)\pi/2] = (-1)^k \cos a \\ \cot[a + (2k + 1)\pi/2] = -\tan a, \\ \tan[a + (2k + 1)\pi/2] = -\cot a \end{cases}$$

are seldom needed but can be derived from the addition formulas for cos and sin [see below] and the fact that

$$(c)\qquad \cos[(2k + 1)\pi/2] = 0 \text{ and } \sin[(2k + 1)\pi/2] = (-1)^k.$$

The first equation in (c) is an immediate consequence of the first part of (a), but the second [which is related to (1)] is new. The latter need be memorized only if you wish to be able to obtain (b) easily from the addition formulas.

Some of the remaining properties of the circular functions are best summarized in a pair of tables:

a	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos a$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0
$\sin a$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\tan a$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	╳

(9)

	$0, \pi/2$	$\pi/2, \pi$	$\pi, 3\pi/2$	$3\pi/2, 2\pi$
$\cos a$	> 0, ↓	< 0, ↑	< 0, ↓	> 0, ↑
$\sin a$	> 0, ↑	> 0, ↓	< 0, ↓	< 0, ↑
$\tan a$	> 0, ↑	< 0, ↑	> 0, ↑	< 0, ↑

The content of the second table is easily read off from sketches of graphs of cos, sin, and tan. Using the data in the first table and (6)–(8), it is easy to find the values of the functions for other special arguments. [$\sin(2\pi/3) = \sin(\pi/6 + \pi/2) = \cos(\pi/6) = \sqrt{3}/2$. Practice other computations of this sort.]

In the text, several of the preceding results were derived with the help of the first of the subtraction formulas:

$$(10)\quad \begin{cases} \cos(a-b) = \cos a \cos b + \sin a \sin b \\ \sin(a-b) = \sin a \cos b - \cos a \sin b \\ \tan(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}\ [\text{neither } a, b, \text{ nor } a-b \text{ an odd multiple of } \pi] \end{cases}$$

It is quite important to memorize these formulas although, as you have seen, the second can be derived with the help of the first, and the third is a rather easy consequence of the first two.

From (10) and (4) it is very easy to derive the addition formulas:

$$(\mathrm{d})\quad \begin{cases} \cos(a+b) = \cos a \cos b - \sin a \sin b \\ \sin(a+b) = \sin a \cos b + \cos a \sin b \\ \tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}\ [\text{neither } a, b, \text{ nor } a+b \text{ an odd multiple of } \pi/2] \end{cases}$$

—just substitute '$-b$' for 'b' in (10). If you derive (d) from (10) a few times you will probably find that you have memorized all six formulas.

As previously remarked, the general reduction formulas (b) for cos and sin are easy consequences of (c) and (d). For example,

$$\begin{aligned} \cos[a + (2k+1)\pi/2] &= \cos a \cos[(2k+1)\pi/2] \\ &\quad - \sin a \sin[(2k+1)\pi/2] \\ &= \cos a \cdot 0 - \sin a \cdot (-1)^k \\ &= -(-1)^k \sin a. \end{aligned}$$

Practicing the preceding and the corresponding computation for $\sin[a + (2k+1)\pi/2]$ will serve to help you memorize the procedure and, perhaps, the results.

It is not necessary to memorize the formulas of Theorems 19–13 and 19–14:

$$(\mathrm{e})\quad \begin{cases} \cos a \cos b = [\cos(a-b) + \cos(a+b)]/2 \\ \sin a \sin b = [\cos(a-b) - \cos(a+b)]/2 \\ \sin a \cos b = [\sin(a-b) + \sin(a+b)]/2 \\ \cos a \sin b = -[\sin(a-b) - \sin(a+b)]/2 \end{cases}$$

and:

$$(\mathrm{f})\quad \begin{cases} \cos d + \cos c = 2 \cos[(c+d)/2] \cos[(c-d)/2] \\ \cos d - \cos c = -2 \sin[(c+d)/2] \sin[(c-d)/2] \\ \sin d + \sin c = 2 \sin[(c+d)/2] \cos[(c-d)/2] \\ \sin d - \sin c = -2 \sin[(c+d)/2] \sin[(c-d)/2] \end{cases}$$

You should, however, practice deriving those of the first set from the subtraction and addition formulas (10) and (d). And you should practice obtaining instances for those in the second set in the way illustrated on page 420.

Although they are easy consequences of the addition laws (d) [and the first of the Pythagorean identities (3)], it is well worthwhile to memorize the doubling formulas:

$$(11)\quad \begin{cases} \cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a \\ \sin 2a = 2 \sin a \cos a \\ \tan 2a = \dfrac{2 \tan a}{1 - \tan^2 a}\ [a \text{ not an odd multiple of } \pi/4 \text{ or of } \pi/2] \end{cases}$$

In fact, you need to know these "both backwards and forwards". You need to know that '$\cos^2 a - \sin^2 a$' can be simplified to '$\cos 2a$' as well as that '$2\cos^2(a/3) - 1$' can be simplified to '$\cos(2a/3)$' and that '$5 \sin a \cos a$' can be simplified to '$(5/2) \sin 2a$'. When you see '$2 \sin^2(a/2)$' you should think of '$1 - \cos a$'. It will be worth your while to spend quite a bit of time making up such variations of (11).

Finally, as far as the circular functions themselves are concerned, we have the halving formulas:

$$(12)\quad \begin{cases} \cos\dfrac{a}{2} = \operatorname{sgn}\left(\cos\dfrac{a}{2}\right)\sqrt{\dfrac{1+\cos a}{2}} \\ \sin\dfrac{a}{2} = \operatorname{sgn}\left(\sin\dfrac{a}{2}\right)\sqrt{\dfrac{1-\cos a}{2}} \\ \tan\dfrac{a}{2} = \operatorname{sgn}\left(\tan\dfrac{a}{2}\right)\sqrt{\dfrac{1-\cos a}{1+\cos a}}\quad [a \text{ not an odd multiple of } \pi] \end{cases}$$

and Definition 19–5 and Theorem 19–31. These are all closely related to the doubling formulas (11) and can be learned together with them. The best procedure is to derive the latter from (11) several times.

In addition to the "ordinary" circular functions we have, also, the degree-functions. If, however, you are thoroughly acquainted with the foregoing, all you need know of the degree-functions is their definition:

(13) If f is one of the circular functions then $°f(a) = f(\pi a/180)$.

Finally, you need to know the relation of the circular functions to the functions of angles which you studied in earlier chapters:

(14) $\cos \angle A = \cos m(\angle A)$, $\sin \angle A = \sin m(\angle A)$, $\tan \angle A = \tan m(\angle A)$ $[m(\angle A) \neq \pi/2]$

and the cosine, sine, and tangent laws:

$$(15)\quad \begin{cases} \text{In } \triangle ABC, \\ c^2 = a^2 + b^2 - 2ab\cos\gamma, \\ \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c} \\ \dfrac{a-b}{a+b} = \dfrac{\tan[(\alpha-\beta)/2]}{\tan[(\alpha+\beta)/2]}. \end{cases}$$

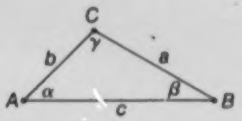

Fig. 19–8

Exercise

Reread this section several times and practice deriving the lettered results from the numbered ones.

19.08 Identities

As you know, the result of substituting numerals for the variables in a sentence is again a sentence and may be either true or false. For example, all such numerical instances obtainable from:

$$(1)\qquad (a-b)(a+b) = a^2 - b^2$$

are true and, in fact, the displayed sentence is itself true [since we interpret it as an assertion about all values of the variables]. As another example, consider:

$$(2)\qquad \cos c \cdot \tan c = \sin c$$

Since the domain of tan consists, by definition, only of those real numbers which are not odd multiples of $\pi/2$, the sentence (2) has meaningless instances, such as '$\cos(\pi/2)\tan(\pi/2) = \sin(\pi/2)$'. Sentence (2) has, however, no false numerical instances—each of its numerical instances is either true or meaningless.

Sentences which, like (1) and (2), have no false numerical instances are called *identities*.

In order to agree with the rest of the world as to what sentences are identities we shall, for this section only, forget that we have assumed that '/0' denotes some real number and, so, that multiplication by /0 is meaningful [though generally useless]. We shall, instead, assume that expressions which indicate multiplication by /0 are meaningless. With this convention the sentence:

$$(3)\qquad \frac{a^2-9}{a-3} = a + 3$$

is an identity since, with multiplication by /0 treated as meaningless its instance '$(3^2 - 9)/(3 - 3) = 3 + 3$' is meaningless rather than false.

The purpose of this section is to acquaint you with a variety of identities involving the circular functions, and with methods of establishing that a given sentence is an identity. To begin with, we may note that the equations occurring in the various definitions and theorems we have already proved are all identities. Those which, in the definitions and theorems, are unrestricted [for example, the subtraction and addition laws for cos and sin] are, of course, true and have only true instances. And the restrictions on other sentences [such as those on the subtraction and addition laws for tan] serve only to rule out meaningless instances.

The main reason for studying identities is to learn how to transform a given expression into an equivalent one which is better suited to the purpose at hand. For example, the identities in Theorem 19–13 are useful when one is faced with a product and wishes it were a sum or a difference; those in Theorem 19–14 are useful in the contrary case.

The procedure for showing that a sentence is an identity may be illustrated by applying it to (2):

$$\begin{aligned} \cos c \cdot \tan c &= \cos c \cdot \frac{\sin c}{\cos c} && \text{[by definition]} \\ &= \sin c && \text{[by algebra]} \end{aligned}$$

Note that the use of the definition and the algebraic step are valid only if $\cos c \neq 0$—that is, only if c is not an odd multiple of $\pi/2$. But, these are just the values of 'c' for which (2) is meaningful. So, we have shown that each numerical instance of (2) is true or meaningless—that is, that (2) has no false numerical instances. Hence, (2) is an identity.

As another example, consider:

$$(4)\qquad \frac{\tan a}{\sin a} = \sec a$$

Note that 'tan a' and 'sec a' are meaningful only if $\cos a \neq 0$ and that the division indicated in the left side of (4) is meaningful only if $\sin a \neq 0$. So, we wish to show that each numerical instance of (4) is true unless it is one for which $\cos a = 0$ or $\sin a = 0$.

$$\begin{aligned} \frac{\tan a}{\sin a} &= \frac{\sin a}{\cos a} \div \sin a && \text{[by definition]} \\ &= \frac{1}{\cos a} && \text{[by algebra]} \\ &= \sec a && \text{[by definition]} \end{aligned}$$

Since the two uses of definitions are valid if $\cos a \neq 0$ and the algebra step is valid unless $\sin a = 0$, we have accomplished what we set out to do.

To continue our study of identities we begin by noting that there are a number of identities which, like (2) and (4), follow by simple algebra from the definitions of tan, cot, sec, and csc. These can be described easily by reference to Fig. 19–9. Starting with any one of the six expressions in the figure one can obtain five identities by the procedure outlined in (i)–(iv):

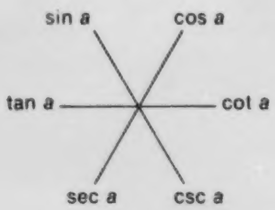

Fig. 19–9

(i) Equate the product of the given expression with the one opposite it to '1'. [$\tan a \cdot \cot a = 1$]

(ii) Equate the given expression to the reciprocal of the one opposite it. [$\cos a = /\sec a$]

(iii) Equate the given expression to the product of its neighbors. [$\sec a = \tan a \cdot \csc a$]

(iv) Equate the given expression to the quotient of either of its neighbors by this neighbor's other neighbor. [$\csc a = \sec a/\tan a$, $\csc a = \cot a/\cos a$]

Exercises

Part A

1. Two of the identities of type (ii) and two of those of type (iv) are definitions. Which are they?
2. Establish the identities in examples (i) and (ii).
3. Write out the thirty identities of types (i)–(iv).
4. Establish each of the six identities of type (iii).
5. Establish the ten identities of type (iv) which are not definitions.

Suggestions for the exercises of section 19.08:

(i) Part A and the preceding discussion should be teacher directed.
(ii) Parts B and C may be assigned as homework.
(iii) Part D and the preceding discussion should be presented by the teacher.
(iv) Part E may be assigned for homework.
(v) Parts F and G constitute a combined class-homework assignment.

Answers for Part A

1. $\sec a = /\cos a$, $\csc a = /\sin a$; $\tan a = \sin a/\cos a$, $\cot a = \cos a/\sin a$

2. $\tan a \cdot \cot a = \dfrac{\sin a}{\cos a} \cdot \dfrac{\cos a}{\sin a} = 1$, for $\cos a \neq 0$ and $\sin a \neq 0$.

   $\cos a = //\cos a = /\sec a$, for $\cos a \neq 0$.

3. (i) $\cos a \cdot \sec a = 1$, $\sin a \cdot \csc a = 1$, $\tan a \cdot \cot a = 1$,
   $\sec a \cdot \cos a = 1$, $\csc a \cdot \sin a = 1$, $\cot a \cdot \tan a = 1$

   (ii) $\cos a = /\sec a$, $\sin a = /\csc a$, $\tan a = /\cot a$,
   $\sec a = /\cos a$, $\csc a = /\sin a$, $\cot a = /\tan a$

   (iii) $\cos a = \cot a \cdot \sin a$, $\sin a = \cos a \cdot \tan a$, $\tan a = \sin a \cdot \sec a$,
   $\sec a = \tan a \cdot \csc a$, $\csc a = \sec a \cdot \cot a$, $\cot a = \csc a \cdot \cos a$

   (iv) $\cos a = \cot a/\csc a$, $\sin a = \cos a/\cot a$, $\tan a = \sin a/\cos a$,
   $\sec a = \tan a/\sin a$, $\csc a = \sec a/\tan a$, $\cot a = \csc a/\sec a$,
   $\cos a = \sin a/\tan a$, $\sin a = \tan a/\sec a$, $\tan a = \sec a/\csc a$,
   $\sec a = \csc a/\cot a$, $\csc a = \cot a/\cos a$, $\cot a = \cos a/\sin a$

4. $\cot a \cdot \sin a = \dfrac{\cos a}{\sin a} \cdot \sin a = \cos a$ [for $\sin a \neq 0$]

   $\cos a \cdot \tan a = \cos a \cdot \dfrac{\sin a}{\cos a} = \sin a$ [for $\cos a \neq 0$]

   $\sin a \cdot \sec a = \sin a/\cos a = \tan a$ [for $\cos a \neq 0$]

   $\tan a \cdot \csc a = (\sin a/\cos a)/\sin a = \sin a/\sin a/\cos a = 1 \cdot \sec a$
   $= \sec a$ [for $\cos a \neq 0 \neq \sin a$]

   $\sec a \cdot \cot a = /\cos a(\cos a/\sin a) = /\sin a = \csc a$ [for $\cos a \neq 0 \neq \sin a$]

   $\csc a \cdot \cos a = /\sin a \cdot \cos a = \cos a \cdot /\sin a = \cot a$ [for $\sin a \neq 0$]

5. $\cot a/\csc a = (\cos a/\sin a)/\csc a = \cos a/(\sin a \csc a)$
   $= \cos a/(\sin a/\sin a) = \cos a/1 = \cos a$

   $\cos a/\cot a = \cos a/(\cos a/\sin a) = \cos a/\cos a//\sin a = 1 \cdot \sin a = \sin a$

   $\tan a/\sin a = (\sin a/\cos a)/\sin a = (\sin a/\sin a)/\cos a = 1 \cdot \sec a = \sec a$

   $\sec a/\tan a = /\cos a/(\sin a/\cos a) = /\cos a//\cos a/\sin a = 1 \cdot \csc a = \csc a$

   $\csc a/\sec a = /\sin a//\cos a = \cos a/\sin a = \cot a$

   $\sin a/\tan a = \sin a/(\sin a/\cos a) = \sin a/\sin a//\cos a = 1 \cdot \cos a = \cos a$

   $\tan a/\sec a = \sin a/\cos a//\cos a = \sin a \cdot 1 = \sin a$

   $\sec a/\csc a = /\cos a//\sin a = \sin a/\cos a = \tan a$

   $\csc a/\cot a = /\sin a/(\cos a/\sin a) = /\sin a//\sin a/\cos a$
   $= 1 \cdot \sec a = \sec a$

   $\cot a/\cos a = \cos a/\sin a/\cos a = \cos a/\cos a/\sin a$
   $= 1 \cdot /\sin a = \csc a$

[Note how, by a consistent use of reciprocals each of the preceding computations becomes a problem in multiplication. The properties of reciprocation used are that $/a/b = /(ab)$, $a/a = 1$, and $//a = a$, subject to the restrictions that a and b are nonzero. These restrictions translate, in the parts of Exercise 5, to '$\cos a \neq 0 \neq \sin a$' except in two cases: in the first derivation the only restriction needed is '$\sin a \neq 0$' and in the seventh the only restriction needed is '$\cos a \neq 0$'. Students should take note of these restrictions and check to see that, in each case, they serve to throw out only meaningless instances.]

Answers for Part B

[The complete proof of (5) is:

$$\begin{aligned}\sin a \tan a &= \sin a(\sin a/\cos a) = \sin^2 a/\cos a\\ &= (1 - \cos^2 a)/\cos a = /\cos a - \cos^2 a/\cos a\\ &= \sec a - \cos a\end{aligned}$$

The only restriction needed is that $\cos a$ be nonzero and instances of (5) for which this restriction is not satisfied are meaningless. So, (5) is an identity.]

1. $$\begin{aligned}\cos a \cot a &= \cos a(\cos a/\sin a) = \cos^2 a/\sin a\\ &= \cos^2 a/\sin a = (1 - \sin^2 a)/\sin a\\ &= /\sin a - \sin^2 a/\sin a = \csc a - \sin a\end{aligned}$$

The only restriction is that $\sin a$ be nonzero. But instances of the given equation for which $\sin a = 0$ are meaningless.

2. $$\begin{aligned}\sec a \cdot \csc a &= /\cos a/\sin a = (\sin^2 a + \cos^2 a)/\cos a/\sin a\\ &= \sin^2 a/\sin a/\cos a + \cos^2 a/\cos a/\sin a\\ &= \sin a/\cos a + \cos a/\sin a\\ &= \tan a + \cot a, \quad \text{for } \cos a \neq 0 \neq \sin a.\end{aligned}$$

[Students will no doubt find it easier to transform the right side of the given equation into its left side. The permission to do this implied by a statement on page 424 stays in force until Part E on page 452.]

3. $$\frac{\cos a}{1 + \sin a} = \frac{\cos^2 a}{(1 + \sin a)\cos a} = \frac{1 - \sin^2 a}{(1 + \sin a)\cos a} = \frac{1 - \sin a}{\cos a} \quad \text{or:}$$

$$\frac{\cos a}{1 + \sin a} = \frac{\cos a(1 - \sin a)}{1 - \sin^2 a} = \frac{\cos a(1 - \sin a)}{\cos^2 a} = \frac{1 - \sin a}{\cos a}$$

[One trick is to introduce a factor which makes the side you are working on somehow similar to the side you are trying to attain to. In this exercise this can be done in two ways: get a '$\cos a$' in the denominator or get a '$1 - \sin a$' in the numerator.] In the first argument the restrictions are that $\cos a \neq 0$ [first step] and that $\sin a \neq -1$ [third step]. In the second argument the restrictions are that $\sin a \neq 1$ [first step] and that $\cos a \neq 0$ [third step]. In each argument the restrictions boil down to '$\cos a \neq 0$' and this is the restriction needed to rule out meaningless instances.]

*

The identities of types (iii) and (iv) are most easily established by transforming their right sides into their left sides.

Part B

The identities we have dealt with so far are based on the definitions of tan, sec, cot, and csc. In addition to these definitions we have the Pythagorean identities:

$$\begin{aligned}\cos^2 a + \sin^2 a &= 1\\ \sec^2 a - \tan^2 a &= 1\\ \csc^2 a - \cot^2 a &= 1\end{aligned}$$

to work with. Using these we can establish such identities as:

$$\sin a \cdot \tan a = \sec a - \cos a \tag{5}$$

[$\sin a \cdot \tan a = \sin^2 a/\cos a = (1 - \cos^2 a)/\cos a = \sec a - \cos a$. Explain the missing steps and explain why this argument shows that each meaningful numerical instance of (5) is true.]

Show that each of the following is an identity.

1. $\cos a \cdot \cot a = \csc a - \sin a$
2. $\sec a \cdot \csc a = \tan a + \cot a$
3. $\dfrac{\cos a}{1 + \sin a} = \dfrac{1 - \sin a}{\cos a}$
4. $\dfrac{\sin a}{1 + \cos a} = \dfrac{1 - \cos a}{\sin a}$
5. $\dfrac{\cos a}{1 + \sin a} = \sec a - \tan a$
6. $\dfrac{\sin a}{1 - \cos a} = \csc a + \cot a$

Part C

The identity (5) and that of Exercise 1 of Part B may seem to be related to one another as may, also, the identities of Exercises 3 and 4 of Part B. The second and third of the Pythagorean identities are related to one another in this same way as are:

$\cos a \cdot \tan a = \sin a$ and: $\sin a \cdot \cot a = \cos a$,
$\sec a = \tan a/\sin a$ and: $\csc a = \cot a/\cos a$, and
$\sec a \cdot \csc a = \tan a + \cot a$ and: $\csc a \cdot \sec a = \cot a + \tan a$

1. Guess an easy way to transform a given identity into a sentence which is also an identity.
2. Use your guess to obtain another identity from the one of Exercise 5 in Part B. Show that the new sentence you obtain actually is an identity.
3. Repeat Exercise 2 with Exercise 6 of Part B rather than Exercise 5.

*4. Prove that your way of getting identities works.

*

Answers for Part B [cont.]

4. $\frac{\sin a}{1 + \cos a} = \frac{\sin^2 a}{(1 + \cos a)\sin a} = \frac{1 - \cos^2 a}{(1 + \cos a)\sin a} = \frac{1 - \cos a}{\sin a}$

[As in Exercise 3, there is another equally simple route.]

5. By Exercise 3, $\frac{\cos a}{1 + \sin a} = \frac{1 - \sin a}{\cos a} = /\cos a - \sin a/\cos a$

$= \sec a - \tan a$.

6. $\frac{\sin a}{1 - \cos a} = \frac{\sin^2 a}{(1 - \cos a)\sin a} = \frac{1 - \cos^2 a}{(1 - \cos a)\sin a} = \frac{1 + \cos a}{\sin a}$

$= /\sin a + \cos a/\sin a = \csc a + \cot a$

Answers for Part C

1. Interchange 'sin' and 'cos', 'tan' and 'cot', and 'sec' and 'csc'.

2. $\frac{\sin a}{1 + \cos a} = \csc a - \cot a$

By Exercise 4, $\frac{\sin a}{1 + \cos a} = \frac{1 - \cos a}{\sin a} = /\sin a - \cos a/\sin a$

$= \csc a - \cot a$. The only restriction on the preceding argument is that $\sin a \neq 0$, and this is precisely the restriction needed to rule out meaningless instances. [Note that if $\sin a \neq 0$ then $1 + \cos a \neq 0$.]

3. $\frac{\cos a}{1 - \sin a} = \sec a + \tan a$

$\frac{\cos a}{1 - \sin a} = \frac{\cos^2 a}{(1 - \sin a)\cos a} = \frac{1 - \sin^2 a}{(1 - \sin a)\cos a} = \frac{1 + \sin a}{\cos a}$

$= /\cos a + \sin a/\cos a = \sec a + \tan a$

4. [See the text following these exercises.]

Two functions which are mentioned on the same horizontal line of Fig. 19–9 are said to be cofunctions of one another. For example, cos is the cofunction of sin and sin is the cofunction of cos. We have already noted that if $a + b = \pi/2$ then

$$\cos a = \sin b \text{ and, of course,}$$
$$\sin a = \cos b.$$

We have also noticed that if $a + b = \pi/2$ then

$$\cot a = \tan b \qquad [a \text{ not an even multiple of } \pi/2].$$

Since, for $a + b = \pi/2$, a is not an even multiple of $\pi/2$ if and only if b is not an odd multiple of $\pi/2$ it is easy to see that, if $a + b = \pi/2$,

$$\tan a = \cot b \qquad [a \text{ not an odd multiple of } \pi/2].$$

Similar results hold for csc and sec [since they hold for sin and cos]. [Explain.] Since $a + (\pi/2 - a) = \pi/2$ it follows that if f and g are cofunctions then:

$$(*) \qquad f(a) = g(\pi/2 - a)$$

is an identity. It is this that justifies the guess you probably made in Exercise 1 of Part C.

To see how (*) can be used to get new identities from old ones, consider the identity:

$$(6) \qquad \frac{\cos a}{1 + \sin a} = \sec a - \tan a$$

which you established in Exercise 5 of Part B. If we replace each function mentioned in (6) by its cofunction we obtain a new sentence:

$$(7) \qquad \frac{\sin a}{1 + \cos a} = \csc a - \cot a$$

Once having proved that (6) is an identity we can show that (7) is an identity as follows:

$$\begin{aligned} \frac{\sin a}{1 + \cos a} &= \frac{\cos(\pi/2 - a)}{1 + \sin(\pi/2 - a)} && [\text{by } (*)] \\ &= \sec(\pi/2 - a) - \tan(\pi/2 - a) && [\text{by } (6)] \\ &= \csc a - \cot a && [\text{by } (*)] \end{aligned}$$

For easy reference we shall call the procedure of transforming (6) into (7)—or (7) into (6)—*the cofunction transformation.*

Part D

1. (a) Apply the cofunction transformation to the identity you established in Exercise 6 of Part B.
   (b) Use (*) and the identity of Exercise 6 to show that the sentence you obtained in part (a) is an identity.
2. As far as possible, arrange the thirty identities you worked with in Part A in pairs such that each member of a pair can be obtained from the other by the cofunction transformation.
3. (a) Establish the identity:

$$\frac{\cos a}{1 - \sin a} = \frac{1 + \sin a}{\cos a}$$

   (b) Obtain a new identity by applying the cofunction transformation to the one in part (a).

Part E

Show that each of the following is an identity. [In order to learn how to transform expressions into equivalent expressions of a specified form, you should work each exercise by using known identities and algebra to transform the *left* side of the given equation into the right. Complete the exercise by determining the values of 'a' for which the given equation has meaningful numerical instances and making sure that the steps you have taken are legitimate for each such value of 'a'.]

1. $\frac{1}{\sin a} - \sin a = \frac{\cos a}{\tan a}$
2. $\frac{\tan^2 a}{1 + \tan^2 a} = \sin^2 a$
3. $\cot a + \tan a = \frac{1}{\cos a \cdot \sin a}$
4. $\cot a \cdot \cos a = \frac{\cos^2 a}{\sin a}$
5. $\frac{\tan a - \cot a}{\tan a + \cot a} = \sin^2 a - \cos^2 a$
6. $\cos^2 a - \sin^2 a = 2\cos^2 a - 1$
7. $\frac{\cot a + \cos a}{1 + \sin a} = \cot a$
8. $\frac{\sin a + \tan a}{\sec a + 1} = \sin a$
9. $(\cos a + \sin a)^2 = 1 + 2\cos a \cdot \sin a$
10. $\cos^4 a - \sin^4 a = (\cos a - \sin a)(\cos a + \sin a)$
11. $\frac{1}{1 + \sin a} + \frac{1}{1 - \sin a} = 2\sec^2 a$

Answers for Part D

1. (a) $\frac{\cos a}{1 - \sin a} = \sec a + \tan a$

   (b) $\frac{\cos a}{1 - \sin a} = \frac{\sin(\pi/2 - a)}{1 - \cos(\pi/2 - a)} = \csc(\pi/2 - \ ) + \cot(\pi/2 - a) = \sec a + \tan a$

2. $(\cos a \cdot \sec a = 1,\ \sin a \cdot \csc a = 1)$, $(\tan a \cdot \cot a = 1,\ \cot a \cdot \tan a = 1)$,
   $(\sec a \cdot \cos a = 1,\ \csc a \cdot \sin a = 1)$, $(\cos a = /\sec a,\ \sin a = /\csc a)$,
   $(\tan a = /\cot a,\ \cot a = /\tan a)$, $(\sec a = /\cos a,\ \csc a = /\sin a)$,
   $(\cos a = \cot a \cdot \sin a,\ \sin a = \tan a \cdot \cos a)$,
   $(\tan a = \sin a \cdot \sec a,\ \cot a = \cos a \cdot \csc a)$,
   $(\sec a = \tan a \cdot \csc a,\ \csc a = \cot a \cdot \sec a)$ [In the last 3 we have used CPM to get a match.]
   $(\cos a = \cot a/\csc a,\ \sin a = \tan a/\sec a)$,
   $(\sin a = \cos a/\cot a,\ \cos a = \sin a/\tan a)$,
   $(\tan a = \sin a/\cos a,\ \cot a = \cos a/\sin a)$,
   $(\sec a = \tan a/\sin a,\ \csc a = \cot a/\cos a)$,
   $(\csc a = \sec a/\tan a,\ \sec a = \csc a/\cot a)$,
   $(\cot a = \csc a/\sec a,\ \tan a = \sec a/\csc a)$

3. (a) $\frac{\cos a}{1 - \sin a} = \frac{\cos^2 a}{(1 - \sin a)\cos a} = \frac{1 - \sin^2 a}{(1 - \sin a)\cos a} = \frac{1 + \sin a}{\cos a}$

   (b) $\frac{\sin a}{1 - \cos a} = \frac{1 + \cos a}{\sin a}$

Answers for Part E

1. $\frac{1}{\sin a} - \sin a = \frac{1 - \sin^2 a}{\sin a} = \frac{\cos^2 a}{\sin a} = \frac{\cos a}{\sin a/\cos a} = \frac{\cos a}{\tan a}$

   [In more detail, the third step may be developed into:

   $$\cos^2 a/\sin a = \cos a \cdot /\sin a \cdot \cos a = \cos a(/\sin a//\cos a)$$
   $$= \cos a/(\sin a/\cos a)]$$

   The required restrictions are that $\sin a \neq 0$ and that $\cos a \neq 0$. These are the restrictions under which $\sin a \neq 0$ and $\tan a$ is defined and is not 0. So, the equation is an identity.

2. $\frac{\tan^2 a}{1 + \tan^2 a} = \frac{\tan^2 a}{\sec^2 a} = (\tan a \cdot \cos a)^2 = \sin^2 a$. The only restriction to the argument is that $\cos a \neq 0$ and this is the only restriction needed to ensure that $\tan^2 a$ is not meaningless.

[To save space we shall omit the discussion of restrictions except in cases where such a discussion may be difficult.]

3. $\cot a + \tan a = \frac{\cos a}{\sin a} + \frac{\sin a}{\cos a} = \frac{\cos^2 a + \sin^2 a}{\sin a \cdot \cos a} = \frac{1}{\cos a \cdot \sin a}$

4. $\cot a \cdot \cos a = \frac{\cos a}{\sin a} \cdot \cos a = \frac{\cos^2 a}{\sin a}$

TC 452 (2)

Answers for Part E [cont.]

5. $$\frac{\tan a - \cot a}{\tan a + \cot a} = \frac{\dfrac{\sin a}{\cos a} - \dfrac{\cos a}{\sin a}}{\dfrac{\sin a}{\cos a} + \dfrac{\cos a}{\sin a}} = \frac{(\sin^2 a - \cos^2 a)/(\sin a \cdot \cos a)}{(\sin^2 a + \cos^2 a)/(\sin a \cdot \cos a)}$$
$$= \sin^2 a - \cos^2 a$$

An alternative route:

$$\frac{\tan a - \cot a}{\tan a + \cot a} = \frac{(\tan a - \cot a)\tan a}{(\tan a + \cot a)\tan a} = \frac{\tan^2 a - 1}{\tan^2 a + 1} = \frac{\tan^2 a - 1}{\sec^2 a} = \text{etc.}$$

To avoid meaningless instances we must have $\cos a \neq 0 \neq \sin a$. These are sufficient since it implies that $\tan a \neq 0$ and, so, that $\tan a + /\tan a \neq 0$. These restrictions are also sufficient to justify the algebraic steps in either argument.

6. $\cos^2 a - \sin^2 a = \cos^2 a - (1 - \cos^2 a) = 2\cos^2 a - 1$

7. $$\frac{\cot a + \cos a}{1 + \sin a} = \frac{\cot a + \cot a \cdot \sin a}{1 + \sin a} = \frac{\cot a(1 + \sin a)}{1 + \sin a} = \cot a.$$

[Restrictions: $\sin a \neq -1$ and $\sin a \neq 0$]

8. $$\frac{\sin a + \tan a}{\sec a + 1} = \frac{\sin a + \sin a \cdot \sec a}{\sec a + 1} = \frac{\sin a(1 + \sec a)}{\sec a + 1} = \sin a$$

9. $(\cos a + \sin a)^2 = \cos^2 a + \sin^2 a + 2\cos a \cdot \sin a = 1 + 2\cos a \cdot \sin a$

10. $\cos^4 a - \sin^4 a = (\cos^2 a - \sin^2 a)(\cos^2 a + \sin^2 a) = \cos^2 a - \sin^2 a$
$= (\cos a - \sin a)(\cos a + \sin a)$

11. $$\frac{1}{1 + \sin a} + \frac{1}{1 - \sin a} = \frac{1 - \sin a + 1 + \sin a}{1 - \sin^2 a} = \frac{2}{\cos^2 a} = 2\sec^2 a$$

TC 453 (1)

12. $$\frac{1 - \cos^2 a}{\cos a} = \frac{\sin^2 a}{\cos a} = \sin a \cdot \frac{\sin a}{\cos a} = \sin a \cdot \tan a$$

13. $$\cot a + \tan a = \frac{\cos a}{\sin a} + \frac{\sin a}{\cos a} = \frac{\cos^2 a + \sin^2 a}{\sin a \cdot \cos a} = \frac{1}{\cos a \cdot \sin a}$$
$$= \sec a \cdot \csc a$$

14. $$\frac{1 + \cot^2 a}{\cot^2 a} = \tan^2 a + 1 = \sec^2 a$$

15. $$\frac{\tan^2 a + 1}{\cot^2 a + 1} = \frac{\sec^2 a}{\csc^2 a} = \frac{\sin^2 a}{\cos^2 a} = \tan^2 a$$

16. $$(\sec a - \tan a)^2 = \left(\frac{1}{\cos a} - \frac{\sin a}{\cos a}\right)^2 = \frac{(1 - \sin a)^2}{\cos^2 a} = \frac{(1 - \sin a)^2}{1 - \sin^2 a}$$
$$= \frac{(1 - \sin a)^2}{(1 - \sin a)(1 + \sin a)} = \frac{1 - \sin a}{1 + \sin a}$$

17. $$\frac{\sin a - \cos a}{\sin a + \cos a} = \frac{(\sin a - \cos a)/\cos a}{(\sin a + \cos a)/\cos a} = \frac{\tan a - 1}{\tan a + 1}$$

18. $$\frac{\cos^2 a}{1 - \sin a} = \frac{1 - \sin^2 a}{1 - \sin a} = 1 + \sin a$$

12. $\dfrac{1 - \cos^2 a}{\cos a} = \sin a \cdot \tan a$

13. $\cot a + \tan a = \sec a \cdot \csc a$

14. $\dfrac{1 + \cot^2 a}{\cot^2 a} = \sec^2 a$

15. $\dfrac{\tan^2 a + 1}{\cot^2 a + 1} = \tan^2 a$

16. $(\sec a - \tan a)^2 = \dfrac{1 - \sin a}{1 + \sin a}$

17. $\dfrac{\sin a - \cos a}{\sin a + \cos a} = \dfrac{\tan a - 1}{\tan a + 1}$

18. $\dfrac{\cos^2 a}{1 - \sin a} = 1 + \sin a$

19. $\cos^4 a + \sin^4 a = 1 - 2\sin^2 a \cdot \cos^2 a$

20. $\dfrac{\sec a}{\sec a - \tan a} = \sec a\,(\sec a + \tan a)$

21. $\dfrac{\sec a - \cos a}{\sec a + \cos a} = \dfrac{\tan^2 a}{1 + \sec^2 a}$

22. $\tan^2 a - \sin^2 a = \tan^2 a \cdot \sin^2 a$

23. $\sec^2 a + \csc^2 a = (\tan a + \cot a)^2$

24. $\sec^2 a - \tan^2 a = \csc^2 a - \cot^2 a$

25. $\tan^2 a + \cos^2 a = \sec^2 a - \sin^2 a$

26. $\tan^2 a + \cos^2 a = 1 + \sin^2 a \cdot \tan^2 a$

*

The identities we have dealt with up to now are based on the definitions of tan, sec, cot, and csc and on the Pythagorean identities. We can, however, make use of other identities such as the subtraction and addition laws, the doubling formulas and the halving formulas. It should be noted that, for identities based on these results, the cofunction transformation no longer transforms an identity into an identity. If, for example, we apply the transformation to the identity:

(8) $$2\sin a \cdot \cos a = \sin 2a$$

we obtain:

(8′) $$2\cos a \cdot \sin a = \cos 2a$$

which is not an identity. [To see that (8′) is not an identity, consider the numerical instance obtained by substituting '0' for 'a'.] To see why

Answers for Part E [cont.]

19. $$\begin{aligned}\cos^4 a + \sin^4 a &= (\cos^4 a + \sin^4 a + 2\sin^2 a\cos^2 a) - 2\sin^2 a\cdot\cos^2 a\\ &= (\cos^2 a + \sin^2 a)^2 - 2\sin^2 a\cdot\cos^2 a\\ &= 1 - 2\sin^2 a\cdot\cos^2 a\end{aligned}$$

20. $$\frac{\sec a}{\sec a - \tan a} = \frac{\sec a(\sec a + \tan a)}{\sec^2 a - \tan^2 a} = \sec a(\sec a + \tan a)$$

21. $$\frac{\sec a - \cos a}{\sec a + \cos a} = \frac{(\sec a - \cos a)\sec a}{(\sec a + \cos a)\sec a} = \frac{\sec^2 a - 1}{\sec^2 a + 1} = \frac{\tan^2 a}{1 + \sec^2 a}$$

22. $$\begin{aligned}\tan^2 a - \sin^2 a &= \frac{\sin^2 a}{\cos^2 a} - \sin^2 a = \frac{\sin^2 a(1 - \cos^2 a)}{\cos^2 a}\\ &= \tan^2 a(1 - \cos^2 a) = \tan^2 a\cdot\sin^2 a\end{aligned}$$

23. $$\begin{aligned}\sec^2 a + \csc^2 a &= 1 + \tan^2 a + 1 + \cot^2 a = \tan^2 a + \cot^2 a\\ &\quad + 2\tan a\cdot\cot a = (\tan a + \cot a)^2\end{aligned}$$

24. $\sec^2 a - \tan^2 a = 1 = \csc^2 a - \cot^2 a$

25. $\tan^2 a + \cos^2 a = \tan^2 a + 1 - (1 - \cos^2 a) = \sec^2 a - \sin^2 a$

26. $$\begin{aligned}\tan^2 a + \cos^2 a &= \tan^2 a + (1 - \sin^2 a) = 1 + (\tan^2 a - \sin^2 a)\\ &= 1 + \sin^2 a\cdot\tan^2 a,\text{ by Exercise 22.}\end{aligned}$$

TC 454 (1)

Answers for Part F

1. By Exercise 19 of Part E, $\cos^4 a + \sin^4 a = 1 - 2\sin^2 a\cos^2 a$
$= 1 - \dfrac{(2\sin a\cdot\cos a)^2}{2} = 1 - \dfrac{\sin^2 2a}{2}$.

2. $\cos^4 a - \sin^4 a = (\cos^2 a - \sin^2 a)(\cos^2 a + \sin^2 a) = \cos 2a$

3. $(\cos a + \sin a)^2 = \cos^2 a + \sin^2 a + 2\sin a\cdot\cos a = 1 + \sin 2a$

4. $$\cos^2(\tfrac{\pi}{4} - a) = \cos^2[(\tfrac{\pi}{2} - 2a)/2] = \frac{1 + \cos(\frac{\pi}{2} - 2a)}{2} = \frac{1 + \sin 2a}{2}$$

5. $$\sin^2(\tfrac{\pi}{4} - a) = \sin^2[(\tfrac{\pi}{2} - 2a)/2] = \frac{1 - \cos(\frac{\pi}{2} - 2a)}{2} = \frac{1 - \sin 2a}{2}$$

6. $$\begin{aligned}\tan(\tfrac{\pi}{4} + a) - \tan(\tfrac{\pi}{4} - a) &= \frac{\tan(\pi/4) + \tan a}{1 - \tan(\pi/4)\cdot\tan a} - \frac{\tan(\pi/4) - \tan a}{1 + \tan(\pi/4)\cdot\tan a}\\ &= \frac{1 + \tan a}{1 - \tan a} - \frac{1 - \tan a}{1 + \tan a} = \frac{(1 + \tan a)^2 - (1 - \tan a)^2}{1 - \tan^2 a}\\ &= \frac{4\tan a}{1 - \tan^2 a} = 2\tan 2a\end{aligned}$$

7. $$\begin{aligned}\cos^2 a - \cos 2a &= \cos^2 a - (\cos^2 a - \sin^2 a) = \sin^2 a = 1 - \cos^2 a\\ &= 1 - \cot^2 a\cdot\sin^2 a\end{aligned}$$

8. $$\begin{aligned}\cos 3a = \cos(2a + a) &= \cos 2a\cdot\cos a - \sin 2a\cdot\sin a\\ &= (\cos^2 a - \sin^2 a)\cos a - 2\sin a\cdot\cos a\cdot\sin a\\ &= \cos^3 a - \sin^2 a\cdot\cos a - 2\sin^2 a\cdot\cos a\\ &= \cos^3 a - 3(1 - \cos^2 a)\cos a = 4\cos^3 a - 3\cos a\end{aligned}$$

the transformation does not work, let's apply (*) on page 451 to the left side of this second sentence and then make use of (8).

$$\begin{aligned}2\cos a\cdot\sin a &= 2\sin(\pi/2 - a)\cos(\pi/2 - a) &&\text{[by (*)]}\\ &= \sin[2(\pi/2 - a)] &&\text{[by (8)]}\\ &= \sin(\pi - 2a)\\ &= \sin 2a\end{aligned}$$

We do obtain an identity but it is not (8′).

Part F

Show that each of the following is an identity. [As in Part E, transform left sides into right sides.]

1. $\cos^4 a + \sin^4 a = 1 - \dfrac{\sin^2 2a}{2}$ [*Hint*: Recall Exercise 19 of Part E.]
2. $\cos^4 a - \sin^4 a = \cos 2a$
3. $(\cos a + \sin a)^2 = 1 + \sin 2a$
4. $\cos^2\left(\dfrac{\pi}{4} - a\right) = \dfrac{1 + \sin 2a}{2}$
5. $\sin^2\left(\dfrac{\pi}{4} - a\right) = \dfrac{1 - \sin 2a}{2}$
6. $\tan\left(\dfrac{\pi}{4} + a\right) - \tan\left(\dfrac{\pi}{4} - a\right) = 2\tan 2a$
7. $\cos^2 a - \cos 2a = 1 - \cot^2 a\cdot\sin^2 a$
8. $\cos 3a = 4\cos^3 a - 3\cos a$
9. $\sin 3a = 3\sin a - 4\sin^3 a$
10. $\sin^2 2a + 4\sin^4 a = 4\sin^2 a$
11. $\sin^2 a(1 + \tan^2 a) = \tan^2 a$
12. $\dfrac{\sin a - \sin b}{\cos a + \cos b} = \tan\left(\dfrac{a - b}{2}\right)$
13. $\tan a - \tan b = \dfrac{\sin(a - b)}{\cos a\cos b}$
14. $\sin a + \sin 3a + \sin 5a + \sin 7a = 4\cos a\cos 2a\sin 4a$
[*Hint*: Recall Theorem 19–14.]

Part G

It is sometimes desirable to transform an expression of the form:

$$\sin^{2p} a\cdot\cos^{2q} a \qquad [\text{where } p, q \in Nn]$$

into an equivalent expression which involves neither exponents nor products of values of cos and sin. This can be accomplished by using the doubling formulas and Theorem 19–13. For example, Answers for Part F [cont.]

9. $\sin 3a = \sin(2a + a) = \sin 2a \cdot \cos a + \cos 2a \cdot \sin a$
$= 2 \sin a \cdot \cos^2 a + \cos^2 a \cdot \sin a - \sin^3 a$
$= 3 \sin a(1 - \sin^2 a) - \sin^3 a$
$= 3 \sin a - 4 \sin^3 a$

10. $\sin^2 2a + 4 \sin^4 a = 4 \sin^2 a \cdot \cos^2 a + 4 \sin^4 a$
$= 4 \sin^2 a(\cos^2 a + \sin^2 a) = 4 \sin^2 a$

11. $\sin^2 a(1 + \tan^2 a) = \sin^2 a \cdot \sec^2 a = \sin^2 a/\cos^2 a = \tan^2 a$

12. $\dfrac{\sin a - \sin b}{\cos a + \cos b} = \dfrac{-2 \cos[(b + a)/2] \sin[(b - a)/2]}{2 \cos[(b + a)/2] \cos[(b - a)/2]} = -\tan[(b - a)/2]$
$= \tan[(a - b)/2]$

13. $\tan a - \tan b = \dfrac{\sin a}{\cos a} - \dfrac{\sin b}{\cos b} = \dfrac{\sin a \cdot \cos b - \cos a \cdot \sin b}{\cos a \cdot \cos b} = \dfrac{\sin(a - b)}{\cos a \cdot \cos b}$

14. $\sin a + \sin 3a + \sin 5a + \sin 7a = 2 \sin 2a \cdot \cos a + 2 \sin 6a \cdot \cos a$
$= 2 \cos a(\sin 2a + \sin 6a)$
$= 2 \cos a \cdot 2 \sin 4a \cdot \cos 2a$
$= 4 \cos a \cdot \cos 2a \cdot \sin 4a$

TC 455

Answers for Part G

1. $(1 - \cos 4a)/8$
2. $(10 - 15 \cos 2a + 6 \cos 4a - \cos 6a)/32$
3. $(2 + \cos 2a - 2 \cos 4a - \cos 6a)/32$
4. $(5 - 4 \cos 2a - 4 \cos 4a + 4 \cos 6a - \cos 8a)/128$

Suggestions for the exercises of section 19.09:
(i) Use Part A to illustrate the discussion preceding it.
(ii) Part B may be assigned as homework, after appropriate examples from the discussion on page 457.
(iii) The discussion on pages 458-459, and Part C, should be teacher directed. This applies to pages 460-462.
(iv) Part D may be assigned for homework.
(v) Part E should be teacher directed.

$$\begin{aligned}
\sin^4 a \cos^2 a &= (\sin a \cos a)^2 \sin^2 a \\
&= (\tfrac{1}{2} \sin 2a)^2 [\tfrac{1}{2} (1 - \cos 2a)] \\
&= \tfrac{1}{8} \sin^2 2a(1 - \cos 2a) \\
&= \tfrac{1}{8} [\tfrac{1}{2} (1 - \cos 4a)](1 - \cos 2a) \\
&= \tfrac{1}{16} [1 - \cos 2a - \cos 4a + \cos 2a \cos 4a] \\
&= \tfrac{1}{16}[1 - \cos 2a - \cos 4a + \tfrac{1}{2} (\cos (-2a) + \cos 6a)] \\
&= \tfrac{1}{32}[2 - \cos 2a - 2 \cos 4a + \cos 6a].
\end{aligned}$$

Transform each of the following into an equivalent expression without either exponents or indicated products of values of cos and sin.

1. $\sin^2 a \cos^2 a$
2. $\sin^6 a$
3. $\sin^2 a \cos^4 a$
4. $\sin^6 a \cos^2 a$

19.09 Some Applications of Circular Functions

In Chapter 13 we have introduced the notion of direction numbers of a line l with respect to a given orthonormal coordinate system. [See page 110.] Recall that such direction numbers are the components, with respect to the corresponding orthonormal basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ for $\mathscr{T}$, of any non-$\vec{0}$ vector in $[l]$. In particular, if P and Q are two points of l whose coordinates are (x_1, x_2, x_3) and (y_1, y_2, y_3), respectively, then $(y_1 - x_1, y_2 - x_2, y_3 - x_3)$ are direction numbers of l. If $\vec{v}$ is one of the unit vectors in $[l]$ with components (v_1, v_2, v_3) and $Q = P + \vec{v}$ it follows that (v_1, v_2, v_3) are direction numbers of l. Now, since $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is orthonormal it follows that $v_1 = \vec{v} \cdot \vec{u}_1$, $v_2 = \vec{v} \cdot \vec{u}_2$, and $v_3 = \vec{v} \cdot \vec{u}_3$. So, v_1, for example, is the cosine of an angle one of whose sides has the sense of $\vec{v}$ and whose other side has the sense of $\vec{u}_1$. All such angles are congruent and have the same measure—say, α_1. Hence, $v_1 = \cos \alpha_1$. Similarly [see Fig. 19–10], $v_2 = \cos \alpha_2$ and $v_3 = \cos \alpha_3$. For this reason the numbers (v_1, v_2, v_3) are called *direction cosines* for l. They are the cosines of "the angles between" l, as oriented by the choice of $\vec{v}$, and the positively oriented coordinate axes. Choosing, in place of $\vec{v}$, the other unit vector in $[l]$ and proceeding as above leads to another sequence of direction cosines for l. These are just the opposites of those

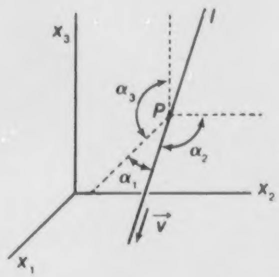

Fig. 19–10

Answers for Part A

1. (a) $(3/\sqrt{22}, 3/\sqrt{22}, 2/\sqrt{22})$ (b) $(1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6})$
   (c) $(-1/\sqrt{35}, 3/\sqrt{35}, -5/\sqrt{35})$ (d) $(0, -1, 0)$
2. (a) $(3/\sqrt{22}, 3/\sqrt{22}, 2/\sqrt{22})$ (b) $(1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6})$
   (c) $(1/\sqrt{35}, -3/\sqrt{35}, 5/\sqrt{35})$ (d) $(0, 1, 0)$

   [This is a convention for orienting lines which is frequently used in analytic geometry. It is sometimes expressed by saying that the upward sense on a line is positive unless the line is horizontal, in which case the rightward sense is positive unless the line is parallel to the x-axis, in which case the forward sense is positive.]
3. [The phrase 'makes the same angle with' is a conventional way of saying 'makes congruent angles with rays having the same sense as'.]

   The direction numbers of such a line are $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ or $(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$.
4. $(2/7, 3/7, -6/7)$

*

Sample Quiz

1. Show that, for each $k \in I$, $2 \sin kx \cos x = \sin(k+1)x + \sin(k-1)x$.
2. Express $\tan 3x$ in terms of '$\sin x$' and '$\cos x$'.
3. Tell whether cot is an even function or an odd function or neither. Explain your answer.
4. Find the values of $\cos p$ and $\sin p$ given that $\cot p = -7/24$ and $2\pi \le p \le -\pi$.
5. Determine whether or not the following is an identity:
$$\frac{\csc x - \sin x}{\csc x + \sin x} = \frac{\cot^2 x}{\csc^2 x + 1}$$
6. For what values of 'x' is the equation in Exercise 5 meaningless?

Key to Sample Quiz

1. $$\begin{aligned} 2 \sin kx \cos x &= \sin kx \cos x + \sin kx \cos x \\ &= (\sin kx \cos x + \cos kx \sin x) + (\sin kx \cos x - \cos kx \sin x) \\ &= \sin(kx + x) + \sin(kx - x) \\ &= \sin(k+1)x + \sin(k-1)x \end{aligned}$$
2. $\tan 3x = \sin 3x/\cos 3x = [3 \sin x - 4 \sin^3 x]/[4 \cos^3 x - 3 \cos x]$
3. cot is an odd function, for $\cot(-x) = \cos(-x)/\sin(-x) = -\cos x/\sin x = -\cot x$.
4. $\cos p = -7/25$ and $\sin p = 24/25$
5. It is an identity, for
$$\frac{\csc x - \sin x}{\csc x + \sin x} = \frac{(\csc x - \sin x)\csc x}{(\csc x + \sin x)\csc x} = \frac{\csc^2 x - 1}{\csc^2 x + 1} = \frac{\cot^2 x}{\csc^2 x + 1}.$$
6. The even multiples of $\pi/2$. [Note that neither denominator ever has the value zero. So, to say that the equation is meaningless when the denominators have value 0 does not tell the whole story.]

obtained above. [Why?] Since either set of direction cosines of l are the components, with respect to an orthonormal basis, of a unit vector, we have, in either case,

$$(1) \qquad \cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 = 1.$$

In case l is an oriented line it is natural to choose $\vec{v}$ to be the unit vector in the positive sense of l and define

$$(2) \qquad \cos \alpha_1 = \vec{v} \cdot \vec{u}_1,\ \cos \alpha_2 = \vec{v} \cdot \vec{u}_2, \text{ and } \cos \alpha_3 = \vec{v} \cdot \vec{u}_3.$$

With this convention, each oriented line has just one sequence of direction cosines and the oppositely oriented line has the opposites of these numbers as its direction cosines.

Suppose that P and Q are two points of l with coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3), respectively. Suppose, also, that we choose to orient l so that $Q - P$ is in the positive sense. Since the components of $Q - P$ are $(y_1 - x_1, y_2 - x_2, y_3 - x_3)$ it follows that the components of the unit vector in the positive sense are

$$\left(\frac{y_1 - x_1}{d}, \frac{y_2 - x_2}{d}, \frac{y_3 - x_3}{d}\right),$$

where $d = \|Q - P\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2}$. So, the direction cosines of the oriented line l are given by:

$$(3) \qquad \cos \alpha_1 = \frac{y_1 - x_1}{d},\ \cos \alpha_2 = \frac{y_2 - x_2}{d},\ \cos \alpha_3 = \frac{y_3 - x_3}{d}$$

Exercises

Part A

1. In each of the following you are given the coordinates of a point P and a point Q. Find the direction cosines of $\overleftrightarrow{PQ}$ when this line is oriented so that the positive sense is that of $Q - P$.
   (a) P: (5, 2, 2), Q: (8, 5, 4) (b) P: (1, 1, 1), Q: (2, 0, 3)
   (c) P: (−3, −4, 1), Q: (−4, −1, −4) (d) P: (−1, 5, −5), Q: (−1, 2, −5)
2. For each of the lines $\overleftrightarrow{PQ}$ of Exercise 1, find its direction cosines when it is oriented in such a way that $\cos \alpha_3 > 0$, or, if $\cos \alpha_3 = 0$, so that $\cos \alpha_2 < 0$.
3. Find the direction cosines of a line which makes the same angle with each of the coordinate axes. [*Hint*: Use (1). How many answers are there?]
4. A line has direction numbers (−2, −3, 6) and is oriented so that its second direction cosine is positive. What are its direction cosines?

*

Consider two oriented lines, l and m, with $\vec{v}$ and $\vec{w}$ as the unit vectors in their positive senses. The oriented line l has direction cosines given by (2) and m has direction cosines

$$(4) \qquad \cos \beta_1 = \vec{w} \cdot \vec{u}_1, \ \cos \beta_2 = \vec{w} \cdot \vec{u}_2, \text{ and } \cos \beta_3 = \vec{w} \cdot \vec{u}_3.$$

As in the case of l,

$$(5) \qquad \cos^2 \beta_1 + \cos^2 \beta_2 + \cos^2 \beta_3 = 1.$$

It is customary to say that "the angle between" the oriented lines l and m is any angle whose sides have the senses of $\vec{v}$ and $\vec{w}$. Our problem is to find the cosine of any such angle in terms of the direction cosines of the two lines. Now, one such angle is $\angle AOB$, where O is the origin of the coordinate system and A and B have coordinates $(\cos \alpha_1, \cos \alpha_2, \cos \alpha_3)$ and $(\cos \beta_1, \cos \beta_2, \cos \beta_3)$, respectively. [Explain.] We can use the law of cosines to find $\cos \angle AOB$:

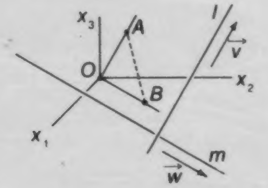

Fig. 19–11

$$(6) \qquad \cos \angle AOB = \frac{OA^2 + OB^2 - AB^2}{2OA \cdot OB}$$

By (1) and (5), $OA = OB = 1$. Also,

$$\begin{aligned} AB^2 &= (\cos \beta_1 - \cos \alpha_1)^2 + (\cos \beta_2 - \cos \alpha_2)^2 + (\cos \beta_3 - \cos \alpha_3)^2 \\ &= \cos^2 \beta_1 + \cos^2 \beta_2 + \cos^2 \beta_3 + \cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 \\ &\quad - 2(\cos \beta_1 \cos \alpha_1 + \cos \beta_2 \cos \alpha_2 + \cos \beta_3 \cos \alpha_3) \\ &= 2[1 - (\cos \beta_1 \cos \alpha_1 + \cos \beta_2 \cos \alpha_2 + \cos \beta_3 \cos \alpha_3)]. \end{aligned}$$

Substituting into (6) we find that

$$(7) \qquad \cos \angle AOB = \cos \beta_1 \cos \alpha_1 + \cos \beta_2 \cos \alpha_2 + \cos \beta_3 \cos \alpha_3.$$

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The sense of $\overrightarrow{OA}$ is that of the vector A - O which has components $(\cos \alpha_1, \cos \alpha_2, \cos \alpha_3)$. Similarly, the sense of $\overrightarrow{OB}$ is that of B - O whose components are $(\cos \beta_1, \cos \beta_2, \cos \beta_3)$.

*

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Answers for Part B

1. 2/63 [≐ 0.0317]
2. 88.2°
3. Since $7 \cdot 2 + -2 \cdot 1 + 4 \cdot -3 = 0$, vectors in the directions of the two lines are orthogonal. So, the lines are perpendicular. [Note that to test for perpendicularity it is not necessary to find components of unit vectors in the directions of the given line. In analytic geometry terms, it is enough to know any direction numbers of the two lines — it is not necessary to find direction cosines.]

Part B

1. Two lines have direction numbers (2, −6, 3) and (8, 1, −4), respectively, and are oriented so that their third direction cosines are positive. Find the cosine of the angle between these oriented lines.
2. Find the degree-measure of the angle between the oriented lines of Exercise 1.
3. Show that lines with direction numbers (7, −2, 4) and (2, 1, −3) are perpendicular.

*

In Chapter 13 we dealt with figures in a given plane π by introducing orthonormal (x, y)-coordinates based on an origin $O \in \pi$ and an orthonormal basis $(\vec{u}, \vec{v})$ for $[\pi]$. If we consider the chosen basis to be positively sensed we have chosen an orientation of π and can speak of sensed angles as being positively or negatively sensed.

Given a line $l \subseteq \pi$, the *inclination* α of l with respect to a given coordinate system is the least number which is the measure of a null or positively sensed angle whose initial side has for its sense the positive sense of the oriented x-axis and whose terminal side has one of the senses in $[l]$. Evidently $0 \leq \alpha < \pi$.

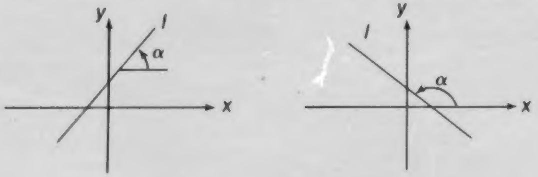


Fig. 19–12

Suppose, now, that P_1 and P_2 are two points of l with coordinates (x_1, y_1) and (x_2, y_2), respectively, and are such that $y_2 \geq y_1$. It follows that, if $d = P_1P_2$ and α is the inclination of l,

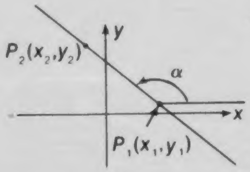


Fig. 19–13

$$x_2 = x_1 + d \cos \alpha \text{ and } y_2 = y_1 + d \sin \alpha.$$

Assuming that $x_2 \neq x_1$ – that is, that $\alpha \neq \pi/2$ – it follows that

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{8}$$

Since $(y_1 - y_2)/(x_1 - x_2) = (y_2 - y_1)/(x_2 - x_1)$, (8) holds whether or not the restriction that $y_2 \geq y_1$ is satisfied. Recalling that the right side of (8) is the slope of l [with respect to the given coordinate system] we have that

(9) the slope of a nonvertical line l is
the tangent of the inclination of l.

[Recall that the slope of a vertical line, like 'tan $(\pi/2)$' is not defined.]

Two intersecting lines, l_1 and l_2, contain four angles but, since vertical angles have the same measure, there are just two correct answers to the question "What is the measure of an angle contained in $l_1 \cup l_2$?" Clearly, for any intersecting lines l_1 and l_2 the sum of the answers to this question will be π. To obtain a unique answer we define "the" angle from l_1 to l_2 to be the smallest positively sensed angle whose initial side is contained in l_1 and whose terminal side is contained in l_2. [There are actually two such angles, but they have the same measure.] If θ [thay'ta] is the measure of this angle then it is clear from Fig. 19–14 that if $\alpha_1 < \alpha_2$ then $\theta = \alpha_2 - \alpha_1$, while if $\alpha_1 > \alpha_2$ then $\alpha_1 - \alpha_2 = \pi - \theta$. Since tan has the period π it follows, in either case, that $\tan \theta = \tan (\alpha_2 - \alpha_1)$. So, if m_1 is the slope of l_1 and m_2 is the slope of l_2 then, unless $l_1 \perp l_2$, it follows by the subtraction law for tan that

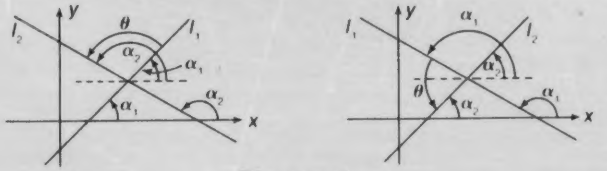


Fig. 19–14

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{10}$$

In deriving (10) it has been tacitly assumed that neither l_1 nor l_2 is vertical and that $l_1 \nparallel l_2$, as well as that $l_1 \not\perp l_2$. Equation (10) gives the correct value, 0, for $\tan \theta$ in case $l_1 \parallel l_2$. As we already know, $l_1 \perp l_2$ if and only if $m_1 m_2 = -1$. And, if $l_1 \perp l_2$ then $\theta = \pi/2$ and 'tan θ' is not defined.

Part C

1. (a) Find the measure of the angle from the line l_1 whose equation is '$x + 3y + 1 = 0$' to the line l_2 whose equation is '$x - 2y = 3$'.
   (b) What is the measure of the angle from l_2 to l_1?
2. Find an equation for a line l, through the point whose coordinates are $(-1, 2)$, such that the measure of the angle from the line whose equation is '$4x - 3y + 5 = 0$' to l is $3\pi/4$.
3. The sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of $\triangle ABC$ are contained in lines with equations '$x + 8y - 2 = 0$', '$2x - 3y + 5 = 0$', and '$2x + 5y + 10 = 0$', respectively.
   (a) Find $\tan \angle A$, $\tan \angle B$, and $\tan \angle C$.
   (b) Check that the measure of an exterior angle at A is the sum of the measures of $\angle B$ and $\angle C$. [*Hint*: Use the addition law for tan.]

*

According to the congruence theorems for triangles [Theorem 16–7] if one is given the measures of certain "parts"—sides or angles—of a triangle, he should be able to compute the measures of the other parts. You have solved various problems of this kind already. Now we shall discuss this kind of problem systematically.

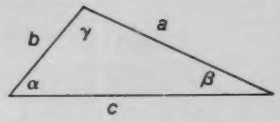


Fig. 19–15

There are four cases to be considered:

- I Given the measures of three sides of a triangle, find the measures of its angles.
- II Given the measures of two sides and of the included angle, find the measure of the other side and the measures of the other angles.
- III Given the measure of one side and the measures of the including angles, find the measures of the other two sides and the measure of the other angle.
- IV Given the measures of two sides and the measure of the angle opposite one of them, find the measures of the third side and of the other angles.

Case I: Given a, b, and c, the cosines of the angles of the triangle can be found by using the law of cosines. Then, the measures of the angles can be found from the tables. [If numbers are given for 'a', 'b', and 'c' such that no triangle has these as side-measures—say, $a = 1$, $b = 1$, and $c = 3$—then the computations will lead to values for the cosines of the angles which are not between -1 and 1.] You should check your

Answers for Part C

1. (a) $\pi/4$ [$\tan\theta = (1/2 + 1/3)/(1 - /2 \cdot /3) = 1$]
   (b) $3\pi/4$
2. $x - 7y + 15 = 0$ [$(m - 4/3)/(1 + 4m/3) = -1$ if and only if $m = 1/7$.]
3. (a) $\tan\angle A = 11/42$, $\tan\angle B = 19/22$, $\tan\angle C = -16/11$
   (b) $\tan(\beta + \gamma) = (19/22 - 16/11)/[1 + (19/22)(16/11)] = -11/42 = -\tan\alpha$, where α, β, γ are the measures of $\angle A$, $\angle B$, and $\angle C$, respectively.

[In part (a) students may need a figure to decide which slope is m_2 and which is m_1 in applying (10) on page 459.]

work by using the fact that the sum of the measures of the angles is π [or that the sum of their degree-measures is 180].

Case II: Given a, b, and γ the cosine law can be used to compute c. Then, as in case I, the cosine law can be used to compute α and β. Alternately, once c is computed, the sine law can be used to compute α and β:

$$\sin\alpha = \frac{a\sin\gamma}{c}, \sin\beta = \frac{b\sin\gamma}{c}$$

[This alternative procedure can also be used in case I once the measure of one of the angles has been computed.]

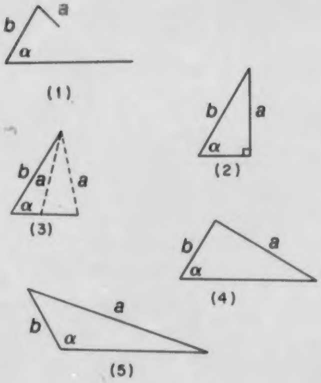
A third procedure in case II is to use the fact that $\alpha + \beta = \pi - \gamma$ [or, if degree-measures are used, $180 - \gamma$] and to use Theorem 19–28 to compute $\alpha - \beta$. From this and $\alpha + \beta$ one can find α and β. Then, to compute c, one can use the sine law, $c = a \sin\gamma/\sin\alpha$.

Case III: Given a, β, and γ one can easily compute α. Then, use the sine law to compute b and c.

Case IV: Given a, b, and α there are four possibilities in case $0 < \alpha < \pi/2$:

(1) $a < b \sin\alpha$, in which case there is no solution.
(2) $a = b \sin\alpha$, in which case the triangle is a right triangle with right angle at B.
(3) $b \sin\alpha < a < b$ in which case there are two solutions.
(4) $b \leq a$, in which case there is just one solution.

In case $\pi/2 \leq \alpha < \pi$ then there is a solution only if $b < a$, and in this case there is just one solution.



(1) (2) (3) (4) (5)

Fig. 19–16

In spite of this variety of subcases, the procedure for solving a case IV triangle is the same in all of them. If there is a solution then

$$(*) \qquad \sin\beta = \frac{b\sin\alpha}{a}.$$

If there is no solution you will find that $b \sin\alpha/a > 1$ [subcase (1)], and your work is done.

If $b \sin \alpha/a = 1$ [subcase (2)] then $\beta = \pi/2$, $\gamma = \pi/2 - \alpha$, and $c = b \cos \alpha$.

If $b \sin \alpha/a < 1$ then there are two values of 'β' which satisfy (*). One of these values, β_1, is between 0 and $\pi/2$; the other, β_2, is $\pi - \beta_1$. You are certain of one solution of the triangle:

$$\beta = \beta_1, \gamma = \gamma_1 = \pi - (\alpha + \beta), c = a \sin \gamma/\sin \alpha$$

If $a \geq b$ then this is the only solution. If $a < b$ then there is a second solution:

$$\beta = \beta_2, \gamma = \gamma_2 = \pi - (\alpha + \beta_2), c = a \sin \gamma_2/\sin \alpha$$

[Rather than trying to remember that the second solution arises when $a < b$, it is simpler to compute γ_2. The second solution will exist if and only if $\gamma_2 > 0$.]

Part D

Solve each of the following triangles—that is, find the measures of the sides and angles whose measures are not given. [Use degree-measure for angles.]

1. $a = 3$, $b = 10$, and $c = 8$

2. $a = 3$, $b = 2$, and γ is 60 [That is, $\angle C$ is a 60° angle.]

3. $\alpha = 60$, $\beta = 45$, and $c = 5$

4. $a = 4$, $b = 10$, and $\alpha = 30$

5. $a = 5$, $b = 10$, and $\alpha = 30$

6. $a = 8$, $b = 10$, and $\alpha = 30$

7. $a = 20$, $b = 10$, and $\alpha = 30$

8. $a = 20$, $b = 10$, and $\alpha = 150$

Part E

In preparation for the next section we need some properties of cos and sin which follow from Theorem 19–14 and the corollary to Theorem 19–17. One of the questions we need to answer is: When is it the case that $\cos a = \cos b$? Now, by Theorem 19–14,

$$\cos a = \cos b \longrightarrow (\sin [(a + b)/2] = 0 \text{ or } \sin [(a - b)/2] = 0).$$

By the corollary to Theorem 19–17,

$$\begin{aligned} \sin [(a + b)/2] = 0 &\longleftrightarrow \exists_{k \in I}\, \frac{a + b}{2} = k\pi \\ &\longleftrightarrow \exists_{k \in I}\, a = 2k\pi - b. \end{aligned}$$

Similarly, $\sin [(a - b)/2] = 0$ if and only if there is an integer k such that $a = 2k\pi + b$. Hence,

$$\cos a = \cos b \longleftrightarrow \exists_{k \in I}\, (a = 2k\pi - b \text{ or } a = 2k\pi + b).$$

Answers for Part D

1. $\alpha \doteq 14.4$, $\beta \doteq 124.2$, $\gamma \doteq 41.4$
2. $c \doteq 2.646$, $\alpha \doteq 79.1$, $\beta \doteq 40.9$
3. $\gamma = 75$, $b \doteq 3.659$, $a \doteq 4.482$
4. [no solution]
5. $\beta = 90$, $\gamma = 60$, $c = 5\sqrt{3} \doteq 8.662$
6. $\beta_1 \doteq 38.7$, $\gamma_1 \doteq 111.3$, $c_1 \doteq 14.91$;
   $\beta_2 \doteq 141.3$, $\gamma_2 \doteq 8.7$, $c_2 \doteq 2.421$
7. $\beta \doteq 14.5$, $\gamma \doteq 35.5$, $c \doteq 23.23$
8. $\beta \doteq 14.5$, $\gamma \doteq 15.5$, $c \doteq 10.69$

This result should seem intuitively obvious if you look at a graph of cos. It is, ultimately, a consequence of the fact that cos is even and has period 2π.

1. **Show that $\sin a = \sin b$ if and only if $\exists_{k\in I}\, a = k\pi + (-1)^k b$.**
2. **Show that $\tan a = \tan b$ if and only if $\exists_{k\in I}\, a = k\pi + b$ [*Hint*: For the only if-part, use the subtraction law for tan; for the if-part use the fact that tan has period π.]**

*

Theorem 19-32

(a) $\cos a = \cos b \longleftrightarrow \exists_{k\in I}\,(a = 2k\pi - b \text{ or } a = 2k\pi + b)$

(b) $\sin a = \sin b \longleftrightarrow \exists_{k\in I}\, a = k\pi + (-1)^k b$

(c) $\tan a = \tan b \longleftrightarrow \exists_{k\in I}\, a = k\pi + b$
[neither a nor b an odd multiple of $\pi/2$]

(d) $\cot a = \cot b \longleftrightarrow \exists_{k\in I}\, a = k\pi + b$
[neither a nor b an even multiple of $\pi/2$]

19.10 The Inverse Circular Functions

Recall that a function f has an inverse if and only if the converse of f is a function—that is, if and only if f does not have the same value for any two of its arguments. If this condition is satisfied then the converse of f is called *the inverse of f* and is sometimes denoted by 'f^{-1}'. The notion of the inverse of a function is discussed in Section 1.03, pages 18 and 19 of Volume 1 and, also, on pages 24–26 of Volume 1. To prepare the way for introducing some new functions—the inverse circular functions—we shall review a similar problem dealing with the squaring function, $\{(x, y)\colon y = x^2\}$. To make clear the similarity of the two problems we shall give the squaring function a name, 'sq':

$$\text{sq} = \{(x, y)\colon y = x^2\}$$

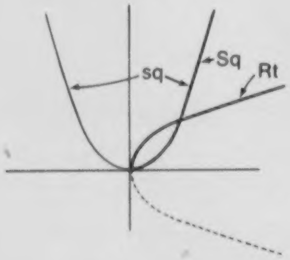


Fig. 19-17

Answers for Part E

1. $\sin a = \sin b \Longleftrightarrow (\cos[(a + b)/2] = 0 \text{ or } \sin[(a - b)/2] = 0)$

$$\Longleftrightarrow \left(\exists_{k\in I}\ \frac{a+b}{2} = \frac{(2k-1)\pi}{2} \text{ or } \exists_{k\in I}\ \frac{a-b}{2} = k\pi\right)$$

$$\Longleftrightarrow (\exists_{k\in I}\ a = -b + (2k-1)\pi \text{ or } \exists_{k\in I}\ a = b + 2k\pi)$$

$$\Longleftrightarrow \exists_{k\in I}\ a = k\pi + (-1)^k b$$

2. Suppose that $\tan a = \tan b$. It follows that neither a nor b is an odd multiple of $\pi/2$. Also, $a - b$ is not an odd multiple of $\pi/2$ for, if so, $\tan a \tan b = -1$ and $\tan a \neq \tan b$. So, the subtraction formula for tan holds and $\tan(a - b) = 0$. So, $\sin(a - b) = 0$ and, for some $k \in I$, $a = k\pi + b$. Hence, if $\tan a = \tan b$ then $\exists_{k\in I}\ a = k\pi + b$. On the other hand, if $a = k\pi + b$ where $k \in I$ and neither a nor b is an odd multiple of $\pi/2$ then, since tan has period π, $\tan a = \tan b$.

[A similar argument applies to the cotangent, giving the result stated in Theorem 19-32(d). However, it is probably simpler to treat first the case in which neither a nor b is any multiple of $\pi/2$ and note that, in this case, $\cot a = \cot b$ if and only if $/\cot a = /\cot b$ and that this holds if and only if $\tan a = \tan b$. Then, complete the discussion by noting that if a or b is an odd multiple of $\pi/2$ then $\cot a = 0 = \cot b$ and there is a $k \in I$ such that $a = k\pi + b$.]

The reason for introducing the peculiar notations 'sq', 'Sq', and 'Rt' in place of exponents and radical signs is that this forces students to think about what is being said and, so, appreciate better the statements (1) - (6). Statements analogous to these can — and will — be made for each of the functions cos, sin, tan, and cot. For example, (7) - (12) on page 465 are the corresponding statements for cos. In them, 'cos' plays the role of 'sq', 'Cos' that of 'Sq', and 'Arccos' that of 'Rt'.

Recall that (2) is a consequence of the fact that Sq is an increasing function. Similarly, (8) on page 465 is a consequence of the fact that Cos is a decreasing function.

* * *

Suggestions for the exercises of section 19.10:

(i) With appropriate examples, Parts A and B may be used for homework.
(ii) Part C may be used for homework.
(iii) Part D should be teacher supervised.

[What is sq(2)? sq(−3)? sq(3)? What is the domain of sq?] We shall also deal with a subset, Sq, of sq:

$$\text{Sq} = \{(x, y): x \geq 0 \text{ and } y = x^2\}$$

[What is the domain of Sq?] Since sq(−3) = sq(3) it is clear that sq does not have an inverse. In fact, we have proved in the background exercises for the introduction [pages 6–8] that

(1) $$\text{sq}(a) = \text{sq}(b) \Longleftrightarrow (a = b \text{ or } a = -b).$$

We also proved that

(2) $$\text{sq}(a) = \text{sq}(b) \Longrightarrow a = b \qquad [a \geq 0, b \geq 0]$$

and this tells us that the function Sq does have an inverse. This inverse is the principal square rooting function and, for the present, we shall call it 'Rt'. Its graph is shown in Fig. 19–17, and it may be described as follows:

$$\text{Rt} = \{(x, y): y \geq 0 \text{ and } \text{sq}(y) = x\}$$

In other words,

(3) Rt(a) is the number z such that $z \geq 0$ and sq(z) = a.

In consequence, we have:

(4) $$\text{sq}(\text{Rt}(a)) = a \qquad [a \geq 0]$$

and:

(5) $$\text{Rt}(\text{sq}(a)) = a \qquad [a \geq 0]$$

Finally, because of (1) and (4), we have:

(6) $$\text{sq}(a) = c \Longleftrightarrow (a = \text{Rt}(c) \text{ or } a = -\text{Rt}(c)) \qquad [c \geq 0]$$

[Replace 'c' by 'sq(Rt(c))', by (4), and apply (1).]

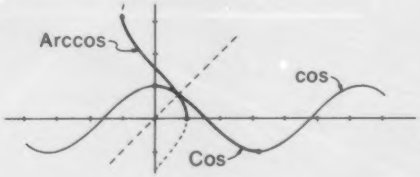


Fig. 19–18

All that we have said about sq can now be paralleled by statements about cos. Paralleling (1) we have, by Theorem 19–32(a), that

(7) $$\cos a = \cos b \Longleftrightarrow \exists_{k \in I}\, (a = 2k\pi - b \text{ or } a = 2k\pi + b).$$

In particular, cos does not have an inverse. Since cos is decreasing in the segment $\overline{0, \pi}$ we have that

(8) $$\cos a = \cos b \Longrightarrow a = b \qquad [0 \leq a \leq \pi, 0 \leq b \leq \pi].$$

Hence, if we define a function Cos by:

$$\text{Cos} = \{(x, y): 0 \leq x \leq \pi \text{ and } y = \cos x\}$$

it follows that this function Cos does have an inverse. We might call this inverse 'Cos^{-1}', but it is customary to call it 'Arccos'. [This comes from the phrase 'the arc whose cosine is'.] Note that the domain of Arccos is the range of Cos and, so, is $\{x: -1 \leq x \leq 1\}$. By definition,

$$\text{Arccos} = \{(x, y): 0 \leq y \leq \pi \text{ and } \cos y = x\}.$$

In other words:

(9) Arccos a is the z such that $0 \leq z \leq \pi$ and $\cos z = a$.

In consequence, we have:

(10) $$\cos(\text{Arccos } a) = a \qquad [-1 \leq a \leq 1]$$

and:

(11) $$\text{Arccos}(\cos a) = a \qquad [0 \leq a \leq \pi]$$

[Note the restriction!] Finally, because of (7) and (10), we have:

(12) $$\cos a = c \Longleftrightarrow \exists_{k \in I}\, (a = 2k\pi - \text{Arccos } c \text{ or } a = 2k\pi + \text{Arccos } c) \qquad [-1 \leq c \leq 1]$$

[Replace 'c' by 'cos (Arccos c)', by (10), and apply (7).]

We shall later collect (9)–(12) into a definition [Definition 19–7] and three theorems [Theorems 19–33 through 19–35]. These will contain other parts like (9)–(12) referring to sin, tan, and cot.

Exercises

Part A

1. Evaluate each of the following.
   (a) Arccos $(\frac{1}{2})$ (b) Arccos $(\sqrt{2}/2)$ (c) Arccos 0
   (d) Arccos (-1) (e) Arccos $(-\sqrt{2}/2)$ (f) Arccos $(-\sqrt{3}/2)$
   (g) Arccos $[\cos(\pi/2)]$ (h) Arccos $[\cos(-\pi/2)]$ (i) Arccos $[\sin(\pi/2)]$
2. Solve each of the following equations. [*Hint*: Use (12), and check against a graph of cos.]
   (a) $\cos x = 1$ (b) $\cos x = \frac{1}{2}$ (c) $\cos x = 5$
   (d) $\cos y = \sqrt{3}/2$ (e) $\cos t = 0$ (f) $\mathrm{Cos}\, x = -\frac{1}{2}$
   (g) Arccos $x = 0$ (h) Arccos $a = 1$ (i) Arccos $b = 4$
3. Show that, for $0 \le a \le \pi$ and $-1 \le b \le 1$,

$$\cos a = b \longleftrightarrow a = \mathrm{Arccos}\, b.$$

There are analogues of (1)–(6) and (7)–(12) for the sine function.

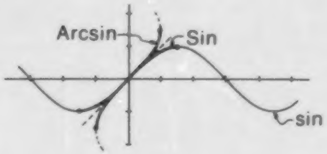


Fig. 19–19

This function, like cos, does not have an inverse. In fact, by Theorem 19–32(b) we have:

$$(13) \qquad \sin a = \sin b \longleftrightarrow \exists_{k \in I}\, a = k\pi + (-1)^k b$$

But, since sin is increasing on the segment $\overline{-\pi/2, \pi/2}$, we also have:

$$(14) \qquad \sin a = \sin b \longrightarrow a = b \qquad [-\pi/2 \le a \le \pi/2,\ -\pi \le b \le \pi/2]$$

Hence, if we define a function Sin by:

$$\mathrm{Sin} = \{(x, y)\colon -\pi/2 \le x \le \pi/2 \text{ and } y = \sin x\}$$

it follows that this function Sin does have an inverse. We call this inverse 'Arcsin'. [What is the domain of Arcsin?] By definition,

$$\mathrm{Arcsin} = \{(x, y)\colon -\pi/2 \le y \le \pi/2 \text{ and } \sin y = x\}.$$

In other words:

(15) Arcsin a is the z such that $-\pi/2 \le z \le \pi/2$ and $\sin z = a$.

Answers for Part A

1. (a) $\pi/3$ (b) $\pi/4$ (c) $\pi/2$
   (d) π (e) $3\pi/4$ (f) $5\pi/6$
   (g) $\pi/2$ (h) $\pi/2$ (i) 0
2. (a) $\{x\colon \exists_{k \in I}\, x = 2k\pi\}$
   (b) $\{x\colon \exists_{k \in I}\, (x = (6k - 1)\pi/3 \text{ or } x = (6k + 1)\pi/3)\}$
   (c) $\emptyset$
   (d) $\{x\colon \exists_{k \in I}\, (x = (12k - 1)\pi/6 \text{ or } x = (12k + 1)\pi/6)\}$
   (e) $\{x\colon \exists_{k \in I}\, (x = (4k - 1)\pi/2 \text{ or } x = (4k + 1)\pi/2)\}$
   (f) $2\pi/3$ (g) $\pi/2$ (h) 0 (i) [no solution]
3. For $-1 \le b \le 1$, Arccos b is the z such that $0 \le z \le \pi$ and $\cos z = b$. So, for $0 \le a \le \pi$ and $-1 \le b \le 1$, $\cos a = b$ if and only if $a =$ Arccos b.

*

The domain of Arcsin is $\{x\colon |x| \le 1\}$.

In consequence, we have:

$$(16)\qquad \sin(\text{Arcsin}\, a) = a \qquad [-1 \le a \le 1]$$

and:

$$(17)\qquad \text{Arcsin}(\sin a) = a \qquad [-\pi/2 \le a \le \pi/2]$$

[Note the restriction!] Finally, because of (16) and (13), we have:

$$(18)\qquad \sin a = c \Longleftrightarrow \exists_{k \in I}\, a = k\pi + (-1)^k\, \text{Arcsin}\, c \; [-1 \le c \le 1]$$

[Replace 'c' by 'sin (Arcsin c)' and use (13).]

Part B

1. Evaluate each of the following.
   (a) Arcsin ($\frac{1}{2}$) (b) Arcsin ($\sqrt{2}/2$) (c) Arcsin 0
   (d) Arcsin (-1) (e) Arcsin ($-\sqrt{2}/2$) (f) Arcsin ($-\sqrt{3}/2$)
   (g) Arcsin [cos ($\pi/2$)] (h) Arcsin [cos ($-\pi/2$)] (i) Arcsin [sin ($\pi/2$)]
2. Compare your answers for Exercise 1 with those for Exercise 1 of Part A.
3. Solve each of the following equations.
   (a) $\sin x = 1$ (b) $\sin x = \frac{1}{2}$ (c) $\sin x = 2$
   (d) $\sin y = \sqrt{3}/2$ (e) $\sin t = 0$ (f) $\text{Sin}\, x = -\frac{1}{2}$
   (g) $\text{Arcsin}\, x = 0$ (h) $\text{Arcsin}\, a = 1$ (i) $\text{Arcsin}\, b = -2$
4. (a) Solve '$\sin x = \cos y$' for 'x'. [*Hint*: cos y = sin?]
   (b) Solve '$\sin x = \cos x$'.
5. Show that, for $-\pi/2 \le a \le \pi/2$ and $-1 \le b \le 1$,

$$\sin a = b \Longleftrightarrow a = \text{Arcsin}\, b.$$

Like cos and sin, tan and cot do not themselves have inverses but each has subsets which have inverses. Two such subsets, Tan of tan and Cot of cot, are pictured in Fig. 19–20.

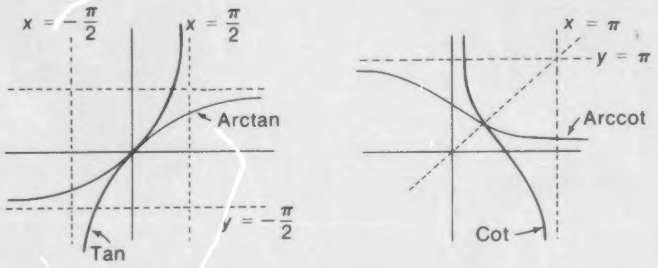


Fig. 19–20

Answers for Part B

1. (a) $\pi/6$ (b) $\pi/4$ (c) 0
   (d) $-\pi/2$ (e) $-\pi/4$ (f) $-\pi/3$
   (g) 0 (h) 0 (i) $\pi/2$
2. The comparison suggests that, for a such that $|a| \le 1$, $\text{Arccos}\, a + \text{Arcsin}\, a = \pi/2$. [See Theorem 19-36(a) on page 470.]
3. (a) $\{x\colon \exists_{k \in I}\, x = (2k + (-1)^k)\pi/2\}$
   (b) $\{x\colon \exists_{k \in I}\, x = (6k + (-1)^k)\pi/6\}$
   (c) $\emptyset$
   (d) $\{x\colon \exists_{k \in I}\, x = (3k + (-1)^k)\pi/3\}$
   (e) $\{x\colon \exists_{k \in I}\, x = k\pi\}$
   (f) $-\pi/3$ (g) 0 (h) $\pi/2$ (i) [no solution]
4. (a) $\exists_{k \in I}\, x = (2k + (-1)^k)\pi/2 - (-1)^k y$
   (b) $\exists_{k \in I}\, x = (4k + 1)\pi/4$
5. For $-1 \le b \le 1$, Arcsin b is the z such that $-\pi/2 \le z \le \pi/2$ and $\sin z = b$. So, for $-\pi/2 \le a \le \pi/2$ and $-1 \le b \le 1$, $\sin a = b$ if and only if $a = \text{Arcsin}\, b$.

In each case we have analogues of (1)–(6). In the case of tan they are:

(19) $\tan a = \tan b \iff \exists_{k \in I}\ a = k\pi + b$ [a and b not odd multiples of $\pi/2$]

(20) $\tan a = \tan b \implies a = b$ [$-\pi/2 < a < \pi/2,\ -\pi/2 < b < \pi/2$]

$$\text{Tan} = \{(x, y)\colon -\pi/2 < x < \pi/2 \text{ and } y = \tan x\}$$

(21) Arctan a is the z such that $-\pi/2 < z < \pi/2$ and $\tan z = a$.

(22) $\tan(\text{Arctan } a) = a$

(23) $\text{Arctan}(\tan a) = a$ [$-\pi/2 < a < \pi/2$]

(24) $\tan a = c \iff \exists_{k \in I}\ a = k\pi + \text{Arctan } c$

In the case of cot we have:

(25) $\cot a = \cot b \iff \exists_{k \in I}\ a = k\pi + b$ [a and b not even multiples of $\pi/2$]

(26) $\cot a = \cot b \implies a = b$ [$0 < a < \pi,\ 0 < b < \pi$]

$$\text{Cot } a = \{(x, y)\colon 0 < x < \pi \text{ and } y = \cot x\}$$

(27) Arccot a is the z such that $0 < z < \pi$ and $\cot z = a$.

(28) $\cot(\text{Arccot } a) = a$

(29) $\text{Arccot}(\cot a) = a$ [$0 < a < \pi$]

(30) $\cot a = c \iff \exists_{k \in I}\ a = k\pi + \text{Arccot } c$

As promised, we collect some of our results into a definition and some theorems:

Definition 19-7
- (a) Arccos a is the number z such that $0 \le z \le \pi$ and $\cos z = a$.
- (b) Arcsin a is the number z such that $-\pi/2 \le z \le \pi/2$ and $z = a$.
- (c) Arctan a is the number z such that $-\pi/2 < z < \pi/2$ and $\tan z = a$.
- (d) Arccot a is the number z such that $0 < z < \pi$ and $\cot z = a$.

Theorem 19-33
- (a) $\cos(\text{Arccos } a) = a$ [$-1 \le a \le 1$]
- (b) $\sin(\text{Arcsin } a) = a$ [$-1 \le a \le 1$]
- (c) $\tan(\text{Arctan } a) = a$
- (d) $\cot(\text{Arccot } a) = a$

Theorem 19-34
- (a) $\text{Arccos}(\cos a) = a$ [$0 \le a \le \pi$]
- (b) $\text{Arcsin}(\sin a) = a$ [$-\pi/2 \le a \le \pi/2$]
- (c) $\text{Arctan}(\tan a) = a$ [$-\pi/2 < a < \pi/2$]
- (d) $\text{Arccot}(\cot a) = a$ [$0 < a < \pi$]

Theorem 19-35
- (a) $\cos a = c \iff \exists_{k \in I}\ (a = 2k\pi - \text{Arccos } c$ or $a = 2k\pi + \text{Arccos } c)$ [$-1 \le c \le 1$]
- (b) $\sin a = c \iff \exists_{k \in I}\ a = k\pi + (-1)^k \text{Arcsin } c$ [$-1 \le c \le 1$]
- (c) $\tan a = c \iff \exists_{k \in I}\ a = k\pi + \text{Arctan } c$
- (d) $\cot a = c \iff \exists_{k \in I}\ a = k\pi + \text{Arccot } c$

Corollary
- (a) $\cos a = b \iff a = \text{Arccos } b$ [$0 \le a \le \pi,\ -1 \le b \le 1$]
- (b) $\sin a = b \iff a = \text{Arcsin } b$ [$-\pi/2 \le a \le \pi/2,\ -1 \le b \le 1$]
- (c) $\tan a = b \iff a = \text{Arctan } b$ [$-\pi/2 < a < \pi/2$]
- (d) $\cot a = b \iff a = \text{Arccot } b$ [$0 < a < \pi$]

[Parts (a) and (b) of the corollary are Exercise 3 of Part A and Exercise 5 of Part B, respectively. Each part of the corollary follows from the corresponding part of Theorem 19-35.]

Rather than memorize Theorem 19-35 and its corollary you should practice recalling them from sketches of the graphs of the functions in question. Theorem 19-33 and Theorem 19-34 merely formulate the fact that, for example, Arccos is the inverse of Cos, where Cos is a subset of cos whose domain is $\{x\colon 0 \le x \le \pi\}$ and whose range, like that of cos, is $\{x\colon -1 \le x \le 1\}$.

Part C

1. Evaluate each of the following.
   - (a) Arctan (-1)
   - (b) Arccot 0
   - (c) Arctan $\sqrt{3}$
   - (d) Arctan (tan 5.79)
   - (e) Arctan [cot $(\pi/6)$]
   - (f) Arccot [tan $(-\pi/4)$]

Answers for Part C

1. (a) $-\pi/4$ (b) $\pi/2$ (c) $\pi/3$
   (d) $5.79 - 2\pi$ [or: -0.49] (e) $\pi/3$ (f) $3\pi/4$

2. (a) $\{x: \exists_{k \in I}\ x = (4k - 1)\pi/4\}$ (b) $\{x: \exists_{k \in I}\ x = (2k + 1)\pi/2\}$
   (c) $\{x: \exists_{k \in I}\ x = (3k + 1)\pi/3\}$
   (d) $\{x: \exists_{k \in I}\ x = k\pi - 0.49\}$ [or: $\ldots = (k - 2)\pi + 5.79\}$]
   (e) $\{x: \exists_{k \in I}\ x = (3k + 1)\pi/3\}$ (f) $\{x: \exists_{k \in I}\ x = (4k + 3)\pi/4\}$

3. (c) By Theorem 19-35(c), $\tan a = b$ if and only if $\exists_{k \in I}\ a = k\pi + \text{Arctan}\, b$. So, assuming that $\tan a = b$ and that $-\pi/2 < a < \pi/2$ it follows that, for some $k \in I$,

$$-\pi/2 < k\pi + \text{Arctan}\, b < \pi/2$$

that is, that

$$-k\pi - \pi/2 < \text{Arctan}\, b < -k\pi + \pi/2.$$

Since, also, $-\pi/2 < \text{Arctan}\, b < \pi/2$ it follows that

$$-k\pi - \pi/2 < \pi/2 \text{ and } -\pi/2 < -k\pi + \pi/2$$

and, so, that

$$-1 < k < 1.$$

Since $k \in I$ it follows that $k = 0$ and, so, that $a = \text{Arctan}\, b$.

(d) [Similar argument using Theorem 19-35(d) and the fact that $0 < \text{Arccot}\, b < \pi$.]

[Parts (c) and (d) of the corollary can also be established by arguments like those given in answer to Exercise 3 of Part A and Exercise 5 of Part B. Conversely, these exercises can be answered by arguments like that given above for part (c) of the corollary, but these arguments are more complex.]

In the text we show that $\text{Arccos}\, a + \text{Arcsin}\, a = \pi/2$ by showing that $\pi/2 - \text{Arccos}\, a = \text{Arcsin}\, a$. According to part (b) of the corollary to Theorem 19-35 it is sufficient to show that $\sin(\pi/2 - \text{Arccos}\, a) = \sin a$ and $-\pi/2 \leq \pi/2 - \text{Arccos}\, a \leq \pi/2$. This we do. Note that this proof is analogous to proofs concerning principle square roots [the function Rt of page 464.] If, for example, we wish to show that $|a| = \sqrt{a^2}$, it is sufficient to show that $|a|^2 = a^2$ and that $|a| \geq 0$. Or, if we wish to show that, for nonnegative a and b, $\sqrt{a}\sqrt{b} = \sqrt{ab}$, it is sufficient to show that $(\sqrt{a}\sqrt{b})^2 = ab$ and $\sqrt{a}\sqrt{b} \geq 0$. The justification for the argument in the text is that $\text{Arcsin}\, a$ is the number whose sine is a and which is between $-\pi/2$ and $\pi/2$ inclusive. The justification for the similar arguments concerning square roots is that, for $a \geq 0$, $\sqrt{a}$ is the number whose square is a and which is nonnegative.

As noted in the text we could show that $\text{Arccos}\, a + \text{Arcsin}\, a = \pi/2$ by using part (a) of the corollary to Theorem 19-35. [Doing so is Exercise 1 of Part D.] To do this we would try to show that $\pi/2 - \text{Arcsin}\, a = \text{Arccos}\, a$. According to the corollary this will follow if we show that $\cos(\pi/2 - \text{Arcsin}\, a) = a$ and $0 \leq \pi/2 - \text{Arcsin}\, a \leq \pi$.

In proving (*), $\sin(\text{Arccos}\, a) \geq 0$ because $0 \leq \text{Arccos}\, a \leq \pi$ and, for $0 \leq b \leq \pi$, $\sin b \geq 0$.

The proof of (**) is asked for in Exercise 5 of Part D.

---

2. Solve each of the following equations.
   (a) $\tan x = -1$ (b) $\cot x = 0$ (c) $\tan x = \sqrt{3}$
   (d) $\tan a = \tan 5.79$ (e) $\tan b = \cot(\pi/6)$ (f) $\cot c = \tan(-\pi/4)$
3. Establish parts (c) and (d) of the corollary to Theorem 19-35.

*

In comparing your answers for Exercise 1 of Part A and Exercise 1 of Part B you probably came to the conclusion that

$$\text{Arccos}\, a + \text{Arcsin}\, a = \frac{\pi}{2}\ [-1 \leq a \leq 1].$$

We can prove that this is the case by using part (a) or part (b) of the corollary to Theorem 19-35. We choose to use part (b).

$$\text{For } -1 \leq a \leq 1,\ \sin(\pi/2 - \text{Arccos}\, a) = \cos(\text{Arccos}\, a) = a.$$

[Explain.] But, since $0 \leq \text{Arccos}\, a \leq \pi$ it follows that $-\pi/2 \leq \pi/2 - \text{Arccos}\, a \leq \pi/2$. Hence, by part (b) of the corollary to Theorem 19-35,

$$\frac{\pi}{2} - \text{Arccos}\, a = \text{Arcsin}\, a.$$

Consequently, $\text{Arccos}\, a + \text{Arcsin}\, a = \pi/2 \quad [-1 \leq a \leq 1]$.

We have proved part (a) of:

> **Theorem 19-36**
> (a) $\text{Arccos}\, a + \text{Arcsin}\, a = \pi/2 \quad [-1 \leq a \leq 1]$
> (b) $\text{Arctan}\, a + \text{Arccot}\, a = \pi/2$

As a lemma for a later exercise similar to Theorem 19-36, we note that

$$(*) \qquad \sin(\text{Arccos}\, a) = \sqrt{1 - a^2} \qquad [-1 \leq a \leq 1].$$

To prove this it is sufficient to note that, for $-1 \leq a \leq 1$, $\sin(\text{Arccos}\, a) \geq 0$ and that

$$\cos^2(\text{Arccos}\, a) + \sin^2(\text{Arccos}\, a) = 1.$$

The conclusion (*) follows at once when one notes that $\cos^2(\text{Arccos}\, a) = a^2$. Similarly,

$$(**) \qquad \cos(\text{Arcsin}\, a) = \sqrt{1 - a^2} \qquad [-1 \leq a \leq 1].$$

Answers to Part D [cont.]

7. For any c, $-1 < c/\sqrt{1+c^2} < 1$. So,

$$\tan(\text{Arcsin}\frac{c}{\sqrt{1+c^2}}) = \sin(\text{Arcsin}\frac{c}{\sqrt{1+c^2}})/\cos(\text{Arcsin}\frac{c}{\sqrt{1+c^2}})$$

$$= \frac{c}{\sqrt{1+c^2}}\Big/\sqrt{1-\frac{c^2}{1+c^2}} = \frac{c}{\sqrt{1+c^2}}\Big/\frac{1}{\sqrt{1+c^2}} = c.$$

Also, since $c/\sqrt{1+c^2}$ is neither 1 nor -1 but is between -1 and 1 it follows that

$$-\frac{\pi}{2} < \text{Arcsin}\frac{c}{\sqrt{1+c^2}} < \frac{\pi}{2}.$$

Hence, by part (c) of the corollary to Theorem 19-35, $\text{Arcsin}\frac{c}{\sqrt{1+c^2}} = \text{Arctan}\,c$.

8. (a) This follows from Exercise 7 and the fact that, for $-1 < a < 1$, $\sin(\text{Arcsin}\,a) = a$.

(b) $\cos(\text{Arctan}\,c) = \sin(\text{Arctan}\,c)/\tan(\text{Arctan}\,c) = \frac{c}{\sqrt{1+c^2}}/c$ $= 1/\sqrt{1+c^2}$, for $c \neq 0$. In case $c = 0$ $\text{Arctan}\,c = 0$, $\cos(\text{Arctan}\,c) = 1$, and $1/\sqrt{1+c^2} = 1$.

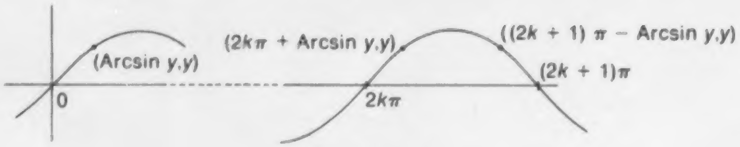


Fig. 19–22

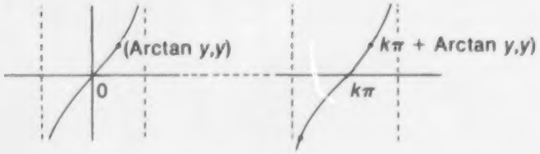


Fig. 19–23

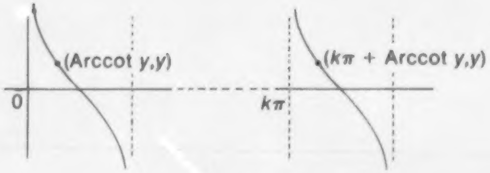


Fig. 19–24

We shall illustrate the methods of solving equations involving the circular functions ["trigonometric equations"] by solving several samples.

Sample 1. Solve for 'x': $2 \sin 2x = \sqrt{3}$

Solution. The given equation is equivalent to '$\sin 2x = \sqrt{3}/2$' and, by Theorem 19–35, this is equivalent to:

$$2x = k\pi + (-1)^k \text{ Arcsin } (\sqrt{3}/2), \text{ for some } k \in I.$$

Since $\text{Arcsin}(\sqrt{3}/2) = \pi/3$, the given equation is equivalent to:

$$(*)\ \exists_{k \in I}\, x = [3k + (-1)^k]\frac{\pi}{6} \qquad \text{[Explain.]}$$

From the solution (*) it appears that the given equation has infinitely many roots – one for each value of 'k'. Some of them are $-11\pi/6$, $-10\pi/6$, $-5\pi/6$, $-4\pi/6$,

Part D

1. Prove part (a) of Theorem 19–36 by using part (a) of the corollary to Theorem 19–35.
2. Prove part (b) of Theorem 19–36.
3. Show that
   (a) $\text{Arcsin } a + \text{Arcsin } (-a) = 0$ and
   (b) $\text{Arccos } a + \text{Arccos } (-a) = \pi$.
4. Show that
   (a) $\text{Arcsin } a = \text{Arccos } \sqrt{1-a^2}$ $[0 \le a \le 1]$ and
   (b) $\text{Arcsin } a = -\text{Arccos } \sqrt{1-a^2}$ $[-1 \le a \le 0]$.
5. Prove (*) with 'sin' and 'cos' interchanged.
6. Show that
   (a) $\text{Arccos } a = \text{Arcsin } \sqrt{1-a^2}$ $[0 \le a \le 1]$ and
   (b) $\text{Arccos } a = \pi - \text{Arcsin } \sqrt{1-a^2}$ $[-1 \le a \le 0]$.
7. Show that
   $$\text{Arctan } c = \text{Arcsin } \frac{c}{\sqrt{1+c^2}}.$$
   [*Hint*: Part of the job is to show that $\tan\left(\text{Arcsin } \frac{c}{\sqrt{1+c^2}}\right) = c$. For this, (**) will be helpful.]
8. Show that
   (a) $\sin(\text{Arctan } c) = c/\sqrt{1+c^2}$ and
   (b) $\cos(\text{Arctan } c) = 1/\sqrt{1+c^2}$.

19.11 Solution of Equations

Theorem 19–35 tells us the solutions [for 'a'] of equations of the forms:

$$\cos a = c,\ \sin a = c,\ \tan a = c,\ \cot a = c$$

The problem of solving many another equation can, as we shall see, be reduced to that of solving one or more equations of one of these types.

Since we shall have frequent use for the results summarized in Theorem 19–35 it is worth having a graphical reminder of what these results are. The following figures should help.

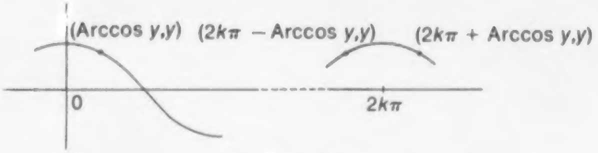


Fig. 19–21

Answers for Part D

1. For $-1 \le a \le 1$, $\cos(\pi/2 - \text{Arcsin } a) = \sin(\text{Arcsin } a) = a$. But, since $-\pi/2 \le \text{Arcsin } a \le \pi/2$ it follows that $0 \le \pi/2 - \text{Arcsin } a \le \pi$. Hence, by part (a) of the corollary to Theorem 19-35, $\pi/2 - \text{Arcsin } a = \text{Arccos } a$. Hence, for $-1 \le a \le 1$,
   $$\text{Arccos } a + \text{Arcsin } a = \pi/2.$$
2. $\tan(\pi/2 - \text{Arccot } a) = \cot(\text{Arccot } a) = a$. But, since $0 < \text{Arccot } a < \pi$ it follows that $-\pi/2 < \pi/2 - \text{Arccot } a < \pi/2$. So, by part (d) of the corollary to Theorem 19-35, $\pi/2 - \text{Arccot } a = \text{Arctan } a$. Hence, $\text{Arctan } a + \text{Arccot } a = \pi/2$. [It is, of course, also possible to prove this by using part (c) of the corollary.]
3. (a) For $-1 \le a \le 1$, $\sin(-\text{Arcsin}(-a)) = -\sin(\text{Arcsin}(-a)) = --a = a$. And, since $-\pi/2 \le \text{Arcsin}(-a) \le \pi/2$, $-\pi/2 \le -\text{Arcsin}(-a) \le \pi/2$. So, by part (b) of the corollary to Theorem 19-35 it follows that $-\text{Arcsin}(-a) = \text{Arcsin } a$. Hence, for $-1 \le a \le 1$, $\text{Arcsin } a + \text{Arcsin}(-a) = 0$.
   (b) For $-1 \le a \le 1$, $\cos(\pi - \text{Arccos}(-a)) = -\cos(\text{Arccos}(-a)) = --a = a$. And, since $0 \le \text{Arccos}(-a) \le \pi$, $0 \le \pi - \text{Arccos}(-a) \le \pi$. So, by part (a) of the corollary to Theorem 19-35 it follows that $\pi - \text{Arccos}(-a) = \text{Arccos } a$. Hence, for $-1 \le a \le 1$, $\text{Arccos } a + \text{Arccos}(-a) = \pi$.
4. For $-1 \le a \le 1$, $\sin(\text{Arccos}\sqrt{1-a^2}) = |a| = a$, for $a \ge 0$, and $\sin(-\text{Arccos}\sqrt{1-a^2}) = -\sin(\text{Arccos}\sqrt{1-a^2}) = -|a| = a$, for $a \le 0$. Also, since $\sqrt{1-a^2} \ge 0$, $0 \le \text{Arccos}\sqrt{1-a^2} \le \pi/2$ and $-\pi/2 \le -\text{Arccos}\sqrt{1-a^2} \le 0$. So, by part (b) of the corollary to Theorem 19-35, $\text{Arccos}\sqrt{1-a^2} = \text{Arcsin } a$ for $0 \le a \le 1$, and $-\text{Arccos}\sqrt{1-a^2} = \text{Arcsin } a$ for $-1 \le a \le 0$.
5. For $-1 \le a \le 1$, $\cos(\text{Arcsin } a) \ge 0$. [For, if $-\pi/2 \le b \le \pi/2$ then $\cos b \ge 0$.] Also,
   $$\cos^2(\text{Arcsin } a) + \sin^2(\text{Arcsin } a) = 1$$
   and $\sin(\text{Arcsin } a) = a$. Hence, $\cos^2(\text{Arcsin } a) = 1 - a^2$ and $\cos(\text{Arcsin } a) = \sqrt{1-a^2}$.
6. For $-1 \le a \le 1$, $\cos(\text{Arcsin}\sqrt{1-a^2}) = \sqrt{1-(1-a^2)} = |a| = a$, for $a \ge 0$, and $\cos(\pi - \text{Arcsin}\sqrt{1-a^2}) = -\cos(\text{Arcsin}\sqrt{1-a^2}) = -|a| = a$, for $a \le 0$. Also, since $\sqrt{1-a^2} \ge 0$, $0 \le \text{Arcsin}\sqrt{1-a^2} \le \pi/2$ and $\pi/2 \le \pi - \text{Arcsin}\sqrt{1-a^2} \le \pi$. So, by part (a) of the corollary to Theorem 19-35, $\text{Arcsin}\sqrt{1-a^2} = \text{Arccos } a$ for $0 \le a \le 1$, and $\pi - \text{Arcsin}\sqrt{1-a^2} = \text{Arccos } a$ for $-1 \le a \le 0$.

$\pi/6$, $2\pi/6$, $7\pi/6$, $8\pi/6$, and $13\pi/6$. [Sometimes you will be asked for only the nonnegative roots less than 2π. For this equation these are $\pi/6$, $\pi/3$, $7\pi/6$, and $4\pi/3$.]

Sample 2. Solve for 'y': $\cos^2 y = \frac{1}{2}$

Solution. The given equation is equivalent to:

$$\cos y = \frac{1}{\sqrt{2}} \text{ or } \cos y = -\frac{1}{\sqrt{2}}$$

Applying Theorem 19–35 to each alternative we find:

$$y = 2k\pi - \frac{\pi}{4} \text{ or } y = 2k\pi + \frac{\pi}{4} \text{ or}$$
$$y = 2k\pi - \frac{3\pi}{4} \text{ or } y = 2k\pi + \frac{3\pi}{4},$$

for some $k \in I$. Since for any k,

$$2k\pi - \frac{3\pi}{4} = (2k - 1)\pi + \frac{\pi}{4} \text{ and}$$

$$2k\pi + \frac{3\pi}{4} = (2k + 1)\pi - \frac{\pi}{4},$$

the first and fourth of our four alternatives, taken together, say that y is some integral multiple, either even or odd, of π, minus $\pi/4$. The second and third say the same thing with 'plus' in place of 'minus'. So, the given equation is equivalent to:

$$\exists_k \left(y = k\pi - \frac{\pi}{4} \text{ or } y = k\pi + \frac{\pi}{4}\right)$$

–for short:

$$(*)\ y = (4k \pm 1)\frac{\pi}{4}, \text{ for some } k \in I.$$

Evidently, the given equation has infinitely many roots–two for each value of 'k'. Those between 0 and 2π are $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. This suggests that the roots are just odd multiples of $\pi/4$. That this is so is easily checked by inspecting (*). [Explain.]

Sample 3. Solve for 'x': $2\cos^2 x - \cos x - 3 = 0$

Solution. For any y, $2y^2 - y - 3 = (2y - 3)(y + 1)$. So, the given equation is equivalent to:

$$\cos x = \frac{3}{2} \text{ or } \cos x = -1$$

Since $|\cos x| \leq 1$, the first alternative yields no solutions and, so, the given equation is equivalent to '$\cos x = -1$'. This last is satisfied if and only if x is an odd multiple of π. So, the solved form of the given equation is:

$$x = (2k - 1)\pi, \text{ for some } k \in I.$$

Sample 4. Solve for 'y': $\sin 2y - \cos y = 0$

Solution. The given equation is equivalent to:

$$2 \sin y \cos y - \cos y = 0$$

and, so, to:

$$\cos y = 0 \text{ or } \sin y = \frac{1}{2}$$

Hence, the given equation is equivalent to:

$$y = (2k - 1)\frac{\pi}{2} \text{ or}$$
$$y = k\pi + (-1)^k\frac{\pi}{6}, \text{ for some } k \in I.$$

Sample 5. Solve for 'x': $\sin x + \sin 3x = 0$

Solution. The given equation is equivalent to:

$$2 \sin 2x \cos x = 0 \qquad \text{[By what theorem?]}$$

and, so, to:

$$\sin 2x = 0 \text{ or } \cos x = 0.$$

The solved form is, then:

$$x = k\pi/2, \text{ for some } k \in I.$$

Sample 6. Solve for 'u': $1 + \cos u = \sqrt{3} \sin u$

Solution 1. The given equation is equivalent to:

$$2\cos^2\left(\frac{u}{2}\right) = \sqrt{3}\cdot 2\sin\left(\frac{u}{2}\right)\cos\left(\frac{u}{2}\right)$$

and, so, to:

$$\cos\left(\frac{u}{2}\right) = 0 \text{ or } \tan\left(\frac{u}{2}\right) = \frac{1}{\sqrt{3}}$$

Hence, the solved form of the given equation is:

$$u = (2k - 1)\pi \text{ or } u = 2k\pi + \frac{\pi}{3}, \text{ for some } k \in I.$$

Solution 2. The given equation is equivalent to:

$$\cos u \cdot 1 - \sin u \cdot \sqrt{3} = -1$$

and, so, to:

$$\sqrt{1+3}\left(\cos u \cdot \frac{1}{2} - \sin u \cdot \frac{\sqrt{3}}{2}\right) = -1.$$

and to:

$$\cos u \cdot \cos\frac{\pi}{3} - \sin u \cdot \sin\frac{\pi}{3} = -\frac{1}{2}$$

Hence, the given equation is equivalent to:

$$\cos\left(u + \frac{\pi}{3}\right) = -\frac{1}{2}$$

and, by Theorem 19–35 to:

$$u + \frac{\pi}{3} = 2k\pi - \frac{2\pi}{3} \text{ or}$$
$$u + \frac{\pi}{3} = 2k\pi + \frac{2\pi}{3}, \text{ for some } k \in I.$$

The technique used in Solution 2 for Sample 6 when combined with the results of Exercise 8 of Part D on page 471 yields the following result:

> Theorem 19–37 For $a > 0$,
>
> $$a\cos x + b\sin x = \sqrt{a^2+b^2}\cos\left(x - \text{Arctan}\,\frac{b}{a}\right).$$

The result in Theorem 19-37 finds many uses in applications of mathematics. Among those is the solution of various problems dealing with maxima and minima, as evidenced by Exercise 2 of Part A. The basis of the proof is that, for $a > 0$ and any b,

$$\frac{a}{\sqrt{a^2+b^2}} = \cos(\text{Arctan}\,\frac{b}{a}), \quad \frac{b}{\sqrt{a^2+b^2}} = \sin(\text{Arctan}\,\frac{b}{a}).$$

From this it follows that, for $a < 0$,

$$\frac{a}{\sqrt{a^2+b^2}} = \cos(\text{Arctan}\,\frac{b}{a} - \pi), \quad \frac{b}{\sqrt{a^2+b^2}} = \sin(\text{Arctan}\,\frac{b}{a} - \pi)$$

Thus it is not only the case that, for any ordered pair (a, b) for which $a \neq 0$, there is a number c such that $a = k\cos c$ and $b = k\sin c$, where $k = \sqrt{a^2+b^2}$, but, also, we know how to find such a number c. [For $a = 0$ and $b \neq 0$ we may take c to be $\pi/2$.]

This procedure of "normalizing" an ordered pair (a, b) is of use in other connections besides that which leads to Theorem 19-37.

TC 476 (1)

Answers for Part A

1. For any a, b, and x [with $(a, b) \neq (0, 0)$]

$$a\cos x + b\sin x = \sqrt{a^2+b^2}(\cos x \cdot \frac{a}{\sqrt{a^2+b^2}} + \sin x \cdot \frac{b}{\sqrt{a^2+b^2}}).$$

For $a > 0$, $\sqrt{a^2+b^2} = a\sqrt{1+b^2/a^2}$ and, so,

$$a\cos x + b\sin x = \sqrt{a^2+b^2}(\cos x \cdot \frac{1}{\sqrt{1+b^2/a^2}} + \sin x \cdot \frac{b/a}{\sqrt{1+b^2/a^2}}).$$

So, by Exercise 8 on page 471,

$$a\cos x + b\sin x = \sqrt{a^2+b^2}[\cos x \cdot \cos(\text{Arctan}\,\frac{b}{a})$$
$$+ \sin x \cdot \sin(\text{Arctan}\,\frac{b}{a})].$$

Hence, by the addition theorem for cos,

$$a\cos x + b\sin x = \sqrt{a^2+b^2}\cos(x - \text{Arctan}\,\frac{b}{a}).$$

2. By Theorem 19-37, $\cos x + \sin x = \sqrt{2}\cos(x - \pi/4)$. This has its greatest value when $x - \pi/4$ is an even multiple of π and has its least value when $x - \pi/4$ is an odd multiple of π. So [with 0 for the even multiple of π and π for the odd multiple of π] $\cos x + \sin x$ has its greatest value at $\pi/4$ and its least value at $5\pi/4$.

Exercises

Part A

1. Prove Theorem 19–37.
2. For what value 'x' between 0 and 2π does the expression '$\cos x + \sin x$' have its greatest value? Its least value?

Part B

Solve each of the following equations for 'x'.

1. $2 \sin^2 x + \cos x - 1 = 0$
2. $\tan x + \cot x = 2$
3. $3 \sin 2x + 2 \cos^2 x = 2$
4. $\sin x + \sin 3x = \cos x$
5. $\tan 3x = \tan 5x$
6. $12 \cos x - 5 \sin x = 13$
7. $\tan 2x = 2 \tan x$
8. $\cos 2x = \cos^2 x$
9. $\cos 5x + \cos 3x = 0$
10. $\sec^2 x = \tan^2 x + 2$
11. $\sin 4x - \sin 2x = \cos 3x$
12. $2 \cos^2 (x/2) + \cos 2x = 0$
13. $\sec x = \tan x - 1$
14. $\sin 2x + \sin x - 2 \cos x - 1 = 0$
15. $\sin x \cos x = 0$
16. $\sin x + \cos 2x = 0$
17. $\sin 2x = \sqrt{2} \cos x$
18. $\cos 3x - \sin x = \cos x$

19.12 Chapter Summary

Vocabulary Summary

the winding function W
reduction formula
doubling formula
identity
cofunction transformation
inverse circular functions
periodic function
circular functions
halving formula
Pythagorean identity
direction cosines for l
trigonometric equations

Definitions

19-1. (a) $W(0) = U$; (b) for $0 < a < 2\pi$, $W(a)$ is the point X such that the measure of the counterclockwise arc $\overset{\frown}{UX}$ is a;
(c) for $a \notin \{x: 0 \le x < 2\pi\}$, $W(a) = W(a - 2\pi[\![a/2\pi]\!])$.

19-2. $(\cos (a), \sin (a)) = W(a)$

19-3. $^\circ\!\cos a = \cos (\pi a/180)$ and $^\circ\!\sin a = \sin (\pi a/180)$.

19-4. (a) $\tan a = \dfrac{\sin a}{\cos a}$ (b) $\sec a = \dfrac{1}{\cos a}$

(c) $\cot a = \dfrac{\cos a}{\sin a}$ (d) $\csc a = \dfrac{1}{\sin a}$

[(a), (b) defined for real numbers which are not odd multiples of $\pi/2$; (c), (d) defined for real numbers which are not even multiples of $\pi/2$.]

19-5. For $\angle A$ not a right angle, $\tan \angle A = \sin \angle A/\cos \angle A$.

Answers for Part B

1. $\exists_{k \in I}\,(x = 2k\pi - \frac{2\pi}{3}$ or $x = 2k\pi$ or $x = 2k\pi + \frac{2\pi}{3})$ [Replace '$\sin^2 x$' by '$1 - \cos^2 x$'.]
2. $\exists_{k \in I}\, x = (4k + 1)\pi/4$ [Equation is equivalent to '$\tan^2 x + 1 = 2 \tan x$'.]
3. $\exists_{k \in I}\,(x = k\pi$ or $x = k\pi + \text{Arctan}\,3)$ [Equation is equivalent to '$\cos 2x + 3 \sin 2x = 1$' and, so, to '$\cos(2x - \text{Arctan}\,3) = 1/\sqrt{10} = \cos(\text{Arctan}\,3)$'. Use Theorem 19-32(a).]
4. $\exists_{k \in I}\,(x = (2k - 1)\pi/2$ or $x = k\pi/2 + (-1)^k \pi/12)$ [Equation equivalent to '$2 \sin 2x \cos x = \cos x$'.]
5. $\exists_{k \in I}\, x = k\pi$ [Use Theorem 19-32(c).]
6. $\exists_{k \in I}\, x = 2k\pi - \text{Arctan}(5/12)$ [Equation equivalent to '$\cos(x + \text{Arctan}\frac{5}{12}) = 1$'.]
7. $\exists_{k \in I}\, x = k\pi$ [Equation equivalent to '$\tan x = 0$ or $1 - \tan^2 x = 1$'—that is, to '$\tan x = 0$'.]
8. $\exists_{k \in I}\, x = k\pi$ [Equation equivalent to '$\cos^2 x = 1$'.]
9. $\exists_{k \in I}\,(x = (2k - 1)\pi/8$ or $x = (2k - 1)\pi/2$ [Equation equivalent to '$\cos 4x \cos x = 0$'.]
10. [no solution] [Recall that, for any x, $\sec^2 x = \tan^2 x + 1$.]
11. $\exists_{k \in I}\, x = (2k - 1)\pi/6$ [Equation equivalent to '$-2 \cos 3x \sin x = \cos 3x$'.]
12. $\exists_{k \in I}\,(x = (2k - 1)\pi/2$ or $x = (3k \pm 1)2\pi/3$ [Equation equivalent to '$2 \cos^2 x + \cos x = 0$'.]
13. $\exists_{k \in I}\,(x = (2k - 1)\pi)$ [Equation equivalent to 'x not an odd multiple of $\pi/2$ and $1 = \sin x - \cos x$'. The latter equation is equivalent to '$1 + \cos x = \sin x$' and, so, to '$2 \cos^2 \frac{x}{2} = 2 \sin \frac{x}{2} \cos \frac{x}{2}$'.]
14. $\exists_{k \in I}\,(x = 2k\pi + \pi/2$ or $x = (3k \pm 1)2\pi/3)$ [Equation equivalent to '$2 \sin x \cos x + \sin x - \cos x - 1 = 0$' and, so, to '$(\sin x - 1)(2 \cos x + 1) = 0$'.]
15. $\exists_{k \in I}\, x = k\pi/2$ [Equation equivalent to '$\sin 2x = 0$'.]
16. $\exists_{k \in I}\,(x = (6k - (-1)^k)\pi/6$ or $x = 2k\pi + \pi/2)$ [Equation equivalent to '$\sin x + 1 - 2 \sin^2 x = 0$'.]
17. $\exists_{k \in I}\,(x = (2k - 1)\pi/2$ or $x = k\pi + (-1)^k \pi/4$ [Equation equivalent to '$2 \sin x \cos x = \sqrt{2} \cos x$'.]
18. $\exists_{k \in I}\,(x = k\pi$ or $x = k\pi/2 - (-1)^k \pi/12)$ [Equation is equivalent to '$\cos 3x - \cos x = \sin x$' and, so, to '$-2 \sin 2x \sin x = \sin x$'.]

19-6. crc $a = \tan \frac{a}{2}$ [a not an odd multiple of π]

19-7. **(a)** Arccos a is the number z such that $0 \leq z \leq \pi$ and $\cos z = a$.
(b) Arcsin a is the number z such that $-\pi/2 \leq z \leq \pi/2$ and $\sin z = a$.
(c) Arctan a is the number z such that $-\pi/2 < z < \pi/2$ and $\tan z = a$.
(d) Arccot a is the number z such that $0 < z < \pi$ and $\cot z = a$.

Other Theorems

Lemma 1. For any a, $W(a) = W(a - 2\pi[\![a/2\pi]\!])$.
19-1. W is periodic with period 2π.
Lemma 2. For $0 \leq a < b < 2\pi$, one of the arcs with endpoints $W(a)$ and $W(b)$ has the measure $b - a$.
19-2. For $0 \leq a < b < 2\pi$, $d(W(a), W(b)) = d(U, W(b - a))$.
19-3. For $0 < c < 2\pi$, $UW(-c) = UW(c)$.
Corollary. For $0 \leq c < 2\pi$ and $0 \leq d < 2\pi$, $W(d)W(c) = UW(c - d)$.
19-4. $\cos \angle A = \cos m^{\perp}(\angle A)$ and $\sin^{\perp}(\angle A) = \sin(m^{\perp}(\angle A))$.
Corollary. $\cos \angle A = \cos(m(\angle A))$ and $\sin \angle A = \sin(m(\angle A))$.
19-5. $\cos^2 a + \sin^2 a = 1$
19-6. cos and sin are periodic with period 2π—that is, $\cos(a + 2\pi) = \cos a$ and $\sin(a + 2\pi) = \sin a$.
Corollary. For any $k \in I$, $\cos(a + 2k\pi) = \cos a$ and $\sin(a + 2k\pi) = \sin a$.
19-7. If, in $\triangle ABC$, α, β, and γ are the radian-measures of $\angle A$, $\angle B$, and $\angle C$, and a, b, and c are the measures of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, then $c^2 = a^2 + b^2 - 2ab \cos \gamma$ [cosine law] and $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$ [sine law].
19-8. [The Subtraction and Addition Laws for cos and sin]
(a) $\cos(a - b) = \cos a \cos b + \sin a \sin b$
(b) $\cos(a + b) = \cos a \cos b - \sin a \sin b$
(c) $\sin(a - b) = \sin a \cos b - \cos a \sin b$
(d) $\sin(a + b) = \sin a \cos b + \cos a \sin b$
19-9. cos is even and sin is odd.
19-10. **(a)** $\cos(\pi/2 - a) = \sin a$ **(b)** $\sin(\pi/2 - a) = \cos a$
Corollary. $a + b = \pi/2 \longrightarrow \cos b = \sin a$
19-11. **(a)** $\cos 2a = \cos^2 a - \sin^2 a$ **(b)** $\sin 2a = 2 \sin a \cos a$
Corollary. **(a)** $\cos 2a = 2\cos^2 a - 1$ **(b)** $\cos 2a = 1 - 2\sin^2 a$
19-12. **(a)** $\cos(a + \pi) = -\cos a = \cos(a - \pi)$
(b) $\sin(a + \pi) = -\sin a = \sin(a - \pi)$

19-13. **(a)** $\cos a \cos b = [\cos(a - b) + \cos(a + b)]/2$
(b) $\sin a \sin b = [\cos(a - b) - \cos(a + b)]/2$
(c) $\sin a \cos b = [\sin(a - b) + \sin(a + b)]/2$
(d) $\cos a \sin b = -[\sin(a - b) - \sin(a + b)]/2$
19-14. **(a)** $\cos d + \cos c = 2 \cos[(c + d)/2] \cos[(c - d)/2]$
(b) $\cos d - \cos c = 2 \sin[(c + d)/2] \sin[(c - d)/2]$
(c) $\sin d + \sin c = 2 \sin[(c + d)/2] \cos[(c - d)/2]$
(d) $\sin d - \sin c = -2 \cos[(c + d)/2] \sin[(c - d)/2]$
19-15. For $k \in I$, **(a)** $\cos(a + k\pi) = (-1)^k \cos a$, and
(b) $\sin(a + k\pi) = (-1)^k \sin a$.
19-16. For $k \in I$, **(a)** $\cos[a + (2k + 1)\pi/2] = -(-1)^k \sin a$, and
(b) $\sin[a + (2k + 1)\pi/2] = (-1)^k \cos a$.
Corollary. For $k \in I$, **(a)** $\cos k\pi = (-1)^k$, **(b)** $\cos(2k + 1)\pi/2 = 0$, **(c)** $\sin k\pi = 0$, and **(d)** $\sin(2k + 1)\pi/2 = (-1)^k$.
19-17. **(a)** $\cos a = 0 \longrightarrow \exists_{k \in I}\, a = (2k + 1)\pi/2$
(b) $\sin a = 0 \longrightarrow \exists_{k \in I}\, a = k\pi$
Corollary. **(a)** $\cos a = 0 \longleftrightarrow \exists_{k \in I}\, a = (2k + 1)\pi/2$
(b) $\sin a = 0 \longleftrightarrow \exists_{k \in I}\, a = k\pi$
19-18. **(a)** $(2k - 1)\pi/2 < a < (2k + 1)\pi/2 \longrightarrow \operatorname{sgn}(\cos a) = (-1)^k$
(b) $k\pi < a < (k + 1)\pi \longrightarrow \operatorname{sgn}(\sin a) = (-1)^k$
19-19. **(a)** cos is decreasing for $0 \leq a \leq \pi$, and
(b) sin is increasing for $-\pi/2 \leq a \leq \pi/2$.
19-20. $\cos a = \cos(180a/\pi)^\circ$ and $\sin a = \sin(180a/\pi)^\circ$.
19-21. **(a)** $\sec^2 a - \tan^2 a = 1$ [a not an odd multiple of $\pi/2$]
(b) $\csc^2 a - \cot^2 a = 1$ [a not an even multiple of $\pi/2$]
19-22. tan and cot are periodic with period π; sec and csc are periodic with period 2π.
19-23. tan, cot, and csc are odd functions; sec is an even function.
19-24. **(a)** $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$
[a, b, $a - b$ not odd multiples of $\pi/2$]
(b) $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$
[a, b, $a + b$ not odd multiples of $\pi/2$]
Corollary. For a not an odd multiple of $\pi/4$ or of $\pi/2$, $\tan 2a = \dfrac{2 \tan a}{1 - \tan^2 a}$.
19-25. For $k \in I$ and a not an even multiple of $\pi/2$,
$\tan[a + (2k + 1)\pi/2] = -\cot a$.
Corollary. **(a)** $\tan(\pi/2 - a) = \cot a$ [a not an even multiple of $\pi/2$]
(b) $\cot(\pi/2 - a) = \tan a$ [a not an odd multiple of $\pi/2$]
19-26. **(a)** $\tan a = 0 \longleftrightarrow a$ is an even multiple of $\pi/2$
(b) $\cot a = 0 \longleftrightarrow a$ is an odd multiple of $\pi/2$
19-27. For $\angle A$ not a right angle, $\tan \angle A = \tan(m(\angle A))$.

19-28. If α and β are the radian-measures of two angles of a triangle, and a and b are the measures of the sides opposite these angles, then

$$\frac{a-b}{a+b} = \frac{\tan[(\alpha-\beta)/2]}{\tan[(\alpha+\beta)/2]}.$$

19-29. (a) $\cos\dfrac{a}{2} = \operatorname{sgn}\left(\cos\dfrac{a}{2}\right)\sqrt{\dfrac{1+\cos a}{2}}$

(b) $\sin\dfrac{a}{2} = \operatorname{sgn}\left(\sin\dfrac{a}{2}\right)\sqrt{\dfrac{1-\cos a}{2}}$

(c) $\tan\dfrac{a}{2} = \operatorname{sgn}\left(\tan\dfrac{a}{2}\right)\sqrt{\dfrac{1-\cos a}{1+\cos a}}$ [a not an odd multiple of π]

Corollary. (a) $\cos\dfrac{a}{2} = \sqrt{\dfrac{1+\cos a}{2}}$ $[-\pi \le a \le \pi]$

(b) $\sin\dfrac{a}{2} = \sqrt{\dfrac{1-\cos a}{2}}$ $[0 \le a \le 2\pi]$

(c) $\tan\dfrac{a}{2} = \sqrt{\dfrac{1-\cos a}{1+\cos a}}$ $[0 \le a < \pi]$

19-30. $\tan\dfrac{a}{2} = \dfrac{\sin a}{1+\cos a}$ [a not an odd multiple of π]

19-31. For a not an odd multiple of π, (a) $\operatorname{crc} a = \dfrac{\sin a}{1+\cos a}$, (b) $\cos a = \dfrac{1-\operatorname{crc}^2 a}{1+\operatorname{crc}^2 a}$, and (c) $\sin a = \dfrac{2\operatorname{crc} a}{1+\operatorname{crc}^2 a}$.

19-32. (a) $\cos a = \cos b \Longleftrightarrow \exists_{k \in I}(a = 2k\pi - b \text{ or } a = 2k\pi + b)$
(b) $\sin a = \sin b \Longleftrightarrow \exists_{k \in I}\, a = k\pi + (-1)^k b$
(c) $\tan a = \tan b \Longleftrightarrow \exists_{k \in I}\, a = k\pi + b$ [neither a nor b an odd multiple of $\pi/2$]
(d) $\cot a = \cot b \Longleftrightarrow \exists_{k \in I}\, a = k\pi + b$ [neither a nor b an even multiple of $\pi/2$]

19-33. (a) $\cos(\operatorname{Arccos} a) = a$ $[-1 \le a \le 1]$
(b) $\sin(\operatorname{Arcsin} a) = a$ $[-1 \le a \le 1]$
(c) $\tan(\operatorname{Arctan} a) = a$
(d) $\cot(\operatorname{Arccot} a) = a$

19-34. (a) $\operatorname{Arccos}(\cos a) = a$ $[0 \le a \le \pi]$
(b) $\operatorname{Arcsin}(\sin a) = a$ $[-\pi/2 \le a \le \pi/2]$
(c) $\operatorname{Arctan}(\tan a) = a$ $[-\pi/2 < a < \pi/2]$
(d) $\operatorname{Arccot}(\cot a) = a$ $[0 < a < \pi]$

19-35. (a) $\cos a = c \Longleftrightarrow \exists_{k \in I}(a = 2k\pi - \operatorname{Arccos} c \text{ or } a = 2k\pi + \operatorname{Arccos} c)$ $[-1 \le c \le 1]$
(b) $\sin a = c \Longleftrightarrow \exists_{k \in I}\, a = k\pi + (-1)^k \operatorname{Arcsin} c$ $[-1 \le c \le 1]$
(c) $\tan a = c \Longleftrightarrow \exists_{k \in I}\, a = k\pi + \operatorname{Arctan} c$
(d) $\cot a = c \Longleftrightarrow \exists_{k \in I}\, a = k\pi + \operatorname{Arccot} c$

Corollary. (a) $\cos a = b \Longleftrightarrow a = \operatorname{Arccos} b$ $[0 \le a \le \pi, -1 \le b \le 1]$
(b) $\sin a = b \Longleftrightarrow a = \operatorname{Arcsin} b$ $[-\pi/2 \le a \le \pi/2, -1 \le b \le 1]$
(c) $\tan a = b \Longleftrightarrow a = \operatorname{Arctan} b$ $[-\pi/2 < a < \pi/2]$
(d) $\cot a = b \Longleftrightarrow a = \operatorname{Arccot} b$ $[0 < a < \pi]$

19-36. (a) $\operatorname{Arccos} a + \operatorname{Arcsin} a = \pi/2$ (b) $\operatorname{Arctan} a + \operatorname{Arccot} a = \pi/2$

19-37. For $a > 0$, $a\cos x + b\sin x = \sqrt{a^2+b^2}\cos\left(x - \operatorname{Arctan}\dfrac{b}{a}\right)$.

Chapter Test

1. Compute each of the following without making use of tables.
   (a) $\sin(-25\pi/2)$ (b) $\cos(13\pi/3)$ (c) $\tan(-17\pi/6)$
   (d) $\cos 9\pi$ (e) $\operatorname{Arccos}(-1)$ (f) $\operatorname{Arcsin} 1$
   (g) $\operatorname{Arctan} 1$ (h) $\operatorname{Arccot}(-1)$ (i) $\sin(\operatorname{Arccos} 1)$
   (j) $\cos(\operatorname{Arctan}(-1))$ (k) $\sec(-15\pi/4)$ (l) $\csc(17\pi/6)$
2. Given that $\sin a = \frac{3}{5}$ and $\cos b = -\frac{5}{13}$, find the following. [Give all possible answers.]
   (a) $\sin(a+b)$ (b) $\cos(a-b)$
3. Establish these identities.
   (a) $\dfrac{\cot^2 x}{1+\cot^2 x} = \cos^2 x$ (b) $1 - \tan^2 x\cos^2 x = \cos^2 x$
   (c) $\cos^2\left(\dfrac{\pi}{4} - x\right) = \dfrac{1+\sin 2x}{2}$
   (d) $\sin(x+y) - \sin(x-y) = 2\cos x\sin y$
4. Simplify each of the following, expressing your results in terms of '$\cos x$'.
   (a) $\cos x + \sin x \cdot \tan x$ (b) $1 + \tan^2 x$
   (c) $\sin 2x/(2\sin x)$ (d) $(1 - \tan^2 x)/\cos 2x$
5. Solve these equations and give all the solutions between 0 and 2π, inclusive.
   (a) $2\cos^2 x - 3\cos x + 1 = 0$ (b) $\sin x + \cos x = 1$
6. In each of the following, you are given an equation which describes a function related to the circular functions. In each case, give the period and range of values.
   (a) $y = 3\sin 2x$ (b) $y = -2\cos 4x$ (c) $y = \tan 3x$

Background Topic

In the Background Topic for Chapter 13 we discussed the field of complex numbers. This is the set $\mathscr{R} \times \mathscr{R}$ of ordered pairs of real numbers subject to the definitions:

(1) $(a, b) + (c, d) = (a + c, b + d)$, $0 = (0, 0)$, $-(a, b) = (-a, -b)$

(2) $\begin{cases}(a, b) \times (c, d) = (ac - bd, ad + bc), \ 1 = (1, 0), \\ /(a, b) = (a/(a^2 + b^2), -b/(a^2 + b^2))\end{cases}$

Answers for Chapter Test

1. (a) -1 (b) $1/2$ (c) $1/\sqrt{3}$ (d) -1
   (e) π (f) $\pi/2$ (g) $\pi/4$ (h) $3\pi/4$
   (i) 0 (j) $1/\sqrt{2}$ (k) $\sqrt{2}$ (ℓ) 2
2. (a) $33/65,\ -63/65$ (b) $56/65,\ 16/65,\ -16/65,\ -56/65$
3. (a) $\frac{\cot^2 x}{1+\cot^2 x} = \frac{\cot^2 x}{\csc^2 x} = \frac{\cos^2 x}{\sin^2 x}\sin^2 x = \cos^2 x$ [x not a multiple of π]
   (b) $1 - \tan^2 x \cos^2 x = 1 - \frac{\sin^2 x}{\cos^2 x}\cos^2 x = 1 - \sin^2 x = \cos^2 x$ [x not an odd multiple of $\pi/2$]
   (c) $\cos^2(\frac{\pi}{4} - x) = \frac{1 + \cos(\frac{\pi}{2} - 2x)}{2} = \frac{1 + \sin 2x}{2}$
   (d) $\sin(x+y) - \sin(x-y) = (\sin x \cos y + \cos x \sin y) - (\sin x \cos y - \cos x \sin y) = 2 \cos x \sin y$
4. (a) /cos x (b) /$\cos^2 x$ (c) cos x (d) /$\cos^2 x$
5. (a) $\exists_{k \in I}\ (x = 2k\pi - \frac{\pi}{3}$ or $x = 2k\pi$ or $x = 2k\pi + \frac{\pi}{3})$; $0,\ \pi/3,\ 5\pi/3$
   (b) $\exists_{k \in I}\ (x = 2k\pi$ or $x = 2k\pi + \pi/2)$; $0,\ \pi/2$
6. (a) period $= \pi$, range $= \{x\colon -3 \le x \le 3\}$
   (b) period $= \pi/2$, range $= \{x\colon -4 \le x \le 4\}$
   (c) period $= \pi/3$, range $= \mathcal{R}$

We also used multiplication of a complex number by a real number:

$$(a, b)c = (ac, bc) \tag{3}$$

and noted that, with:

$$i = (0, 1), \tag{4}$$

$$(a, b) = 1a + ib. \tag{5}$$

Since, for any a and b, $(a, 0) + (b, 0) = (a + b, 0)$, $(a, 0) \times (b, 0) = (ab, 0)$, and $(a, b)c = (a, b) \times (c, 0)$ we found it convenient to "identify" each real-complex number $(a, 0)$ with the corresponding real number a and so, by (5), to write '$a + ib$' in place of '(a, b)'. This convention and the ordinary rules of algebra made it easy to remember the definitions of addition and multiplication in (1) and (2):

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and since, by (2) and (4), $i^2 = -1$:

$$\begin{aligned}(a + ib) \times (c + id) &= ac + i(ad + bc) + i^2(bd)\\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

In addition to the fundamental definitions in (1) and (2), we defined the conjugate of a complex number:

$$\overline{(a, b)} = (a, -b) \tag{6}$$

and the absolute value of a complex number:

$$|(a, b)| = \sqrt{a^2 + b^2} \tag{7}$$

and noted that

$$(a, b) \times \overline{(a, b)} = |(a, b)|^2. \tag{8}$$

It was (8) which motivated the definition of reciprocation in (2) since,

$$(a, b) \times [\overline{(a, b)}/|(a, b)|^2] = 1 \qquad [(a, b) \ne 0].$$

Note, finally, how our '$a + ib$' notation and the notions of absolute value and conjugate simplify the carrying out of divisions:

$$\left.\begin{aligned}(a + ib) \div (c + id) &= \frac{a + ib}{c + id}\\ &= \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id}\end{aligned}\right\} c + id \ne 0$$

$$\begin{aligned}&= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}\\ &= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}\end{aligned}$$

Part A

1. Simplify – that is, fir quivalent expression of the form '$a + ib$' [or, '$a - ib$']:
   - (a) $(2 + i3) + (5 - \ldots$
   - (b) $-(3 - i4)$
   - (c) $(4 - i5) - (7 + i2)$
   - (d) $(3 + i4) \times (2 + i3)$
   - (e) $(2 - i) \times (-3 + i2)$
   - (f) $/(3 + i4)$
   - (g) $(2 + i5) \div (3 - i4)$
   - (h) $(\sqrt{3} + i)^2$
   - (i) $(1 - i)^2$
   - (j) i^3
   - (k) i^5
   - (l) $/i$
2. For each part of Exercise 1, draw a picture like those in Fig. 19–25 showing the "given numbers" and the answer. [For example, for part (a) show $2 + i3$, $5 - i7$, and $(2 + i3) + (5 - i7)$.]

*

In the Background Topic for Chapter 13 the definition of multiplication in (2) was justified by showing that it did have properties such that $\mathscr{R} \times \mathscr{R}$ together with (1) and (2) form a field. Now that we have studied the circular functions we can approach multiplication of complex numbers from a different direction and obtain some motivation for our definition of multiplication.

By Exercise 5 of Part D on page 405 and Theorem 19–4 it follows that, for $a + ib \neq 0$, if $r = |a + ib| = \sqrt{a^2 + b^2}$ then there is exactly one number θ such that $-\pi < \theta \leq \pi$ and

$$a + ib = r \cos \theta + ir \sin \theta. \tag{9}$$

[This number θ is the measure of $\angle UOP$ where $U = 1$, $O = 0$, and $P = (a, b)$.] Of course there are other numbers θ not between $-\pi$ and π for which (9) is satisfied. Any two of these differ by a multiple of 2π,

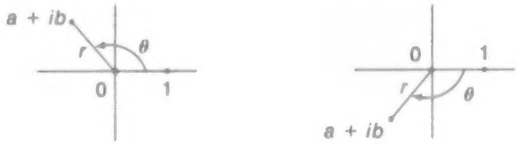


Fig. 19–25

and each of them is called *an argument of* $a + ib$. Recalling the definition of r we see that the arguments of $a + ib$ are just the solutions of:

$$\cos \theta = a/r \text{ and } \sin \theta = b/r, \tag{10}$$

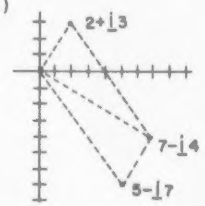
where $r = \sqrt{a^2 + b^2}$. For example, the arguments of $1 + i$ are just the numbers $\pi/4 + 2k\pi$, where $k \in I$. For $r = 0$, $a + ib = 0$ and each

---

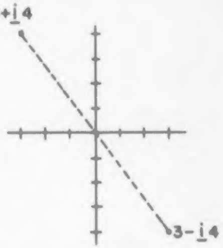
Answers for Part A

1. (a) $7 - \underline{i}4$ (b) $-3 + \underline{i}4$ (c) $-3 - \underline{i}7$ (d) $-6 + \underline{i}17$
   (e) $-4 + \underline{i}7$ (f) $3/5 - \underline{i}(4/5)$ (g) $26/25 + \underline{i}(23/25)$ (h) $2 + \underline{i}(2\sqrt{3})$
   (i) $-\underline{i}2$ (j) $-\underline{i}$ (k) $\underline{i}$ (ℓ) $-\underline{i}$

2. (a)



(b)



[The figures for (c) - (ℓ) are of a similar kind.]

real number satisfies (10) and, so, is an argument of $a + ib$. Using (3) we could rewrite the right side of (9) as

$$a + ib = r\,(\cos\theta + i\sin\theta) \tag{11}$$

An expression of the form of the right side of (11) is called *a polar form* of the complex number to which it refers.

Part B

1. What are the arguments of the complex number *1*? Of -1?
2. What are the arguments of i? Of $-i$?
3. What are the arguments of the real-complex number a
   (a) if $a > 0$? (b) if $a < 0$?
4. What are the arguments of the "pure imaginary" number ib
   (a) if $b > 0$? (b) if $b < 0$?
5. (a) What are the arguments of
   (i) $1 - i$? (ii) $1 + i\sqrt{3}$? (iii) $1 - i\sqrt{3}$?
   (b) Give a polar form for each of the numbers in part (a).
6. Show that the arguments of the conjugate of a given complex number are just the opposites of the arguments of the given complex number. [*Hint*: θ is an argument of $a + ib$ if and only if (10) is satisfied. But, (10) is satisfied if and only if $\cos(-\theta) = a/r$ and $\sin(-\theta) = -b/r$.]
7. Show that the arguments of the opposite of a given complex number are obtained by adding π to the arguments of the given number. [*Hint*: Proceed as in Exercise 1, but, instead of using the fact that cos is even and sin is odd, use the fact that $\cos(\theta + \pi) = -\cos\theta$ and $\sin(\theta + \pi) = -\sin\theta$.]
8. Use the definition of multiplication of complex numbers [and of multiplication of complex numbers by real numbers] and your knowledge of the circular functions to find a polar form for $r(\cos\theta + i\sin\theta) \times s(\cos\phi + i\sin\phi)$.

*

In Exercise 8 you should have found something you knew before — that the absolute value of a product of complex numbers is the product of the absolute values of these numbers — and something new. The new thing is that any sum of arguments of given complex numbers is an argument of their product. Both these results are contained in the result of Exercise 8:

$$\begin{aligned} &r\,(\cos\theta + i\sin\theta) \times s\,(\cos\phi + i\sin\phi) \\ &\quad = (rs)[\cos(\theta + \phi) + i\sin(\theta + \phi)] \end{aligned} \tag{12}$$

Notice that this result gives a very simple proof that multiplication of complex numbers is both commutative and associative. It is com-

Answers for Part B

1. The arguments of 1 are the even multiples of π; those of -1 are the odd multiples of π.
2. The number a is an argument of $\underline{i}$ if, for some $k \in I$, $a = 2k\pi + \pi/2$; a is an argument of $-\underline{i}$ if, for some $k \in I$, $a = (2k - 1)\pi + \pi/2$ [or: $2k\pi - \pi/2$].
3. The arguments of a are the even multiples of π if $a > 0$ and the odd multiples of π if $a < 0$. [See Exercise 1.]
4. The arguments of $\underline{i}b$ are the even multiples of π plus $\pi/2$ if $b > 0$ and are the odd multiples of π plus $\pi/2$ if $b < 0$. [See Exercise 2.]
5. (a) (i) the even multiples of π minus $\pi/4$
   (ii) the even multiples of π plus $\pi/3$
   (iii) the even multiples of π minus $\pi/3$
   (b) (i) $\sqrt{2}\,[\cos(7\pi/4) + \underline{i}\sin(7\pi/4)]$
   (ii) $2\,[\cos(\pi/3) + \underline{i}\sin(\pi/3)]$
   (iii) $2\,[\cos(-\pi/3) + \underline{i}\sin(-\pi/3)]$
   In each case there are other equally correct answers.
6. By the hint, θ is an argument of $a + \underline{i}b$ if and only if $-\theta$ is an argument of $a - \underline{i}b$.
7. Suppose that θ is an argument of $a + \underline{i}b$. This is the case if and only if
   $$(*) \qquad \cos\theta = a/r \ \text{ and } \ \sin\theta = b/r,$$
   with $r = \sqrt{a^2 + b^2}$. Since $\cos(\theta + \pi) = -\cos\theta$ and $\sin(\theta + \pi) = -\sin\theta$, $(*)$ holds if and only if
   $$\cos(\theta + \pi) = -a/r \ \text{ and } \ \sin(\theta + \pi) = -b/r.$$
   But, this last holds if and only if $\theta + \pi$ is an argument of $-(a + \underline{i}b)$. Hence, the arguments of the opposite of a given complex number may be obtained by adding π to the arguments of the given number.
8. By the definition of multiplication given in (2) on page 480 it follows that
   $$\begin{aligned} &(r\cos\theta + \underline{i}r\sin\theta) \times (s\cos\phi + \underline{i}s\sin\phi) \\ &\quad = rs(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ &\qquad + \underline{i}rs(\cos\theta\sin\phi + \sin\theta\cos\phi) \\ &\quad = rs[\cos(\theta + \phi) + \underline{i}\sin(\theta + \phi)]. \end{aligned}$$

TC 484-485

The principal square root of $\underline{i}$ is $1/\sqrt{2} + \underline{i}/\sqrt{2}$ since the square of this number is $\underline{i}$ and its argument, $\pi/4$, is between $-\pi/2$ and $\pi/2$, the latter included.

The square roots of 1 are 1 and -1. Those of -1 are $\underline{i}$ and $-\underline{i}$.

mutative because $rs = sr$ and $\theta + \phi = \phi + \theta$; it is associative because $(rs)t = r(st)$ and $(\theta + \phi) + \chi = \theta + (\phi + \chi)$. [Explain. Pronounce '$\chi$' as 'ki' and '$\phi$' as 'fi'.]

Equation (12) suggests an interesting geometric interpretation of multiplication of complex numbers. Supposing, as we may, that $-\pi < \phi \le \pi$ it follows from (12) that the product of any given complex number by $s\,(\cos\phi + i\sin\phi)$ is the image of the given number under resultant of two transformations. The first of these transformations is the uniform stretching about *0* with the stretching factor s and the second transformation is the rotation about *0* through a sensed angle whose measure is ϕ. [Fig. 19–26 illustrates a case in which the absolute value of the multiplier is 1 and in which, consequently, there is no stretching.]

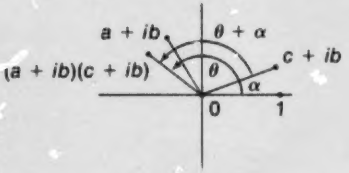


Fig. 19–26

As a special case of (12) we have:

$$[r\,(\cos\theta + i\sin\theta)]^2 = r^2\,(\cos 2\theta + i\sin 2\theta) \tag{13}$$

This special case suggests a result called *de Moivre's Theorem*:

$$[r\cos\theta + i\sin\theta]^n = r^n\,(\cos n\theta + i\sin n\theta),\ \text{for } n \in I^+$$

This follows easily from (12) by mathematical induction.

Returning to (13) we see that this result furnishes us with a way of finding square roots of complex numbers. For example, let's find a number whose square is i. To do so we note that, in polar form,

$$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

By this and (13) it follows that

$$[r\,(\cos\theta + i\sin\theta)]^2 = i$$

if and only if $r^2 = 1$, $\cos 2\theta = \cos(\pi/2)$ and $\sin 2\theta = \sin(\pi/2)$. Since r is the absolute value of the number we are seeking, $r > 0$ and, so, $r = 1$. There are many possible choices for θ, but the most obvious is $\theta = \pi/4$. It follows that

$$1[\cos(\pi/4) + i\sin(\pi/4)] \tag{14}$$

—that is, $(1/\sqrt{2}) + i(1/\sqrt{2})$—is a number whose square is i. [Check this by using the definition of multiplication in (2).] The number (14) is not the only number whose square is i. [What is another?] This is because the equations:

$$\cos 2\theta = \cos\frac{\pi}{2},\ \sin 2\theta = \sin\frac{\pi}{2}$$

have more than one solution. As we know, these equations are satisfied if, for any $k \in I$,

$$2\theta = \frac{\pi}{2} + 2k\pi$$

or, equivalently,

$$\theta = \frac{\pi}{4} + k\pi.$$

So, for any $k \in I$, the number

$$\cos\left(\frac{\pi}{4} + k\pi\right) + i\sin\left(\frac{\pi}{4} + k\pi\right) \tag{15}$$

is a square root of i. Since cos and sin have period 2π, the formula (15) yields the same number for any even value of 'k'. [This number is the one we have already found in (14).] Formula (15) also yields another number for any odd value of 'k'. This number is

$$\cos\left(\frac{\pi}{4} + \pi\right) + i\sin\left(\frac{\pi}{4} + \pi\right)$$

and is merely the opposite of the number given in (14). [Why?] It follows that i has exactly two square roots,

$$\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\ \text{and} - \frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\,. \tag{16}$$

It should be apparent that this is the case not only for i but for any non-*0* complex number. In particular each real [-complex] number other than *0* has exactly two complex square roots, each of which is the opposite of the other. [What are the square roots of 1? Of -1?] Since each complex number has an argument between $-\pi$ and π, π included, it follows that each complex number has a square root whose argument is between $-\pi/2$ and $\pi/2$, $\pi/2$ included. We shall call this square root *the principal square root* of the given complex number and, as in the case of real numbers, use '$\sqrt{\ }$' [without a prefixed opposing sign] in referring to principal square roots. For example, the principal square root of -1 is $\sqrt{-1}$, and this is i. The other square root of -1 is $-\sqrt{-1}$ and this is $-i$. Which of the two numbers in (16) is the principal square root of i?

Part C

1. Show that the notion of the principal square root of a complex number is, when applied to a real-complex number, consistent with our convention that the principal square root of a positive real number is its positive square root.
2. What is the principal square root of the real number b if $b < 0$?
3. What is the principal square root of ib? [*Hint*: You may find it helpful to consider two cases.]
4. What are the two square roots of $1 + i\sqrt{3}$?
5. A number whose cube is a given number is called *a cube root* of the given number. How many cube roots does a given non-*0* complex number have?
6. Find all the cube roots of 1 and show them on a picture [like Fig. 19–25] of the complex plane.
7. Give a short argument to show that any quadratic equation:

$$ax^2 + bx + c = 0 \qquad [a \neq 0],$$

where a, b, and c are complex numbers, has exactly two roots unless $b^2 - 4ac = 0$. Also, discuss the case in which $b^2 - 4ac = 0$. [*Hint*: Recall the quadratic formula on page 148.]

*8. Find the two roots of the equation '$x^2 + 2x - i\sqrt{3} = 0$'. Check your answer.

Answers for Part C

1. If a is a positive real-complex number then it has $a(\cos 0 + \underline{i} \sin 0)$ as a polar form. Its square roots are $\sqrt{a}(\cos 0 + \underline{i} \sin 0)$ and $\sqrt{a}(\cos \pi + \underline{i} \sin \pi)$. Of these, the first has an argument between $-\pi/2$ and $\pi/2$ and, so is the principal square root of the real-complex number a. But, $\sqrt{a}$ is the principal square root of the positive real number a.

2. $\underline{i}\sqrt{-b}$ [For $b < 0$, a polar form of b is $-b(\cos \pi + \underline{i} \sin \pi)$. So, the principal square root of such a complex number is $\sqrt{-b}[\cos(\pi/2) + \underline{i} \sin(\pi/2)]$, and this simplifies to $\underline{i}\sqrt{-b}$.]

3. For $b > 0$, $\underline{i}b = b[\cos(\pi/2) + \underline{i} \sin(\pi/2)]$ and its square roots are $\sqrt{b}[\cos(\pi/4) + \underline{i} \sin(\pi/4)]$ and $\sqrt{b}[\cos(5\pi/4) + \underline{i} \sin(5\pi/4)]$. Of these, it is the first which has an argument between $-\pi/2$ and $\pi/2$ and, so, is the principal square root of $\underline{i}b$.

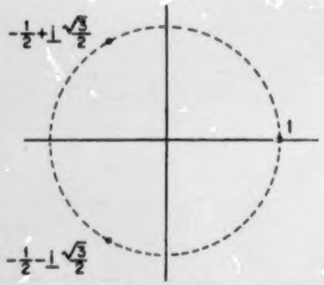
   For $b < 0$ $\underline{i}b = -b[\cos(3\pi/2) + \underline{i} \sin(3\pi/2)]$ and, since $-b > 0$, its square roots are $\sqrt{-b}[\cos(3\pi/4) + \underline{i} \sin(3\pi/4)]$ and $\sqrt{-b}[\cos(7\pi/4) + \underline{i} \sin(7\pi/4)]$. Of these, it is the second which has an argument $[-\pi/4]$ between $-\pi/2$ and $\pi/2$ and, so, is the principle square root of $\underline{i}b$.

   Summarizing, for $b > 0$, the principal square root of $\underline{i}b$ is $\sqrt{b}[/\sqrt{2} + \underline{i}/\sqrt{2}]$ and, for $b < 0$, the principal square root of $\underline{i}b$ is $\sqrt{-b}[/\sqrt{2} - \underline{i}/\sqrt{2}]$.

4. $\sqrt{2}[\cos(\pi/6) + \underline{i} \sin(\pi/6)]$ and $-\sqrt{2}[\cos(\pi/6) + \underline{i} \sin(\pi/6)]$ [The first of these may be rewritten: $\sqrt{6}/2 + \underline{i}\sqrt{2}/2$]

5. A non-$\underline{0}$ complex number has three cube roots.

6. Since $1 = 1(\cos 0 + \underline{i} \sin 0) = 1(\cos 2\pi + \underline{i} \sin 2\pi) = 1(\cos 4\pi + \underline{i} \sin 4\pi)$ it follows that each of the numbers $\cos 0 + \underline{i} \sin 0$, $\cos(2\pi/3) + \underline{i} \sin(2\pi/3)$ and $\cos(4\pi/3) + \underline{i} \sin(4\pi/3)$ is a cube root of 1. So, the cube roots of 1 are 1, $-1/2 + \underline{i}\sqrt{3}/2$, and $-1/2 - \underline{i}\sqrt{3}/2$. Here is a picture showing the cube roots of 1. [The three cube roots of any non-$\underline{0}$ complex number will be the end points of three 120°-arcs of a circle.]


$-\frac{1}{2}+\underline{i}\frac{\sqrt{3}}{2}$
1
$-\frac{1}{2}-\underline{i}\frac{\sqrt{3}}{2}$

Answers for Part C [cont.]

7. As in the case of real numbers, we may "complete the square":

$$ax^2 + bx + c = 0$$

$$(x^2 + \frac{b}{a}x) = -\frac{c}{a}$$

$$(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

The square roots of $(b^2 - 4ac)/(4a^2)$ are $\sqrt{b^2 - 4ac}/(2a)$ and its opposite, where $\sqrt{b^2 - 4ac}$ is the principal square root of $b^2 - 4ac$. So, the given equation is equivalent to:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Hence, if $b^2 - 4ac \neq 0$, the given equation has two roots, while if $b^2 - 4ac = 0$, the only root of the given equation is $-b/(2a)$.

8. By Exercise 7, the roots of the equation are

$$\frac{-2 + \sqrt{4 + \underline{i}4\sqrt{3}}}{2} \text{ and } \frac{-2 - \sqrt{4 + \underline{i}4\sqrt{3}}}{2}$$

or, more simply,

$$-1 + \sqrt{1 + \underline{i}\sqrt{3}} \text{ and } -1 - \sqrt{1 + \underline{i}\sqrt{3}}.$$

Since $1 + \underline{i}\sqrt{3} = 2[\cos(\pi/3) + \underline{i}\sin(\pi/3)]$ it follows that

$$\begin{aligned}\sqrt{1 + \underline{i}\sqrt{3}} &= \sqrt{2}[\cos(\pi/6) + \underline{i}\sin(\pi/6)]\\ &= \sqrt{2}[\sqrt{3}/2 + \underline{i}/2]\\ &= \sqrt{6}/2 + \underline{i}(\sqrt{2}/2).\end{aligned}$$

So, the roots of the given equation are

$$(-1 + \frac{\sqrt{6}}{2}) + \underline{i}\frac{\sqrt{2}}{2} \text{ and } (-1 - \frac{\sqrt{6}}{2}) - \underline{i}\frac{\sqrt{2}}{2}.$$

As a check, squaring either of these and adding its double yields $\underline{i}\sqrt{3}$.

Appendix

A-1 Solid Figures

Although we have discussed the notion of area-measure for plane figures in Chapter 16 we have not had space to treat by vector methods the notion of volume-measure for solid figures or the notion of area-measure for such curved surfaces as cylinders, cones, and spheres. Both of these notions fit into an approach to geometry like ours, but they require the introduction of another kind of multiplication of vectors [in addition to dot multiplication] as well as additional discussion of orientation, including what it means to orient our space $\mathscr{E}$ and the relation of orientation of $\mathscr{E}$ to orientations of planes and lines. Each of these subjects is interesting on its own as well as for its various applications but, since we do not have space to discuss either adequately, we shall give here a discussion of volumes of simple solid figures, and of areas of simple surfaces, which is pretty much independent of what has gone before. We shall, however, make use of some of the notions concerning similarity from Section 16.08.

We shall be concerned mainly with ways of finding the volumes and surface areas of prisms and pyramids, cylinders and cones, and [solid] spherical balls.

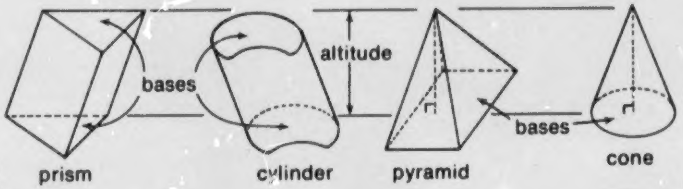


Fig. A–1

The figure shows a triangular prism, a cylinder, a quadrangular pyramid, and a cone.

A prism or *a cylinder* has two *bases* which are congruent regions in two parallel planes. Each base is the image of the other under a translation. The bases of a prism are bounded by polygons and a prism is said to be triangular, quadrangular, etc. according as its bases are triangular regions, quadrangular regions, etc. The only cylinders we shall study to any extent are those with circular regions as bases. Prisms and cylinders can be thought of as unions of parallel segments joining corresponding points of their bases. When these segments are perpendicular to the bases the prism or cylinder is *a right prism* or *a right cylinder*. The union of the segments joining corresponding points of the boundaries of the bases of a prism or cylinder is its *lateral surface*. In the case of a prism, the union of the segments joining corresponding points of two corresponding sides of the bases is *a lateral face* of the prism. The other *faces* of the prism are its bases. Each segment joining corresponding vertices of the bases of a prism is *a lateral edge* of the prism. These together with the sides of the prism's bases are *the edges* of the prism. *The altitude* of a prism or cylinder is the distance between the planes of its bases.

Note that the lateral faces of any prism are bounded by parallelograms. A prism whose bases are also bounded by parallelograms is *a parallelepiped*. A right parallelepiped is *a rectangular solid*, and a rectangular solid all of whose faces are squares is *a cube*.

A pyramid or *a cone* has a single plane region as a base and has a vertex which is a point not belonging to the plane of the base. The base of a pyramid [like those of a prism] is a polygonal region. The only cones we shall study to any extent are those whose bases are circular regions. Pyramids and cones can be thought of as unions of segments joining their vertices to the points of their bases. *The lateral surface* of a pyramid or cone and *the lateral faces*, *lateral edges*, and *edges* of a pyramid are defined much as are those of a prism or cylinder. *The altitude* of a pyramid or cone is the distance between its vertex and the plane of its base.

Note that the lateral faces of any pyramid are triangular regions. A pyramid whose base is also a triangular region is *a tetrahedron*.

A regular pyramid is a pyramid whose base is bounded by a regular polygon whose center is the foot of the perpendicular from the vertex of the pyramid to the plane of its base. The lateral faces of a regular pyramid are bounded by congruent isosceles triangles and the common altitude [measure] of these triangles from the vertex of the pyramid is the pyramid's *slant height*.

A right circular cone is a cone whose base is a circular region whose center is the foot of the perpendicular from the vertex of the cone to the plane of its base. *The slant height* of a right circular cone is the common measure of the segments from its vertex to points of the boundary of its base.

Prisms and pyramids are examples of polyhedra [singular: polyhedron]—that is, of solids whose surfaces are unions of polygonal regions. The faces of a polyhedron are the polygonal regions which make up its surface; its edges and vertices are the sides and vertices of these regions.

Exercises

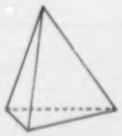
Part A

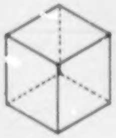
1. (a) What is the least number of faces a polyhedron can have? The least number of edges? The least number of vertices?
   (b) Is there a polyhedron which satisfies all three conditions in part (a)?
2. (a) How many faces has a triangular prism? A triangular pyramid? A quadrangular prism? A cube? A regular quadrangular pyramid?
   (b) Give the number of edges of each solid referred to in part (a).
   (c) Give the number of vertices of each solid referred to in part (a).
3. Find formulas for the number F of faces, E of edges, and V of vertices
   (a) of a prism each of whose bases has n sides, and
   (b) of a pyramid whose base has n sides.

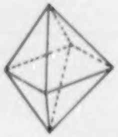
Part B

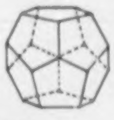
A regular polyhedron is a convex polyhedron whose faces are congruent regular polygonal regions and each two of whose vertices are endpoints of the same number of edges. [*A convex polyhedron* is one whose surface has no "dents"—formally, such a polyhedron is one which contains each segment whose endpoints belong to it.]

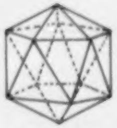
1. You are already acquainted with two kinds of regular polyhedra—one with four triangular faces and one with six rectangular faces. Name them and draw a picture of one of each kind.
2. (a) Why is there at most one regular polyhedron with square faces?
   (b) Why is there no regular polyhedron with hexagonal faces?
   (c) How many kinds of regular polyhedra do you think there might be with triangular faces?
3. In Exercise 1 you were asked to draw a regular polyhedron with four triangular faces. Now, draw one with eight triangular faces.
4. It can be proved that there are just five kinds of regular polyhedra.


a regular tetrahedron


a regular hexahedron


a regular octahedron


a regular dodecahedron


a regular icosahedron

For each kind, tell the number of faces, edges, and vertices which such a polyhedron has. [*Hint*: If you count the number of edges of each face of a polyhedron and add the results, how many times have you counted each edge?]

Answers for Part A

1. (a) 4; 6; 4 [There is at least one vertex. Suppose, then, that A is a vertex. There are at least three faces which have A as a vertex and, so, at least three edges which have A as one end point. In fact, we can choose faces f_1, f_2, and f_3 with A as vertex such that f_1 and f_2 have a side e_1 in common, f_2 and f_3 have a side e_2 in common, and f_3 has another side e_3 with A as vertex. Now, f_1 has a side e_4 which does not have A as an end point, f_2 has a side e_5 which does not have A as an end point, and f_3 has a side e_6 which does not have A as an end point. There is a fourth side, f_4 which has e_4 as a side, and f_4 has at least three vertices, each of which is different from A.]

   (b) A tetrahedron satisfies all three conditions.

2. (a) 5; 4; 6; 6; 5 (b) 9; 6; 12; 12; 8 (c) 6; 4; 8; 8; 5

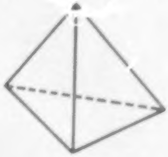
3. (a) n + 2; 3n; 2n (b) n + 1; 2n; n + 1

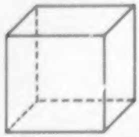
   [Students may note that $F - E + V = 2$. This, which is called Euler's Formula, holds for all convex polyhedra as well as for many others.]

Answers for Part B

[The word 'regular' is used in many ways in mathematics. Note, for example, that although a pyramid is a polyhedron, a regular pyramid need not be a regular polyhedron.]

1. The polyhedra in question are tetrahedra with congruent equilateral faces and cubes.



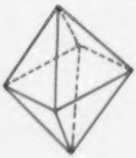


2. (a) There must be at least three faces of a polyhedron "surrounding" each vertex and if the faces are square regions there cannot be more than three.

   (b) Three hexagonal regions surrounding a vertex are coplanar, and there is no room for more than three.

   (c) There is one with three equilateral triangular regions about each vertex, there might be another with four, and another with five, equilateral triangular regions about each vertex. [See Exercise 3.]

3.



4. tetrahedron: 4; 6; 4
   hexahedron: 6; 12; 8
   octahedron: 8; 12; 6
   dodecahedron: 12; 30; 20
   icosahedron: 20; 30; 12

Part C

1. The lateral area of a prism or pyramid is the sum of the areas of its lateral faces. What two numbers should you multiply to obtain the measure of the lateral area of a right prism?
2. The lateral area of a cylinder is that of the plane region obtained by cutting the lateral surface along a segment joining corresponding points of its bases and "unrolling" it. Give a formula for the measure of the lateral area of a right circular cylinder whose base has radius r and whose altitude is h.
3. Give a formula for the total surface area of a cylinder whose dimensions are those given in Exercise 2.
4. What two numbers should you multiply to obtain the measure of the lateral area of a regular pyramid?
5. The lateral area of a cone can be described in analogy with that of a cylinder in Exercise 2. Give a formula for the measure of the lateral area of a right circular cone whose base has radius r and whose slant height is l.
6. Give a formula for the measure of the lateral area of a right circular cone whose base has radius r and whose altitude is h.
7. Suppose that two prisms, cylinders, pyramids, or cones are similar with ratio of similitude k. What is the ratio of the area-measure of their lateral surfaces? Of their total surfaces?

A-2 Cross Sections

A cross section of a prism, cylinder, pyramid, or cone is the intersection of the solid in question with plane parallel to a base of the solid.

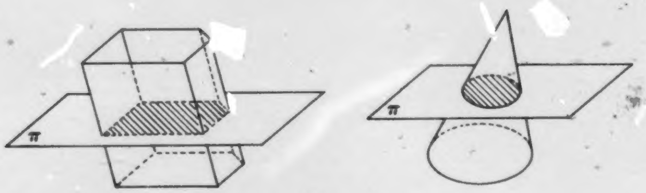


Fig. A–2

We shall consider only cross sections which are nondegenerate—that is, the empty set and the set consisting of the vertex of a pyramid or cone will not be considered to be cross sections.

It is intuitively evident that all cross sections of a prism or a cylinder are congruent and, so, have the same area. In fact, either of two cross sections is the image of the other under a translation. For our work

Answers for Part C

1. The perimeter of a base and the altitude.
2. $2\pi rh$
3. $2\pi rh + 2\pi r^2$ [or: $2\pi r(h + r)$]
4. Half the perimeter of the base and the slant height.
5. $\pi r\ell$
6. $\pi r\sqrt{r^2 + h^2}$
7. k^2; k^2 [For example, in the case of a cylinder, $2\pi(kr)(kh) = k^2(2\pi rh)$ and $2\pi(kr)[(kh) + (kr)] = k^2[2\pi r(h + r)]$.]

TC 491-492

Answers to questions in text.

$\triangle A'F'V$ and $\triangle AFV$ are similar because they are right triangles which share an acute angle. $VA'/VA = d/h$ because $\triangle A'F'V$ and $\triangle AFV$ are similar and $VF'/VF = d/h$.

Answers for Exercises

1. $1\frac{3}{5}$ square inches
2. $3\sqrt{2}$ inches
3. π square feet

with volumes we need to have corresponding information concerning cross sections of pyramids and cones. We shall see that

(1) a cross section of a pyramid or cone of altitude h by a plane π at a distance d from the vertex $[0 < d < h]$ is the image of the base under the uniform stretching [actually, a shrinking] about the vertex V which multiplies distances by d/h.

That this is the case follows at once by considering the similar triangles, $\triangle A'F'V$ and $\triangle AFV$ in the following figure:

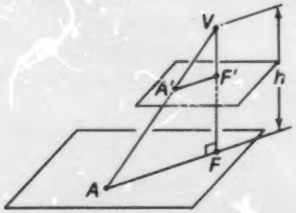


Fig. A–3

[Explain. Why are $\triangle A'F'V$ and $\triangle AFV$ similar? Why is $VA'/VA = d/h$?] The point A' of intersection of $\overrightarrow{VA}$ and π is image of A under the uniform stretching about V which multiplies distances by d/h. It follows by Theorem 16–30 that the area-measure of the image under this mapping of any given triangular region in the plane of the base of our pyramid or cone is d^2/h^2 times the area-measure of the given triangular region. Because of the way we have based areas of regions on areas of triangular regions the same holds for any region in the plane of the base. In particular, it follows from (1) that

(2) the area-measure of the cross section of a pyramid or cone of altitude h by a plane at a distance d from the vertex $[0 < d < h]$ is Kd^2/h^2, where K is the area-measure of the base.

Our principal interest in (2) is that it shows that,

(3) given any two solids each of which is either a pyramid or a cone and both of which have the same altitude and the same base area, any two cross sections of them by planes at the same distance from their bases have the same area.

Exercises

1. A cone has altitude 6 inches and its base has area 10 square inches. What is the area of the cross section of this cone which divides the altitude from its vertex in 2:3?
2. Given the cone of Exercise 1, how far from the vertex must a plane parallel to the base be in order to cut out a cross section of area 5 square inches?
3. A circular disk of paper held parallel to a wall and between the wall and a point source of light casts a circular shadow. What is the area of the shadow if the light is 12 feet from the wall, the cardboard is 6 feet from the wall, and the radius of the cardboard is 6 inches?

A-3 Volumes by Means of Cross Sections

You can build a model of a given solid by gluing together pieces of plywood cut to the size and shape of cross sections of the solid made by planes whose distance apart is the thickness of the plywood. As is illustrated in Fig. A-4 the slices [right prisms] of which such a model is made may stick out too far in some places and not far enough in others. These inaccuracies may be reduced by cutting the slices out of thinner plywood or out of cardboard. [Of course, the thinner the slices are, the more of them you will need to use.] By using sufficiently thin cardboard one can—at least in imagination—construct a model whose volume is as close as he wishes to that of the given solid. Since the volume of the model depends only on the areas of the cross sections which are used and on the thickness of the slices, it appears that the volumes of two solids must be the same if they have the same altitude and their "corresponding" cross sections have the same area. More precisely, we have:

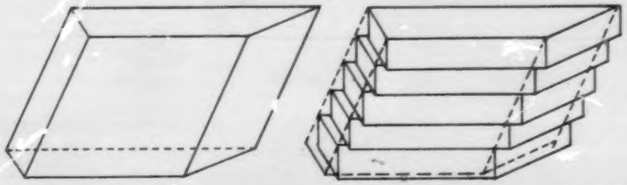


Fig. A–4

Cavalieri's Principle

> Two solids have the same volume if there is a plane π such that each plane which is parallel to π and intersects either of the solids also intersects the other, and the cross sections of the solids by any such plane have the same area.

Cavalieri's Principle is, for us, a postulate which we can apply to *compare* the volumes of certain solids. To use it in *computing* the volumes of solids we need, in addition, a postulate which will enable us to compute the volumes of some solids which can be taken as standards of comparison. For such standards we shall take rectangular solids and, as a second postulate on volume we shall adopt:

> The volume-measure of a rectangular solid is the product of its altitude and the area-measure of its base.

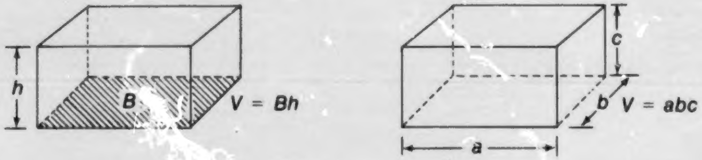


Fig. A–5

Since any face of a rectangular solid may be chosen as a base, we should question whether the formula '$V = Bh$' gives the same value for 'V' no matter which face of the rectangular solid pictured on the left in Fig. A-5 is taken as base. The equivalent formula on the right which expresses the volume-measure in terms of the measures of three sides shows that this is the case. [Explain.]

Using our two postulates it is now easy to see that the same formula which we have adopted for the volume of a rectangular solid applies just as well to any prism or cylinder. For, as is illustrated in Fig. A-6, given any prism or cylinder there is a rectangular solid whose bases have the same area and are in the same planes as are those of the given solid. Since the areas of all cross sections of a prism or cylinder have the same area it follows that a cross section of the given prism or cylinder by a plane π parallel to the base planes has the same area-measure B as does the cross section by π of such a rectangular solid. So, by Cavalieri's Principle, the prism or cylinder has the same volume-measure, Bh, as does the rectangular solid.

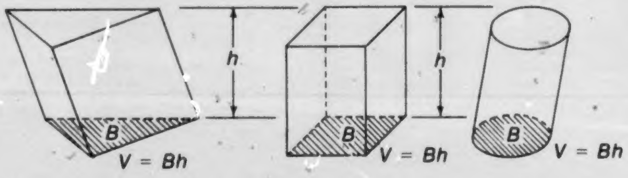


Fig. A–6

In order to obtain a numerical result for the volume-measure of a given prism or cylinder we must, of course, know how to calculate B. For example, the volume-measure of any circular cylinder whose altitude is h and whose base has radius r is $\pi r^2 h$.

The preceding argument is based on the fact that all cross sections of a prism or cylinder have the same area. Using (3) on page 491 instead of this we can show by the same argument that any pyramid or cone has the same volume as a triangular pyramid whose altitude and base area are those of the given pyramid or cone. So, we can express the volume of any pyramid or cone in terms of its altitude and base measure if we can do the same for any triangular pyramid.

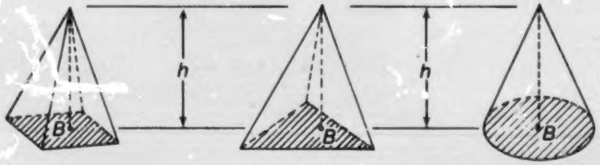


Fig. A–7

To find the volume of a triangular pyramid we note that the triangular pyramid with vertex V and base bounded by $\triangle PQR$ is part of a triangular prism which has the same base and altitude. This triangular prism is the union of three triangular pyramids. One of these is the given pyramid and it and the others are shown, cut apart,

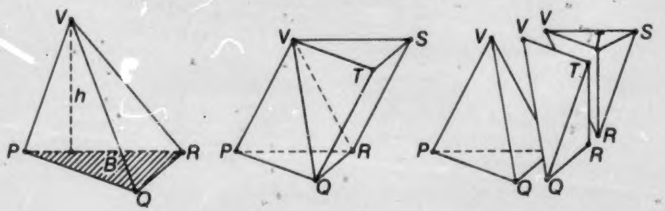


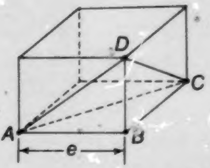
Fig. A–8

in the left-hand picture in Fig. A-8. [The two others are obtained by slicing the quadrangular pyramid with vertex V and base bounded by parallelogram $QRST$ along the plane $\overleftrightarrow{TRV}$.] The two new pyramids may be thought of as having vertex V and bases bounded by $\triangle TRQ$ and $\triangle TRS$. It follows that these pyramids have the same altitude and that their bases have the same area. [Why?] So, by (3) on page 491 and Cavalieri's Principle, they both have the same volume. In addition to this, one of the new pyramids may be thought of as having vertex R and base bounded by $\triangle VTS$. So, this pyramid has the same volume as does the given triangular pyramid. [Explain.] It follows, now, that the volume of the given triangular pyramid is one-third that of the triangular prism. Since the base area of the prism is the same, B, as that of the pyramid and since the two solids have the same altitude, h, it follows that the volume-measure of the triangular pyramid is given by:

$$V = \tfrac{1}{3} Bh$$

And, as shown previously, the same formula must work for any pyramid or cone. For example, the volume-measure of a right circular cone whose altitude is h and whose base has radius r is $\pi r^2h/3$.

Exercises



1. Consider the cube pictured to the right. What is the volume-measure of the solid which remains when the tetrahedron with vertices A, B, C, and D is cut off?
2. A trench 6 feet deep and 50 yards long has a trapezoidal cross section which is 3 feet across at the top and 2 feet across at the bottom. How many cubic yards of earth were removed in digging the ditch?
3. A pyramid has a rhombodial base with diagonals 5 inches and 7 inches long and its volume is 35 cubic inches. What is the height of this pyramid?
4. A right prism with a right triangular base is to be whittled out of a wooden right circular cylinder whose radius is 2 inches and whose length is 1 foot. What is the minimum volume of shavings which will be produced?
5. Suppose that two prisms, cylinders, pyramids, or cones are similar with ratio of similitude k. What is the ratio of the volume-measures of the two solids?

Answers for Exercises

1. $5e^3/6$
2. $250/3$
3. 2 inches
4. $48(\pi - 1)$ cubic inches
5. k^3

A-4 Prismoids

A prismoid is a polyhedron each of whose vertices is contained in one of two parallel planes. A prism is a special kind of prismoid and so is a pyramid. Like a prism, a prismoid [which is not a pyramid] has two bases—its faces in the two parallel planes—and a number of lateral faces each of which is either triangular or trapezoidal. [The prismoid of Fig. A-9 has seven triangular lateral faces and one trapezoidal lateral face.] The cross section of a prismoid made by a plane midway between the planes of its bases is called its *midsection*.

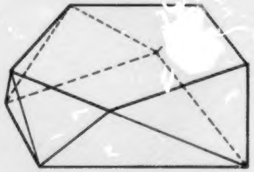


Fig. A–9

As we shall see, the volume of a prismoid whose altitude [the distance between its base planes] is h, and whose bases and midsection have area-measures B_1, B_2, and M, is given by:

$$V = \frac{1}{6}(B_1 + B_2 + 4M)h \tag{*}$$

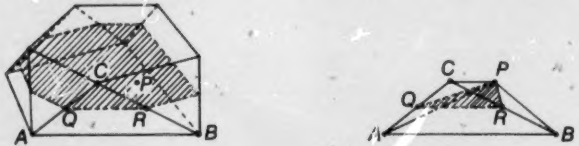


Fig. A–10

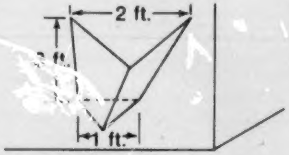
[Note that (*) works for prisms and pyramids. Explain.] To verify this we begin by noting that if P is a point of the midsection of a prismoid then the latter is the union of three solids two of which are pyramids with vertex P and whose bases are those of the prismoid. Since each of these has altitude $h/2$ their volume-measures are $B_1h/6$ and $B_2h/6$, thus accounting for two terms in *the prismoidal formula* (*). The third of the three solids is the union of pyramids each of which has P as vertex and a lateral face of the prismoid as base. [One of these is shown in the right-hand picture in Fig. A-10.] We may assume that each of these pyramids has a triangular base, since a trapezoid can be split along one of its diagonals into triangles. Now, as illustrated in Fig. A-10, each of these triangular pyramids is split, by the portion of the midsection contained in it, into a smaller triangular pyramid and a quadrangular pyramid. And, the volume of the original triangular pyramid is four times that of the smaller. [Why?] Now, the smaller triangular pyramid can be thought of as having a vertex [C] of the prismoid for its vertex and a triangular portion of the midsection [bounded by $\triangle PQR$] as its base. Hence, the volume-measure of this smaller triangular pyramid is one-third of the product of the area-measure of this base by $h/2$. It follows that the volume of the larger triangular pyramid is four times this and that the volume of the union of all such triangular pyramids is $4Mh/6$. This accounts for the third term in the prismoidal formula.

The preceding proof of the prismoidal formula assumes that the point P can be chosen so that the segments joining P to the vertices of the midsection divide the latter into nonoverlapping triangular regions. This will ordinarily be the case, but the formula works even if it is not. The formula also works for prismoidal-like solids whose bases in two parallel planes are regions which are not bounded by polygons.

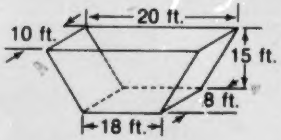
Exercises

Part A

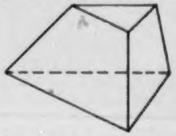
1. A bin whose horizontal cross sections are right isosceles triangles is built onto the side of a barn. Its dimensions are as given in the figure. Compute the volume of the bin.

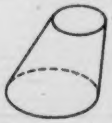


2. A corn-crib has the shape and dimensions shown in the figure. Compute its volume.



Part B





A frustum of a pyramid or cone is a figure consisting of the points of the given solid between [or on] the base and a cross section. A frustum of a pyramid is a prismoid and, by Cavalieri's Principle and (3) on page 491 the prismoidal formula works, also, for a frustum of a cone. The area-measure M of the midsection of a frustum can be expressed in terms of the area-measures B_1 and B_2 of its bases and its

Answers for Part A

1. 7/4 cubic feet   2. 2570 cubic feet

Answers for Part B

1. $B_2 = (\frac{h' - h}{h'})^2 B_1 = (1 - r^2)B_1$, where $r = h/h'$. So,

   $1 - r = \sqrt{B_2/B_1}$, and $r = 1 - \sqrt{B_2/B_1}$.

   $$M = (\frac{h' - h/2}{h'})^2 B_1 = (1 - r/2)^2 B_1$$
   $$= (1 - \frac{1 - \sqrt{B_2/B_1}}{2})^2 B_1 = (\frac{1 + \sqrt{B_2/B_1}}{2})^2 B_1$$
   $$= \frac{1 + B_2/B_1 + 2\sqrt{B_2/B_1}}{4} B_1 = \frac{B_1 + B_2 + 2\sqrt{B_1 B_2}}{4}$$

2. From Exercise 1, $B_1 + B_2 + 4M = 2(B_1 + B_2 + \sqrt{B_1 B_2})$. So, by the prismoidal formula,

   $$V = \frac{1}{3}(B_1 + B_2 + \sqrt{B_1 B_2})h.$$

3. Given a cone, there is a pyramid, of the same altitude, whose base has the same area as does that of the cone. By (3) on page 491, corresponding cross sections of the cone and pyramid have the same area. It follows by Cavalieri's Principle that frustums of the cone and pyramid which have the same altitude also have the same volume. It also follows that corresponding bases of the two frustums have the same area and, so, that application of the frustum formula (*) to either frustum yields the same result. Since, in the case of the pyramidal frustum, this result is the volume-measure of the frustum, and since the volume-measures of the two frustums are the same, it follows that (*) gives the volume-measure of the conical frustum.

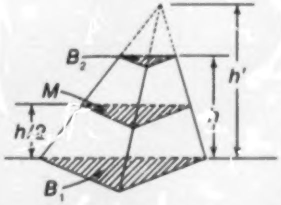
4. Exercise 1 of Part A can be solved by the frustum formula but Exercise 2 cannot. The solid bounded by the bin of Exercise 1 is a frustum of a pyramid whose base is an isosceles right triangle which is 6 feet high. In Exercise 2, the slanting front edges would, if extended, intersect 150 feet below the top of the bin, while the slanting edges on the right side would, if extended, intersect 75 feet below the top of the bin.

5. $\pi R[(\ell + r)^2 - r^2]/(\ell + r)$, where r and R are the radii of the bases $[r < R]$ and ℓ is the slant height. [Note, as a check, that the formula gives expected results in case $r = 0$, $r = R$, or $R = \ell + r$. Another formula, '$\pi\ell(r + R)$' is more difficult to derive.]

altitude h. On showing this you will find that the volume of a frustum is given by:

$$(*) \qquad V = \frac{1}{3}(B_1 + B_2 + \sqrt{B_1 \cdot B_2})h$$

1. Find M for a frustum in terms of B_1 and B_2. [*Hint*: Referring to the figure, let $r = h/h'$. It follows from (2) on page 491 that $B_2 = (1 - r)^2 B_1$ and $M = (1 - r/2)^2 B_1$. (Why?) Solve the first of these equations for 'r' and substitute into the second.]



2. Complete the derivation of the frustum formula (*).
3. Since the frustum of a cone is not a prismoid, the argument in Exercises 1 and 2 does not show that (*) applies to a frustum of a cone. Use Cavalieri's Principle to show that it does.
4. Which of the problems of Part A can be solved by the frustum formula? Explain.
*5. Find a formula for the area-measure of the lateral surface of a frustum of a right circular cone. [*Hint*: Use the radii of the two bases and the slant height of the frustum.]

A-5 Volumes of Balls and Areas of Spheres

By definition, *a ball* consists of a sphere together with the points inside it. We can use Cavalieri's Principle to find the volume of a ball.

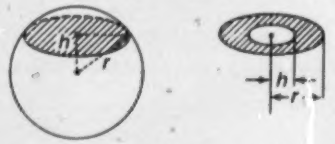


Fig. A–11

To do so we note first that, given a ball of radius r, the area-measure of a cross section at the distance h from its center is $\pi(\sqrt{r^2 - h^2})^2$ or, equivalently, $\pi r^2 - \pi h^2$. This is also the area-measure of the circular ring shown at the left in Fig. A-11. It follows that we can use Cavalieri's Principle to find the volume of the ball if we can find a figure of altitude $2r$ whose volume we know and whose cross sections are circular rings of the appropriate sizes. Fig. A-12 shows how to do this.

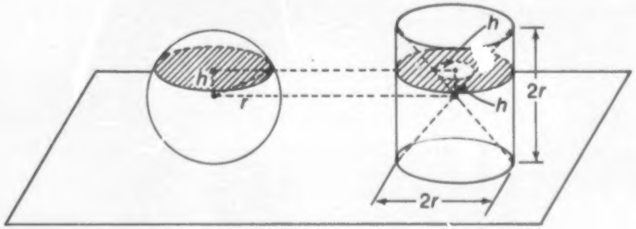


Fig. A–12

So, it turns out, by Cavalieri's Principle, that the volume of a ball is the difference between the volume of a cylinder and the volume of the union of two cones of the dimensions shown in Fig. A-12. So, if V is the volume-measure of the ball,

$$V = \pi r^2(2r) - 2\left(\frac{1}{3}\pi r^2 \cdot r\right)$$

and, so,

$$V = \frac{4}{3}\pi r^3. \tag{1}$$

From this formula we can make an educated guess as to the area-measure S of the sphere which is the surface of a ball of radius r. To do so we may think of such a ball as being the union of pyramid-like solids whose vertices are the center of the ball and whose bases are curved regions on the surface of the ball. From the similarity of these solids to pyramids it is reasonable to guess that the volume of one of them is one-third the product of the area-measure of its base by its altitude r. Assuming this to be the case it follows that

$$V = \frac{1}{3}Sr. \tag{2}$$

Comparing this with (1) we obtain

$$S = 4\pi r^2, \tag{3}$$

the correct formula for the area-measure of a sphere.

It is interesting to note that the area-measure of the sphere pictured in Fig. A-12 is the same as the measure of the lateral surface of the cylinder in the same figure. This is a special case of a more general result concerning the area-measure of *a spherical zone*. Such a zone is the portion of a sphere between two parallel planes. Its altitude is the distance h between the planes and, although we shall not show that this is the case, its area-measure S is given by:

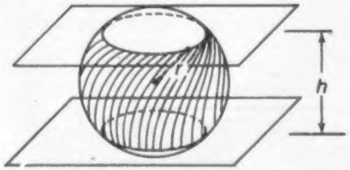


Fig. A–13

$$S = 2\pi rh,$$

where r is the radius of the sphere. It is somewhat surprising to note that any two zones of the same sphere which have the same altitude also have the same area-measure.

Exercises

1. (a) What is the volume-measure of a ball whose diameter is d?
   (b) What is the area-measure of a sphere whose diameter is d?
2. (a) What is the ratio of the volume of a ball inscribed in a cube to the volume of the cube?
   (b) What is the ratio of the volume of a cube inscribed in a ball to the volume of the ball?
3. (a) What is the ratio of the volume of a ball to that of a right circular cylinder circumscribing it?
   (b) What is the ratio of the areas of the surfaces of the two solids in part (a)?
4. *A great circle* of a sphere is a circle on the sphere whose radius is that of the sphere. If the circumference of a great circle is 5 inches, what is the area of the surface of the sphere?
5. (a) If the radius of a sphere is increased by 10%, by what percent is the surface area increased? By what percent is the enclosed volume increased?
   (b) Answer part (a), assuming that it is the diameter of the sphere which is increased by 10%.

Answers for Exercises

1. (a) $\pi d^3/6$ (b) πd^2
2. (a) $\pi/6$ (b) $3/(\pi\sqrt{2})$
3. (a) $2/3$ (b) $2/3$
4. $25/\pi$ square inches
5. (a) 21%; 33.1%
   (b) 21%; 33.1% [Increasing the diameter by 10% is the same as increasing the radius by 10%.]

Theorems from Volume 1

Postulates

1. **(a)** $B - A \in \mathscr{T}$ **(b)** $A + \vec{a} \in \mathscr{E}$
2. **(a)** $A + (B - A) = B$ **(b)** $\vec{a} = (A + \vec{a}) - A$
3. $(B - A) + (C - B) = C - A$
4_0. **(a)** $\vec{a} + \vec{b} \in \mathscr{T}$ **(b)** $\vec{0} \in \mathscr{T}$ **(c)** $-\vec{a} \in \mathscr{T}$ **(d)** $\vec{a}a \in \mathscr{T}$
4_1. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ **4_2.** $\vec{a} + \vec{0} = \vec{a}$
4_3. $\vec{a} + -\vec{a} = \vec{0}$ **4_4.** $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

4‴. $\mathscr{T}$, under function composition, is a commutative group.

4_5. $\vec{a}1 = \vec{a}$ **4_6.** $\vec{a}(b + c) = \vec{a}b + \vec{a}c$
4_7. $(\vec{a} + \vec{b})c = \vec{a}c + \vec{b}c$ **4_8.** $(\vec{a}b)c = \vec{a}(bc)$

4″. $\mathscr{T}$, under function composition, is a vector space over $\mathscr{R}$.

4_9. There are three linearly independent members of $\mathscr{T}$.
4_{10}. There are not four linearly independent members of $\mathscr{T}$.

4′. $\mathscr{T}$, under function composition, is a 3-dimensional vector space over $\mathscr{R}$.

5_0. **(a)** $a + b \in \mathscr{R}$ **(b)** $-a \in \mathscr{R}$ **(c)** $0 \in \mathscr{R}$ **(d)** $a \cdot b \in \mathscr{R}$ **(e)** $1 \in \mathscr{R}$ **(f)** $/a \in \mathscr{R}$
5_1. **(a)** $(a + b) + c = a + (b + c)$ **(b)** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
5_2. **(a)** $a + 0 = a$ **(b)** $a \cdot 1 = a$
5_3. **(a)** $a + -a = 0$ **(b)** $a \cdot /a = 1$ $[a \neq 0]$
5_4. **(a)** $a + b = b + a$ **(b)** $a \cdot b = b \cdot a$
5_5. $0 \neq 1$
5_6. $(a + b) \cdot c = a \cdot c + b \cdot c$
5_7. **(a)** $a - b = a + -b$ **(b)** $a \div b = a \cdot /b$ $[b \neq 0]$
5_8. $a > b$ or $b > a$ $[a \neq b]$
5_9. $a \not> a$
5_{10}. $(a > b$ and $b > c) \longrightarrow a > c$
5_{11}. $a < b \longrightarrow b > a$
5_{12}. **(a)** $a > b \longrightarrow a + c > b + c$
(b) $a > b \longrightarrow a \cdot c > b \cdot c$ $[c > 0]$

5′. $\mathscr{R}$ is an ordered field.

Definitions

3-1. (a) $A - \vec{a} = A + -\vec{a}$ (b) $\vec{a} - \vec{b} = \vec{a} + -\vec{b}$

3-2. (a) $T_O(A) = A - O$ (b) $P_O(\vec{a}) = O + \vec{a}$

5-1. $[\vec{d}] = \{\vec{x}: \exists_x \vec{x} = \vec{d}x\}$

5-2. $[\vec{d}, \vec{e}] = \{\vec{x}: \exists_x \exists_y \vec{x} = \vec{d}x + \vec{e}y\}$

6-1. $\vec{a}$ is a linear combination of $(\vec{a}_1, \ldots, \vec{a}_n)$
$\longleftrightarrow \exists_{x_1} \ldots \exists_{x_n} \vec{a} = \vec{a}_1 x_1 + \ldots + \vec{a}_n x_n$

6-2. $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly dependent
$\longleftrightarrow \exists_{x_1} \ldots \exists_{x_n} (\vec{a}_1 x_1 + \ldots + \vec{a}_n x_n = \vec{0}$ and not $(x_1 = 0, \ldots,$ and $x_n = 0))$

6-3. A sequence is linearly independent if and only if it is not linearly dependent.

6-4. $[\vec{a}_1, \ldots, \vec{a}_n] = \{\vec{x}: \exists_{x_1} \ldots \exists_{x_n} \vec{x} = \vec{a}_1 x_1 + \ldots + \vec{a}_n x_n\}$

7-1. $\{A, B, C\}$ is collinear if and only if $(B - A, C - A)$ is linearly dependent.

7-2. l is a line if and only if (a) l is a subset of $\mathscr{E}$ which contains at least two points, and (b) $\forall_X \forall_Y [(\{X, Y\} \subseteq l$ and $X \neq Y) \longrightarrow \forall_Z [Z \in l \longleftrightarrow \{X, Y, Z\}$ is collinear$]]$

7-3. $\overleftrightarrow{AB} = \{X: \exists_x X = A + (B - A)x\}$ $[= \{X: X - A \in [B - A]\}]$

7-4. $[l] = \{\vec{x}: \exists_Y \exists_Z (Y \in l$ and $Z \in l$ and $\vec{x} = Z - Y)\}$ [Read '$[l]$' as 'the direction of l'. Also, read '$[\vec{a}]$' as 'the direction of arrow a'.]

7-5. (a) $\overleftrightarrow{A[\vec{a}]} = \{X: X - A \in [\vec{a}]\}$
(b) $\overleftrightarrow{A[l]} = \{X: X - A \in [l]\}$

7-6. $l \parallel m$ if and only if $[l] = [m]$.

7-7. For $\mathscr{X} \subseteq \mathscr{E}$, $\mathscr{X} + \vec{a} = \{X: \exists_Y (Y \in \mathscr{X}$ and $X = Y + \vec{a})\}$.

7-8. $[\vec{a}]^+ = \{\vec{x}: \vec{x} \neq \vec{0}$ and $\exists_{x > 0} \vec{x} = \vec{a}x\}$ [Read '$[\vec{a}]^+$' as 'the sense of $\vec{a}$'.]

7-9. (a) $\overrightarrow{A[\vec{a}]^+} = \{X: X - A \in [\vec{a}]^+\}$
(b) $\overrightarrow{AB} = \overrightarrow{A[B - A]^+}$ [Read '$\overrightarrow{AB}$' as 'arrow AB'; when $A \neq B$, it is proper to read '$\overrightarrow{AB}$' as 'half-line AB'.]

7-10. (a) $\overset{\bullet\!\!\longrightarrow}{A[\vec{a}]^+} = \{A\} \cup \overrightarrow{A[\vec{a}]^+}$
(b) $\overset{\bullet\!\!\longrightarrow}{AB} = \{A\} \cup \overrightarrow{AB}$

7-11. (a) $-\overrightarrow{A[\vec{a}]^+} = \overrightarrow{A[-\vec{a}]^+}$
(b) $-\overset{\bullet\!\!\longrightarrow}{A[\vec{a}]^+} = \overset{\bullet\!\!\longrightarrow}{A[-\vec{a}]^+}$

7-12. (a) $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$
(b) $\overset{\bullet\!\!-\!\!\bullet}{AB} = \{A, B\} \cup \overline{AB}$

7-13. A first set is parallel to a second set if and only if they are nondegenerate subsets of parallel lines.

7-14. $\vec{a}:\vec{b} = c \longleftrightarrow \vec{a} = \vec{b}c$ $[[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}]$

7-15. M is the midpoint of $\overset{\bullet\!\!-\!\!\bullet}{AB} \longleftrightarrow M - A = B - M$

8-1. For $P \in \overleftrightarrow{AB}$, $A \neq P \neq B$ and $a \neq 0 \neq b$, P divides the interval from A to B in $a : b$ if and only if $(P - A) : (B - P) = a/b$.

8-2. (a) $PQR = \overset{\bullet\!\!-\!\!\bullet}{PQ} \cup \overset{\bullet\!\!-\!\!\bullet}{QR} \cup \overset{\bullet\!\!-\!\!\bullet}{RP}$
(b) PQR is a triangle $\longleftrightarrow \{P, Q, R\}$ is noncollinear

8-3. The *median* of a triangle from a given vertex is the interval whose end points are the given vertex and the midpoint of the opposite side.

8-4. (a) $PQRS = \overset{\bullet\!\!-\!\!\bullet}{PQ} \cup \overset{\bullet\!\!-\!\!\bullet}{QR} \cup \overset{\bullet\!\!-\!\!\bullet}{RS} \cup \overset{\bullet\!\!-\!\!\bullet}{SP}$
(b) $PQRS$ is a quadrilateral $\longleftrightarrow$ each of $\{P, Q, R\}$, $\{Q, R, S\}$, $\{R, S, P\}$, $\{S, P, Q\}$ is noncollinear

8-5. (a) A quadrilateral is *simple* if and only if no two of its sides intersect.
(b) A quadrilateral is *convex* if and only if its diagonals intersect.

8-6. (a) A quadrilateral is a *trapezoid* if and only if it is simple and has two parallel sides.
(b) A quadrilateral is a *parallelogram* if and only if its opposite sides are parallel.

9-1. $\{A, B, C, D\}$ is coplanar $\longleftrightarrow (B - A, C - A, D - A)$ is linearly dependent.

9-2. π is a plane if and only if (a) π is a subset of $\mathscr{E}$ which contains at least three noncollinear points, and (b) $\forall_X \forall_Y \forall_Z [(\{X, Y, Z\} \subseteq \pi$ and $\{X, Y, Z\}$ is noncollinear $\longrightarrow \forall_W [W \in \pi \longleftrightarrow \{X, Y, Z, W\}$ is coplanar$]]$

9-3. $\overline{ABC} = \{X: \exists_x \exists_y X = A + (B - A)x + (C - A)y\}$

9-4. $[\pi] = \{\vec{x}: \exists_Y \exists_Z (Y \in \pi$ and $Z \in \pi$ and $\vec{x} = Z - Y)\}$

9-5. (a) $\overline{A[\vec{a}, \vec{b}]} = \{X: X - A \in [\vec{a}, \vec{b}]\}$
(b) $\overline{A[\pi]} = \{X: X - A \in [\pi]\}$

9-6. (a) $\pi_1 \parallel \pi_2 \longleftrightarrow [\pi_1] = [\pi_2]$
(b) $l \parallel \pi \longleftrightarrow [l] \subseteq [\pi]$
(c) $\pi \parallel l \longleftrightarrow l \parallel \pi$

10-1. $(\vec{a}_1, \ldots, \vec{a}_n)$ is a basis for $\mathscr{T}$ if and only if (i) $(\vec{a}_1, \ldots, \vec{a}_n)$ is linearly independent, and (ii) $[\vec{a}_1, \ldots, \vec{a}_n] = \mathscr{T}$.

10-2. If $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis for $\mathscr{T}$ and $\vec{a} = \vec{u}_1 a_1 + \vec{u}_2 a_2 + \vec{u}_3 a_3$ then a_1, a_2, and a_3 are, respectively, the first, second, and third components of $\vec{a}$ with respect to the given basis. Also, (a_1, a_2, a_3) is the *component-triple* of $\vec{a}$ with respect to this basis.

10-3. If $O \in \mathscr{E}$, $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis for $\mathscr{T}$, and $A = O + (\vec{u}_1 a_1 + \vec{u}_2 a_2 + \vec{u}_3 a_3)$ then a_1, a_2, and a_3 are, respectively, the first, second, and third coordinates of A with respect to the given point and given basis. Also, (a_1, a_2, a_3) is the coordinate-triple of A with respect to this point and basis.

Other Theorems

2-1. $A + \overrightarrow{a} = B \longleftrightarrow \overrightarrow{a} = B - A$

2-2. $A + \overrightarrow{a} = A + \overrightarrow{b} \longleftrightarrow \overrightarrow{a} = \overrightarrow{b}$

2-3. $A - C = B - C \longleftrightarrow A = B$

2-4. $\overrightarrow{a} + \overrightarrow{b} = [(A + \overrightarrow{a}) + \overrightarrow{b}] - A$

2-5. (a) $\overrightarrow{a} + \overrightarrow{b} \in \mathscr{T}$ (b) $A + (\overrightarrow{a} + \overrightarrow{b}) = (A + \overrightarrow{a}) + \overrightarrow{b}$

3-1. (a) $A + \overrightarrow{0} = A$ (b) $A - A = \overrightarrow{0}$

3-2. (a) $A + \overrightarrow{a} = A \longleftrightarrow \overrightarrow{a} = \overrightarrow{0}$ (b) $B = A \longleftrightarrow B - A = \overrightarrow{0}$

3-3. $A + \overrightarrow{a} = B + \overrightarrow{a} \longleftrightarrow A = B$

3-4. $C - A = C - B \longleftrightarrow A = B$

3-5. (a) $-(B - A) = A - B$ (b) $-\overrightarrow{a} = A - (A + \overrightarrow{a})$

3-6. $(A + \overrightarrow{a}) + \overrightarrow{b} = A \longleftrightarrow \overrightarrow{b} = -\overrightarrow{a}$

3-7. $(B - C) - (A - C) = B - A$

3-8. $(A - B) + \overrightarrow{c} = (A + \overrightarrow{c}) - B$

3-9. $A + (B - C) = B + (A - C)$

3-10. $(A - B) + \overrightarrow{c} = A - (B - \overrightarrow{c})$

3-11. $\overrightarrow{a} - (B - C) = C - (B - \overrightarrow{a})$

3-12. $A - (B - C) = C - (B - A)$

3-13. $(A - B) - (C - D) = (A - C) - (B - D)$

Corollary. $A - B = C - D \longleftrightarrow A - C = B - D$

3-14. (a) $P_O \circ T_O = i_{\mathscr{E}}$ (b) $T_O \circ P_O = i_{\mathscr{T}}$

5-1. (a) $\overrightarrow{a0} = \overrightarrow{0}$ (b) $\overrightarrow{0a} = \overrightarrow{0}$

5-2. (a) $\overrightarrow{a} \cdot -a = -(\overrightarrow{aa})$ (b) $-\overrightarrow{a} \cdot a = -(\overrightarrow{aa})$

5-3. (a) $\overrightarrow{a}(a - b) = \overrightarrow{aa} - \overrightarrow{ab}$ (b) $(\overrightarrow{a} - \overrightarrow{b})a = \overrightarrow{aa} - \overrightarrow{ba}$

5-4. (a) $\overrightarrow{ac} = \overrightarrow{bc} \longrightarrow \overrightarrow{a} = \overrightarrow{b}$ $[c \neq 0]$
(b) $\overrightarrow{ca} = \overrightarrow{cb} \longrightarrow a = b$ $[\overrightarrow{c} \neq \overrightarrow{0}]$

5-5. $\overrightarrow{aa} = \overrightarrow{0} \longrightarrow (a = 0 \text{ or } \overrightarrow{a} = \overrightarrow{0})$

6-1. $(\overrightarrow{a})$ is linearly dependent $\longleftrightarrow \overrightarrow{a} = \overrightarrow{0}$.

6-2. For $n \geq 2$, $(\overrightarrow{a}_1, \ldots, \overrightarrow{a}_n)$ is linearly dependent $\longleftrightarrow$ one of the vectors $\overrightarrow{a}_1, \ldots, \overrightarrow{a}_n$ is a linear combination of the others.

6-3. (a) A sequence one of whose terms is $\overrightarrow{0}$ is linearly dependent.
(b) A sequence two of whose terms are equal is linearly dependent.

6-4. If any subsequence of a given sequence is linearly dependent then the given sequence is linearly dependent.

6-5. If any term of a given sequence is a multiple of another term then the given sequence is linearly dependent.

6-6. A permutation of a given sequence is linearly dependent if and only if the given sequence is linearly dependent.

6-7. $(\overrightarrow{a}_1, \ldots, \overrightarrow{a}_n)$ is linearly independent
$\longleftrightarrow \forall_{x_1} \ldots \forall_{x_n} [\overrightarrow{a}_1 x_1 + \ldots + \overrightarrow{a}_n x_n = \overrightarrow{0} \longrightarrow (x_1 = 0, \ldots, \text{and } x_n = 0)]$

6-8. $(\overrightarrow{a}_1, \ldots, \overrightarrow{a}_n)$ is linearly independent
$\longrightarrow (\overrightarrow{a}_1 a_1 + \ldots + \overrightarrow{a}_n a_n = \overrightarrow{a}_1 b_1 + \ldots + \overrightarrow{a}_n b_n \longrightarrow (a_1 = b_1, \ldots, \text{and } a_n = b_n))$

6-9. If a sequence is linearly independent then any of its subsequences is linearly independent.

6-10. Any permutation of a linearly independent sequence is linearly independent.

6-11. Any linearly independent sequence is a sequence of distinct, non-$\overrightarrow{0}$, terms.

6-12. $((\overrightarrow{a}, \overrightarrow{b})$ is linearly independent and $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = \overrightarrow{0})$
$\longrightarrow (\overrightarrow{a}a + \overrightarrow{b}b + \overrightarrow{c}c = \overrightarrow{0} \longleftrightarrow a = b = c)$

6-13. $((\overrightarrow{a}_1, \ldots, \overrightarrow{a}_{n+1})$ is linearly dependent and $(\overrightarrow{a}_1, \ldots, \overrightarrow{a}_n)$ is linearly independent$) \longrightarrow \overrightarrow{a}_{n+1} \in [\overrightarrow{a}_1, \ldots, \overrightarrow{a}_n]$

6-14. $(B - A, C - A)$ is linearly dependent $\longleftrightarrow (C - B, A - B)$ is linearly dependent

7-1. For $A \neq B$, $\overleftrightarrow{AB}$ is the line which contains A and B.

Corollary. There is one and only one line which contains two given points.

Corollary. Two points are contained in at most one line.

Corollary. Two lines have at most one point in common.

7-2. $(\{C, D\} \subseteq \overleftrightarrow{AB} \text{ and } C \neq D) \longrightarrow \overleftrightarrow{AB} = \overleftrightarrow{CD}$

7-3. $(\{A, B\} \subseteq l \text{ and } A \neq B) \longrightarrow [l] = [B - A]$

7-4. $(A \in l \text{ and } \overrightarrow{a} \in [l]) \longrightarrow A + \overrightarrow{a} \in l$

7-5. (a) For $\overrightarrow{a} \neq \overrightarrow{0}$, $\overleftrightarrow{A[a]}$ is the line through A in the direction of $\overrightarrow{a}$.
(b) $\overleftrightarrow{A[l]}$ is the line through A in the direction of l.

Corollary. $A \in l \longleftrightarrow l = \overleftrightarrow{A[l]}$

7-6. There is one and only one line through a given point and parallel to a given line.

7-7. (a) $l \parallel l$ (b) $l \parallel m \longrightarrow m \parallel l$
(c) $(l \parallel m \text{ and } m \parallel n) \longrightarrow l \parallel n$

7-8. A translation maps any line onto a parallel line.

7-9. $\mathscr{K} + \overrightarrow{a} = \{X : X - \overrightarrow{a} \in \mathscr{K}\}$ $[\mathscr{K} \subseteq \mathscr{E}]$

7-10. $\overleftrightarrow{A[l]} + \overrightarrow{a} = \overleftrightarrow{(A + \overrightarrow{a})[l]}$

7-11. $\overleftrightarrow{A[B - C]} \cap \overleftrightarrow{B[A - C]} = \{A + (B - C)\}$ $[\{A, B, C\}$ noncollinear$]$

7-12. $\overleftrightarrow{D[l]} \cap m \neq \emptyset \longleftrightarrow \overleftrightarrow{D[m]} \cap l \neq \emptyset$ $[l \cap m \neq \emptyset]$

7-13. (a) $[\vec{a}]^+ = \{\vec{x}: \exists_{x>0}\, \vec{x} = \vec{a}x\}$ $\quad [\vec{a} \neq \vec{0}]$
(b) $[\vec{0}]^+ = \varnothing$

7-14. $[\vec{a}] = [\vec{a}]^+ \cup \{\vec{0}\} \cup [-\vec{a}]^+$

7-15. $A + (B - A)a = A + (B - A)b \longleftrightarrow a = b$ $\quad [A \neq B]$

7-16. (a) $\overrightarrow{AB} = \{X: \exists_{x>0}\, X = A + (B - A)x\}$ $\quad [A \neq B]$
(b) $\overset{\bullet\!\!\longrightarrow}{AB} = \{X: \exists_{x\geq 0}\, X = A + (B - A)x\}$

7-17. $C \in \overrightarrow{AB} \longrightarrow (\overrightarrow{AC} = \overrightarrow{AB}$ and $\overset{\bullet\!\!\longrightarrow}{AC} = \overset{\bullet\!\!\longrightarrow}{AB})$

7-18. $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD} \longrightarrow A = C$

Corollary. $\overset{\bullet\!\!\longrightarrow}{AB} = \overset{\bullet\!\!\longrightarrow}{CD} \longrightarrow [B - A]^+ = [D - C]^+$

7-19. $\overrightarrow{AB} = \overrightarrow{CD} \longrightarrow A = C$ $\quad [A \neq B]$

Corollary. $\overrightarrow{AB} = \overrightarrow{CD} \longrightarrow [B - A]^+ = [D - C]^+$

7-20. $\overleftrightarrow{AB} = \overset{\bullet\!\!\longrightarrow}{AB} \cup \overset{\bullet\!\!\longrightarrow}{BA}$

7-21. (a) $\overline{AB} = \{X: \exists_x\, (0 < x < 1$ and $X = A + (B - A)x)\}$ $\quad [A \neq B]$
(b) $\overset{\bullet\!\!-\!\!\bullet}{AB} = \{X: \exists_x\, (0 \leq x \leq 1$ and $X = A + (B - A)x)\}$

7-22. (a) $\overline{AB} = \overline{CD} \longrightarrow \{A, C\} = \{B, D\}$ $\quad [A \neq B]$
(b) $\overset{\bullet\!\!-\!\!\bullet}{AB} = \overset{\bullet\!\!-\!\!\bullet}{CD} \longrightarrow \{A, C\} = \{B, D\}$

7-23. (a) $\vec{a} = \vec{b}(\vec{a} : \vec{b})$ $\quad [[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}]$
(b) $\vec{a} = \vec{b}c \longrightarrow \vec{a} : \vec{b} = c$ $\quad [\vec{a} \neq \vec{0}]$

Corollary. (a) $\vec{a} : \vec{b} = a/b \longrightarrow \vec{a}b = \vec{b}a$ $\quad [[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}, b \neq 0]$
(b) $\vec{a}b = \vec{b}a \longrightarrow \vec{a} : \vec{b} = a/b$ $\quad [\vec{a} \neq \vec{0}, b \neq 0]$

7-24. $(\vec{a}a) : (\vec{b}b) = (\vec{a} : \vec{b})(a/b)$ $\quad [[\vec{a}] = [\vec{b}] \neq \{\vec{0}\}, a \neq 0 \neq b]$

7-25. $(\vec{a}a + \vec{b}b) : \vec{c} = (\vec{a} : \vec{c})a + (\vec{b} : \vec{c})b$ $\quad [[\vec{a}] = [\vec{b}] = [\vec{c}] \neq \{\vec{0}\}, \vec{a}a + \vec{b}b \neq \vec{0}]$

8-1. $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} \neq \varnothing \longleftrightarrow C - A \in [B - A, C - D]$

8-2. If $\overset{\bullet\!\!-\!\!\bullet}{AC}$ and $\overset{\bullet\!\!-\!\!\bullet}{BD}$ are noncollinear parallel segments and $(D - B) : (C - A) = r$ then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if $r = 1$ and $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{A + (B - A)/(1 - r)\}$ if $r \neq 1$.

8-3. If A, B, C, D, and P are five points such that $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{P\}$ then (a) $\overset{\bullet\!\!-\!\!\bullet}{AC} \parallel \overset{\bullet\!\!-\!\!\bullet}{BD} \longrightarrow (P - D) : (P - C) = (D - B) : (C - A) = (P - B) : (P - A)$, and (b) $(P - D) : (P - C) = (P - B) : (P - A) \longrightarrow \overset{\bullet\!\!-\!\!\bullet}{AC} \parallel \overset{\bullet\!\!-\!\!\bullet}{BD}$.

Corollary. Under the conditions specified in the theorem, $\overset{\bullet\!\!-\!\!\bullet}{AC} \parallel \overset{\bullet\!\!-\!\!\bullet}{BD} \longleftrightarrow (P - D) : (P - C) = (P - B) : (P - A)$.

8-4. For $A \neq P \neq B$, (a) P divides the interval from A to B in $a : b \longrightarrow P = A + (B - A)\dfrac{a}{a + b}$ $[P \in \overleftrightarrow{AB},\ a \neq 0 \neq b]$ (b) $P = A + (B - A)\dfrac{a}{a + b} \longrightarrow P$ divides the interval from A to B in $a : b$ $\quad [a + b \neq 0]$

Corollary. For $A \neq P \neq B$, (a) P divides the interval from A to B in $s : 1 \longrightarrow P = A + (B - A)\dfrac{s}{s + 1}$ $[P \in \overleftrightarrow{AB},\ s \neq 0]$ (b) $P = A + (B - A)t \longrightarrow P$ divides the interval from A to B in $t : 1 - t$.

8-5. For $P \in \overleftrightarrow{AB}$ and $A \neq P \neq B$, $P \in \overline{AB}$ or $P \in -\overrightarrow{AB}$ or $P \in -\overrightarrow{BA}$ according as the ratio in which P divides the interval from A to B is positive, or between 0 and -1, or less than -1.

8-6. (a) The ratio of two intervals which are intercepted on one of two parallel lines by concurrent transversals of both these lines is the same as that of the corresponding intervals which are intercepted by these transversals on the other.
(b) The ratio of two intervals which are intercepted by parallel lines on one transversal of these lines is the same as that of the corresponding intervals which are intercepted by these lines on any other transversal.

8-7. The interval whose endpoints are the midpoints of two sides of a triangle is parallel to the third side and its ratio to the third side is 1/2.

Corollary. A line through the midpoint of one side of a triangle is parallel to a second side if and only if it contains the midpoint of the third side.

8-8. The three medians of a triangle intersect at a point which divides each of them, from vertex to midpoint of opposite side, in 2 : 1.

8-9. (a) The interval whose end points are the r-points of two sides of a triangle, from their common endpoint, is parallel to the third side, and its ratio to the third side is r.
(b) A line through the r-point of one side of a triangle, from one of its vertices, is parallel to the side opposite that vertex if and only if it contains the r-point, from that vertex, of the third side.

8-10. (a) Intervals from two vertices of a triangle to r-points of the opposite sides [from their common vertex] intersect at the $\dfrac{2r}{r + 1}$-point of the median from that vertex. The point of intersection divides each of the two intervals, from vertex to opposite side in $1 : r$ and divides the median, from vertex to side, in $2r : 1 - r$.
(b) Lines through two vertices of a triangle which intersect at the s-point of the median from the third vertex intersect the opposite sides at their $\dfrac{s}{2 - s}$-points from this vertex.

8-11. If, in $\triangle ABC$ and $\triangle A'B'C'$, $\overline{AB} \parallel \overline{A'B'}$, $\overline{BC} \parallel \overline{B'C'}$, and $\overline{CA} \parallel \overline{C'A'}$ then (a) $(B' - A') : (B - A) = (C' - B') : (C - B) = (A' - C') : (A - C)$, and (b) for $A \neq A'$, $B \neq B'$, and $C \neq C'$, the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are parallel or concurrent.

8-12. [The Twice-Around Theorem] If, in $\triangle ABC$, G and D are in $\overline{BC}$, E and H are in $\overline{CA}$, and I and F are in $\overline{AB}$, and $\overline{DE} \parallel \overline{BA} \parallel \overline{GH}$, $\overline{EF} \parallel \overline{CB} \parallel \overline{HI}$, $\overline{FG} \parallel \overline{AC}$, then $\overline{ID} \parallel \overline{AC}$.

8-13. If, in $\triangle ABP$, D and F are in $\overline{BP}$, C and E are in $\overline{PA}$, $\overline{EF} \parallel \overline{BC}$, and $\overline{CD} \parallel \overline{AF}$, then $\overline{DE} \parallel \overline{AB}$.

8-14. If $\vec{a}$, $\vec{b}$, and $\vec{r}$ are position vectors of A, B, and R [with respect to any point O] then, for $A \neq B$ and $0 \neq r \neq 1$, R is the point which divides the interval from A to B in $r : 1 - r$ if and only if $\vec{r} = \vec{a}(1 - r) + \vec{b}r$.

8-15. $\vec{a}$, $\vec{b}$, and $\vec{c}$ are position vectors of collinear points if and only if there exist numbers x, y, and z, not all 0, such that $\vec{a}x + \vec{b}y + \vec{c}z = \vec{0}$ and $x + y + z = 0$.

Corollary. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are position vectors of noncollinear points, and a, b, and c are numbers such that $a + b + c = 0$, then $\vec{a}a + \vec{b}b + \vec{c}c = \vec{0}$ if and only if [$a = 0$, $b = 0$, and $c = 0$].

8-16. (a) $PQRS$ is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$ if and only if $\overline{PQ}$ and $\overline{RS}$ are noncollinear parallel intervals such that $(Q - P) : (R - S) > 0$.

(b) If, in trapezoid $PQRS$, $\overline{PS} \nparallel \overline{QR}$ then $\overleftrightarrow{PS}$ and $\overleftrightarrow{QR}$ intersect at a point which divides both the interval from P to S and the interval from Q to R in $-(\overline{PQ} : \overline{RS})$.

8-17. (a) A trapezoid is convex.

(b) If $PQRS$ is a trapezoid with bases $\overline{PQ}$ and $\overline{RS}$ then the insection of its diagonals divides each of them, from P to R and from Q to S, respectively, in $\overline{PQ} : \overline{RS}$.

8-18. (a) The midpoints of successive sides of a simple quadrilateral are the successive vertices of a parallelogram.

(b) The intervals joining the midpoints of opposite sides of a simple quadrilateral bisect each other.

8-19. [Menelaus' Theorem and Converse] If, in $\triangle ABC$, $R \in \overleftrightarrow{BC}$, $S \in \overleftrightarrow{CA}$, and $T \in \overleftrightarrow{AB}$ then $\{R, S, T\}$ is collinear if and only if $BR \cdot CS \cdot AT = -(RC \cdot SA \cdot TB)$.

8-20. If, in $\triangle ABC$, $R \in \overleftrightarrow{BC}$, $S \in \overleftrightarrow{CA}$, and $T \in \overleftrightarrow{AB}$ then $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$, and $\overleftrightarrow{CT}$ are concurrent or parallel if and only if $BR \cdot CS \cdot AT = RC \cdot SA \cdot TB$.

9-1. For $\{A, B, C\}$ noncollinear, $\overline{ABC}$ is the plane which contains A, B, and C.

Lemma. $((\vec{c}, \vec{d})$ is linearly independent and $\{\vec{c}, \vec{d}\} \subseteq [\vec{a}, \vec{b}]) \longrightarrow [\vec{c}, \vec{d}] = [\vec{a}, \vec{b}]$

Corollary. Three noncollinear points determine [uniquely] a plane.

9-2. $(\{D, E, F\} \subseteq \overline{ABC}$ and $\{D, E, F\}$ is noncollinear$) \longrightarrow \overline{ABC} = \overline{DEF}$

9-3. A plane contains the line determined by any two of its points.

9-4. A line and a point not on that line determine a plane.

9-5. Two intersecting lines determine a plane.

9-6. Two parallel lines determine a plane.

9-7. Two nonparallel coplanar lines intersect.

9-8. Two lines are parallel if and only if they are coplanar and have no common point.

9-9. $\{A, B, C\}$ is a noncollinear subset of $\pi \longrightarrow [\pi] = [B - A, C - A]$

9-10. $(P \in \pi$ and $\vec{a} \in [\pi]) \longrightarrow P + \vec{a} \in \pi$

9-11. (a) For $(\vec{a}, \vec{b})$ linearly independent, $\overline{A[\vec{a}, \vec{b}]}$ is the plane through A with the bidirection $[\vec{a}, \vec{b}]$.

(b) $\overline{A[\pi]}$ is the plane through A with the direction of π.

9-12. Any translation maps any plane onto a parallel plane.

9-13. There is one and only one plane through a given point and parallel to a given plane.

9-14. There is a plane containing a given line and parallel to a given plane if and only if the given line and plane are parallel.

9-15. There is one and only one plane which contains a given point and is parallel to each of two given nonparallel lines.

9-16. Two lines which are contained in different parallel planes are parallel if and only if they are coplanar.

9-17. The ratio of two intervals which are intercepted by parallel planes on one transversal of these planes is the same as that of the corresponding intervals which are intercepted by these planes on any other transversal.

10-1. There are four noncoplanar points.

10-2. The intersection of two nonparallel planes is a line.

10-3. A line which is parallel to each of two nonparallel planes is parallel to their intersection.

10-4. A line and a plane which are not parallel intersect at a single point.

Corollary 1. A line which is a transversal of one plane is a transversal of any parallel plane.

Corollary 2. Parallel lines are transversals of the same plane.

10-5. A plane which intersects one of two parallel planes intersects the other, and the intersections are parallel lines.

10-6. (a) $\sigma \parallel \pi \longleftrightarrow (\sigma = \pi$ or $\sigma \cap \pi = \emptyset)$

(b) $l \parallel \pi \longleftrightarrow (l \subseteq \pi$ or $l \cap \pi = \emptyset)$

10-7. Each 3-termed linearly independent sequence of translations is a basis for $\mathscr{T}$.

10-8. Each basis for $\mathscr{T}$ is a 3-termed linearly independent sequence of translations.

10-9. $[\vec{a}_1, \vec{a}_2, \vec{a}_3] = \mathscr{T} \longleftrightarrow (\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is linearly independent

10-10. (a) Each non-$\vec{0}$ translation is a term of some basis for $\mathscr{T}$.

(b) Each two linearly independent translations are terms of some basis for $\mathscr{T}$.

10-11. For any basis for $\mathscr{T}$, (a) each component of $\vec{0}$ is 0, (b) each component of $-\vec{a}$ is the opposite of the corresponding component of $\vec{a}$, (c) each component of $\vec{a} + \vec{b}$ is the sum of the corresponding components of $\vec{a}$ and $\vec{b}$, (d) each component of $\overrightarrow{aa}$ is the product of the corresponding component of $\vec{a}$ by a.

10-12. If $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are position vectors of noncoplanar points and $a + b + c = 0$ then $\overrightarrow{aa} + \overrightarrow{bb} + \overrightarrow{cc} + \overrightarrow{dd} = \vec{0}$ if and only if $a = 0, b = 0, c = 0$, and $d = 0$.

10–13. For $(\vec{u}_1, \vec{u}_2)$ linearly independent, $(\vec{u}_1 a_1 + \vec{u}_2 a_2, \vec{u}_1 b_1 + \vec{u}_2 b_2)$ is linearly dependent if and only if $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$.

10–14. For $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ linearly independent, $(\vec{u}_1 a_1 + \vec{u}_2 a_2 + \vec{u}_3 a_3, \vec{u}_1 b_1 + \vec{u}_2 b_2 + \vec{u}_3 b_3)$ is linearly dependent if and only if

$$\left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right) = (0, 0, 0).$$

10–15. Suppose that, with respect to a given coordinate system, the coordinates of A are (a_1, a_2, a_3) and the components of $\vec{p}$ and $\vec{q}$ are (p_1, p_2, p_3) and (q_1, q_2, q_3), respectively. With respect to the given coordinate system

(a) the parametric equations:

$$\begin{cases} x_1 = a_1 + p_1 s + q_1 t \\ x_2 = a_2 + p_2 s + q_2 t \\ x_3 = a_3 + p_3 s + q_3 t \end{cases}$$

describe the set $\overline{A[\vec{p}, \vec{q}]}$; and this set is a plane if and only if

$$(\dagger) \qquad \left(\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}, \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}\right) \neq (0, 0, 0),$$

and

(b) the single equation:

$$(x_1 - a_1)\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} + (x_2 - a_2)\begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} + (x_3 - a_3)\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = 0$$

represents $\overline{A[\vec{p}, \vec{q}]}$ if and only if the condition (†) is satisfied.

10–16. Suppose that, with respect to a given coordinate system, the components of $\vec{m}$ and $\vec{n}$ are (m_1, m_2, m_3) and (n_1, n_2, n_3), respectively. With respect to the given coordinate system, the equations:

$$\text{(i)} \quad \begin{cases} (x_1 - a_1)m_1 + (x_2 - a_2)m_2 + (x_3 - a_3)m_3 = 0 \\ (x_1 - b_1)n_1 + (x_2 - b_2)n_1 + (x_3 - b_3)n_3 = 0 \end{cases}$$

describe nonparallel planes if and only if $(\vec{m}, \vec{n})$ is linearly independent; and, in this case,

$$\text{(ii)} \qquad \left(\begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix}, \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix}, \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}\right)$$

are the components of a non-$\vec{0}$ vector in the direction of the line of intersection of the planes.

Corollary 1. The equations (i) describe parallel planes if and only if $\vec{m}$ and $\vec{n}$ are non-$\vec{0}$ and $(\vec{m}, \vec{n})$ is linearly dependent.

Corollary 2. If $(\vec{m}, \vec{n})$ is linearly independent then the line of intersection of the planes described by the equations (i) is, itself, described by the system:

$$\begin{cases} x_1 = c_1 + \begin{vmatrix} m_2 & m_3 \\ n_2 & n_3 \end{vmatrix} r \\ x_2 = c_2 + \begin{vmatrix} m_3 & m_1 \\ n_3 & n_1 \end{vmatrix} r \\ x_3 = c_3 + \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} r \end{cases}$$

where (c_1, c_2, c_3) are the coordinates of any chosen point C which is common to the two planes.

10–17. Suppose that, with respect to a given coordinate system, the components of $\vec{l}$ and $\vec{p}$ are (l_1, l_2, l_3) and (p_1, p_2, p_3), respectively. With respect to the given coordinate system, the equations:

$$\text{(i)} \qquad l_1 x_1 + l_2 x_2 + l_3 x_3 = e$$

and:

$$\text{(ii)} \qquad \begin{cases} x_1 = a_1 + p_1 r \\ x_2 = a_2 + p_2 r \\ x_3 = a_3 + p_3 r \end{cases}$$

describe a plane and a line which are parallel if and only if $\vec{l}$ and $\vec{p}$ are non-$\vec{0}$ and

$$l_1 p_1 + l_2 p_2 + l_3 p_3 = 0;$$

and, in this case, the line is a subset of the plane if and only if

$$l_1 a_1 + l_2 a_2 + l_3 a_3 = e.$$

Corollary. The equations (i) and (ii) represent a plane and a transversal to this plane if and only if $l_1p_1 + l_2p_2 + l_3p_3 \neq 0$.

10–18. For $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ linearly independent, $(\vec{u}_1a_1 + \vec{u}_2a_2 + \vec{u}_3a_3, \vec{u}_1b_1 + \vec{u}_2b_2 + \vec{u}_3b_3, \vec{u}_1c_1 + \vec{u}_2c_2 + \vec{u}_3c_3)$ is linearly dependent if and only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.

Theorems About Equations

A. The system of equations:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a unique solution if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$; and, in this case, the given system of equations is equivalent to:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Corollary. The system of equations:

$$\begin{cases} a_1x + b_1y = 0 \\ a_2x + b_2y = 0 \end{cases}$$

has a nontrivial solution if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$.

B. For $\left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right) \neq (0, 0, 0)$, $(a_1x + b_1y + c_1z = 0$ and $a_2x + b_2y + c_2z = 0)$ if and only if $\exists_t \left(x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} t, y = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} t\right.$, and $\left. z = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} t\right)$.

C. The system of equations:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

has a unique solution if and only if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ and, in this case, the given system of equations is equivalent to:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Corollary. The system of equations:

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}$$

has a nontrivial solution if and only if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Tabulated Values of Trigonometric Functions

m°	°sin	°cos	°tan	°cot	
0	.0000	1.0000	.0000	–	90
1	.0175	.9998	.0175	57.290	89
2	.0349	.9994	.0349	28.636	88
3	.0523	.9986	.0524	19.081	87
4	.0698	.9976	.0699	14.301	86
5	.0872	.9962	.0875	11.430	85
6	.1045	.9945	.1051	9.5144	84
7	.1219	.9925	.1228	8.1443	83
8	.1392	.9903	.1405	7.1154	82
9	.1564	.9877	.1584	6.3138	81
10	.1736	.9848	.1763	5.6713	80
11	.1908	.9816	.1944	5.1446	79
12	.2079	.9781	.2126	4.7046	78
13	.2250	.9744	.2309	4.3315	77
14	.2419	.9703	.2493	4.0108	76
15	.2588	.9659	.2679	3.7321	75
16	.2756	.9613	.2867	3.4874	74
17	.2924	.9563	.3057	3.2709	73
18	.3090	.9511	.3249	3.0777	72
19	.3256	.9455	.3443	2.9042	71
20	.3420	.9397	.3640	2.7475	70
21	.3584	.9336	.3839	2.6051	69
22	.3746	.9272	.4040	2.4751	68
23	.3907	.9205	.4245	2.3559	67
24	.4067	.9135	.4452	2.2460	66
25	.4226	.9063	.4663	2.1445	65
26	.4384	.8988	.4877	2.0503	64
27	.4540	.8910	.5095	1.9626	63
28	.4695	.8829	.5317	1.8807	62
29	.4848	.8746	.5543	1.8040	61
30	.5000	.8660	.5774	1.7321	60
31	.5150	.8572	.6009	1.6643	59
32	.5299	.8480	.6249	1.6003	58
33	.5446	.8387	.6494	1.5399	57
34	.5592	.8290	.6745	1.4826	56
35	.5736	.8192	.7002	1.4281	55
36	.5878	.8090	.7265	1.3764	54
37	.6018	.7986	.7536	1.3270	53
38	.6157	.7880	.7813	1.2799	52
39	.6293	.7771	.8098	1.2349	51
40	.6428	.7660	.8391	1.1918	50
41	.6561	.7547	.8693	1.1504	49
42	.6691	.7431	.9004	1.1106	48
43	.6820	.7314	.9325	1.0724	47
44	.6947	.7193	.9657	1.0355	46
45	.7071	.7071	1.0000	1.0000	45
	°cos	°sin	°cot	°tan	*m*°

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